Hamiltonian hydrodynamics equations for a quantum fluid in the presence of solitons

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The hydrodynamics of an arbitrary quantum fluid is considered. The complete set of Poisson brackets for quantities describing the state of a quantum fluid in the presence of vortices (solitons) is found by generalizing the relations for vortex-free motion. The hydrodynamic equations of motion are written down, as well as the ensuing conservation laws for the momentum energy and the quantities connected with the intrinsic symmetry group of the system. The degrees of freedom connected with the normal motion are taken into account. The hydrodynamics of superfluid He are considered as an example. The complete set of equations and the conservation laws for rotating He^4 are found. The anisotropic superfluid He^3-A is considered. The variables required for describing the hydrodynamic motion in the presence of continuously distributed solitons, the nondissipative hydrodynamics laws, and the conservation laws are presented.

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The Hamiltonian formalism[1-3] for the description of the hydrodynamics of a quantum superfluid He^3-A and the Hamiltonian formalism for the description of the hydrodynamics of rotating He^4, i.e., in the presence of a continuous distribution of vortices,[4] have been developed in our earlier papers. In the present work, this formalism is generalized to the case of an arbitrary quantum fluid; in particular, the equations of the hydrodynamics of He^3-A in the presence of a continuous distribution of vortices (solitons) are described. A similar ideology has been developed by Dzyaloshinskiĭ and Volovik,[5] who considered the hydrodynamics of spin glasses in the presence of dissipation.

We denote by $\hat{G}_a$ the set of generators which characterize the intrinsic symmetry group of the system. The following relations hold:

$$\hat{G}_a\hat{G}_b-\hat{G}_b\hat{G}_a=t_{ab}{}^c\hat{G}_c. \tag{1}$$

Here t^c_{ab} are structural constant groups satisfying the Jacobi identity.

We assume the system to be a quantum fluid, i.e., we assume the presence of an order parameter ψ. If we disregard the variation of the modulus, then

$$\delta\psi=-i\delta\alpha^a\hat{G}_a\psi$$

always. Correspondingly, we can introduce w by the definition:

$$\nabla_i\psi=iw_i{}^a\hat{G}_a\psi. \tag{2}$$

The curl of the left side of (2) is equal to zero; setting the curl of the right side equal to zero, we find

$$\omega_i{}^a=0,$$

where

$$\omega^a=[\nabla\times\mathbf{w}^a]-{}^1/_2it_{bc}{}^a[\mathbf{w}^b\times\mathbf{w}^c]. \tag{3}$$

Let φ be the canonical conjugate of the variable ψ. Then the density of the quantity corresponding to the generator $\hat{G}_a$ has the form

$$G_a=-i\varphi\hat{G}_a\psi.$$

From this definition, we find the relations for the Poisson brackets:

$$\{G_a(\mathbf{r}_1),\ G_b(\mathbf{r}_2)\}=it_{ab}{}^cG_c\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{4}$$
$$\{G_a(\mathbf{r}_1),\ \psi(\mathbf{r}_2)\}=-i\hat{G}_a\psi\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{5}$$

With account of (5), we find the following from the definition (2):

$$\{G_a(\mathbf{r}_1),\ \mathbf{w}^b(\mathbf{r}_2)\}=\delta_a{}^b\nabla\delta(\mathbf{r}_1-\mathbf{r}_2)-it_{ac}{}^b\mathbf{w}^c\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{6}$$

Similarly, we have $j=-\varphi\nabla\psi$ for the momentum density. We then find

$$\{j_i(\mathbf{r}_1),\ G_a(\mathbf{r}_2)\}=G_a(\mathbf{r}_1)\nabla_i\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{7}$$
$$\{j_k(\mathbf{r}_1),\ j_i(\mathbf{r}_2)\}=j_i(\mathbf{r}_1)\nabla_k\delta(\mathbf{r}_1-\mathbf{r}_2)+\nabla_i(j_k\delta(\mathbf{r}_1-\mathbf{r}_2)), \tag{8}$$
$$\{j_k(\mathbf{r}_1),\ \psi(\mathbf{r}_2)\}=-\nabla_k\psi\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{9}$$

With account of (8) and $\omega^a=0$, we find the following relation from the definition (2):

$$\{j_i(\mathbf{r}_1),\ w_j{}^a(\mathbf{r}_2)\}=\nabla_j(w_i{}^a\delta(\mathbf{r}_1-\mathbf{r}_2))-\nabla_iw_j{}^a\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{10}$$

The state of the quantum fluid is characterized by the order parameter only in the absence of vortices (solitons). In their presence, the "potentials" $\mathbf{w}^a$ (which are the generalization of the superfluid velocity to the case of an arbitrary symmetry group) are no longer connected by the relation (2) with the order parameter and must be considered independently. Correspondingly, the "intensities" $\boldsymbol{\omega}^a$ which characterize the vortex state of the system are not equal to zero. Thus the energy density of the system has the form

$$E=E(G_a,\ j,\ w^a,\ \omega^a,\ \psi).$$

We shall write its differential in the form

$$dE=\mu^a dG_a+\mathbf{v}d\mathbf{j}+\eta_a d\mathbf{w}^a+\lambda_a d\omega^a+\xi d\psi. \quad (11)$$

In order to formulate the equations of motion, we must know the expressions for the mutual Poisson brackets of the quantities on which E depends. In the absence of ω^a, they are given by the expressions (4)–(10). The relations (4), (5), (7)–(9) are directly generalized to the case $\omega^a \neq 0$. The relations (6) and (10) should also be transformed to the case $\omega^a \neq 0$, since it is precisely this generalization that leads to a set of Poisson brackets satisfying the Jacobi identity even at $\omega^a \neq 0$. For ω^a we can find from (6) and (10)

$$\{G_a(\mathbf{r}_1),\ \omega^b(\mathbf{r}_2)\}=-it_{ac}{}^b\omega^c(\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_2), \quad (12)$$

$$\{j_i(\mathbf{r}_1),\ \omega_j{}^a(\mathbf{r}_2)\}=-\delta_{ij}\omega^a(\mathbf{r}_2)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2)+\omega_j{}^a(\mathbf{r}_1)\nabla_i\delta(\mathbf{r}_1-\mathbf{r}_2). \quad (13)$$

The equations of motion are formulated with the help of the Hamiltonian

$$\mathcal{H}=\int d^3r\,E.$$

They have the form

$$\partial G_a/\partial t=\{\mathcal{H},\ G_a\}=-\nabla_i(v_iG_a+\eta_{ia}) +i(\mu^b t_{ba}{}^c G_c+\eta_b t_{ac}{}^b \mathbf{w}^c+\lambda_b t_{ac}{}^b\omega^c+\xi\hat{G}_a\psi), \quad (14)$$

$$\partial\psi/\partial t=\{\mathcal{H},\ \psi\}=-(\mathbf{v}\nabla)\psi-i\mu^a\hat{G}_a\psi, \quad (15)$$

$$\partial w_j{}^a/\partial t=\{\mathcal{H},\ w_j{}^a\}=-\nabla_j\mu^a-i\mu^b t_{bc}{}^a w_j{}^c -w_i{}^a\nabla_j v_i-v_i\nabla_i w_j{}^a, \quad (16)$$

$$\partial j_i/\partial t=\{\mathcal{H},\ j_i\}=-\nabla_k(v_kj_i+\eta_{ak}w_i{}^a-\omega_k{}^a\lambda_{ai}) -(G_a\nabla_i\mu^a+j\nabla_i v+\omega^a\nabla_i\lambda_a-\xi\nabla_i\psi-\eta_a\nabla_i\mathbf{w}^a). \quad (17)$$

From (16), we find

$$\partial\omega^a/\partial t=\{\mathcal{H},\ \omega^a\}=[\nabla\times[\mathbf{v}\times\omega^a]]-i\mu^b t_{bc}{}^a\omega^c-ivt_{bc}{}^a(\mathbf{w}^b\omega^c). \quad (18)$$

The second term in (17) can be incorporated in the gradient of the pressure

$$P=G_a\mu^a+\mathbf{v}\mathbf{j}+\omega^a\lambda_a-E, \quad (19)$$

so that we arrive at the law of momentum conservation.

If the energy density E is invariant relative to $\hat{G}_a$ (here G_a, w^a and ω^a transform according to an associated representation), then the second term on the right side of (14) vanishes and we arrive as we should, at the pure law for the conservation of G_a. From the set of Eqs. (14)–(18), we obtain the law of energy conservation

$$\partial E/\partial t=-\nabla\mathbf{Q}, \quad (20)$$

where the energy flux density is

$$\mathbf{Q}=(P+E)\mathbf{v}+(\mu^a+\mathbf{w}^a\mathbf{v})\eta_a-(\lambda_a\mathbf{v})\omega^a. \quad (21)$$

If the symmetry group of the system breaks up into the direct product of subgroups, then, generally speaking, it is necessary to take into account the dependence of the energy density E on the densities of the momenta that pertain to each subgroup. The rules for the (4)–(10) Poisson brackets and the equations of motion (14)–(18) that follow from them, are taken here to pertain to each subgroup. Moreover, zero temperature was assumed. At non-zero temperature, it is necessary to take into account the dependence of the energy density E on the entropy density s and on the normal momentum density p. They are taken to pertain to a one-parameter subgroup and, in correspondence with (14) and (17), we have the following equations for them:

$$\partial s/\partial t+\nabla(s\mathbf{v}^{(n)})=0, \quad (22)$$

$$\partial p_i/\partial t+\nabla(\mathbf{v}^{(n)}p_i)=-\mathbf{p}\nabla_i\mathbf{v}^{(n)}-s\nabla T, \quad (23)$$

where $\mathbf{v}^{(n)}=\partial E/\partial\mathbf{p}$ is the normal velocity and $T=\partial E/\partial s$ is the temperature.

We consider the case of superfluid He^4. In this case, the intrinsic symmetry group of the system is a one-parameter group of gauge transformations; the role of conserved quantity which corresponds to the generator of the gauge transformations is played here by the mass density ρ. In the presence of vortices, i.e., in rotating He^4, ω has the meaning of the local angular velocity of the rotation. The energy density E is the function

$$E=E(\mathbf{j},\ \rho,\ \omega,\ s,\ \mathbf{p}).$$

By virtue of the Galilean invariance, the total momentum density is

$$\mathbf{g}=\mathbf{j}+\mathbf{p}=\rho\mathbf{v}. \quad (24)$$

Thus $\mathbf{v}$ has the meaning of the mean mass velocity.

In correspondence with (14), (17), and (18), we have

$$\partial\rho/\partial t=-\nabla(\rho\mathbf{v}), \quad (25)$$

$$\partial\mathbf{j}/\partial t+\nabla_i(v_i\mathbf{j})=-j_i\nabla v_i-\rho\nabla\mu+[[\nabla\times\lambda]\times\omega], \quad (26)$$

$$\partial\omega/\partial t=[\nabla\times[\mathbf{v}\omega]]. \quad (27)$$

As is seen from (24) and (25), the mass flux density is identical with the momentum density, as it should be.

Equations (22), (23), (25)–(27) form a complete set of equations describing the hydrodynamics of rotating He^4 and obtained previously by the authors.[4] The law of momentum conservation,

$$\partial g_i/\partial t+\nabla_k\Pi_{ik}=0, \quad (28)$$

corresponds to these equations. Here the stress tensor is

$$\Pi_{ik}=P\delta_{ik}+p_iv_k^{(n)}+j_iv_k-\lambda_i\omega_k \quad (29)$$

and the pressure is determined by the expression (19). The law of energy conservation (20) also holds, with the energy flux density

$$\mathbf{Q}=(\mathbf{v}\mathbf{j})\mathbf{v}+\mu\rho\mathbf{v}+[[\omega\times\mathbf{v}]\times\lambda]+(\mathbf{p}\mathbf{v}^{(n)})\mathbf{v}^{(n)}+Ts\mathbf{v}^{(n)}. \quad (30)$$

We now proceed to consideration of the anisotropic phase of superfluid He^3.[6] In this phase, the structure of the order parameter is more complicated than in He^4; in addition to the gauge transformations, it is necessary to take into account rotations in orbital and spin space. The generators of these transformations are the densities of the orbital and spin angular momenta $\mathbf{L}$ and $\mathbf{S}$. In the absence of vortices, the energy density E depends on the orbital $\mathbf{l}$ and spin $\mathbf{n}$ anistropy vectors and their derivatives, in place of which, in the presence of vortices, it is necessary to take into account the potentials $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$.

The energy depends explicitly on the order parameter through the anisotropy vectors, which we denote by $\xi^{(l)}=\gamma E/\partial \mathbf{l}$ and $\xi^{(m)}=\partial E/\partial \mathbf{m}$. The equations corresponding to the expressions (14)–(17) have the form (for the orbital variables)

$$\partial \mathbf{l}/\partial t+(\mathbf{v}\nabla)\mathbf{l}=[\boldsymbol{\mu}^{(L)}\times \mathbf{l}], \tag{31}$$

$$\partial \mathbf{L}/\partial t+\nabla_i(v_i\mathbf{L}+\boldsymbol{\eta}_i^{(l)})=[\boldsymbol{\mu}^{(L)}\times \mathbf{L}]+[\boldsymbol{\eta}_i^{(l)}\times \mathbf{w}_i^{(l)}]+[\boldsymbol{\lambda}_i^{(l)}\times \boldsymbol{\omega}_i^{(l)}]+[\boldsymbol{\xi}^{(l)}\times \mathbf{l}], \tag{32}$$

$$\partial \mathbf{w}_j^{(l)}/\partial t+(\mathbf{v}\nabla)\mathbf{w}_j^{(l)}=-\nabla_j\boldsymbol{\mu}^{(L)}+[\boldsymbol{\mu}^{(L)}\times \mathbf{w}_j^{(l)}]-\mathbf{w}_i^{(l)}\nabla_j v_i, \tag{33}$$

$$\partial \boldsymbol{\omega}_\alpha^{(l)}/\partial t-[\nabla\times[\mathbf{v}\times\boldsymbol{\omega}_\alpha^{(l)}]]=e_{\alpha\beta\gamma}(\mu_\beta^{(L)}\boldsymbol{\omega}_\gamma^{(l)}+\mathbf{v}(\mathbf{w}_\beta^{(l)}\boldsymbol{\omega}_\gamma^{(l)})), \tag{34}$$

where $\mu^{(L)}=\partial E/\partial \mathbf{L}$. The equations for the orbital subsystem are obtained from (31)–(32) by the substitutions $\mathbf{l}\to\mathbf{n}$ and $\mathbf{L}\to\mathbf{S}$. For completeness of the system, it is necessary to add (22), (23), (25), and (27) to these equations, and also the equation for the superfluid momentum

$$\begin{aligned}\partial j_i/\partial t+\nabla_k(v_k j_i+\boldsymbol{\eta}_k^{(l)}\mathbf{w}_i^{(l)}+\boldsymbol{\eta}_k^{(n)}\mathbf{w}_i^{(n)}-\boldsymbol{\omega}_k\boldsymbol{\lambda}_i-\boldsymbol{\omega}_k^{(l)}\boldsymbol{\lambda}_i^{(l)}-\boldsymbol{\omega}_k^{(n)}\boldsymbol{\lambda}_i^{(n)})\\ =-\mathbf{L}\nabla_i\boldsymbol{\mu}^{(L)}-\mathbf{S}\nabla_i\boldsymbol{\mu}^{(s)}-\rho\nabla_i\mu-\mathbf{j}\nabla_i\mathbf{v}+\boldsymbol{\eta}_j^{(l)}\nabla_i\mathbf{w}_j^{(l)}\\ +\boldsymbol{\eta}_j^{(n)}\nabla_i\mathbf{w}_j^{(n)}-\boldsymbol{\omega}_j\nabla_i\boldsymbol{\lambda}_j-\boldsymbol{\omega}_j^{(l)}\nabla_i\boldsymbol{\lambda}_j^{(l)}\\ -\boldsymbol{\omega}_j^{(n)}\nabla_i\boldsymbol{\lambda}_j^{(n)}+\boldsymbol{\xi}^{(l)}\nabla_i\mathbf{l}+\boldsymbol{\xi}^{(n)}\nabla_i\mathbf{n}.\end{aligned} \tag{35}$$

In consideration of real processes, it is necessary to keep in mind the kinetic terms that must be added to the right side of the foregoing equations. In particular, the kinetic term proportional to $\mu^{(L)}$ plays a significant role on the right side of (32), since it plays the principal role at hydrodynamic frequencies.[7]

The hydrodynamic equations of He^3-A lead to the laws of conservation of momentum and energy. The law of momentum conservation has the form (28) with the stress tensor

$$\Pi_{ik}=P\delta_{ik}+v_k j_i+v_k^{(n)}p_i+\boldsymbol{\eta}_k^{(l)}\mathbf{w}_i^{(l)}+\boldsymbol{\eta}_k^{(n)}\mathbf{w}_i^{(n)}-\boldsymbol{\omega}_k\boldsymbol{\lambda}_i-\boldsymbol{\omega}_k^{(l)}\boldsymbol{\lambda}_i^{(l)}-\boldsymbol{\omega}_k^{(n)}\boldsymbol{\lambda}_i^{(n)}, \tag{36}$$

where the pressure, in correspondence with (19), is

$$P=\mu\rho+\mathbf{L}\boldsymbol{\mu}^{(L)}+\mathbf{S}\boldsymbol{\mu}^{(s)}+sT+\mathbf{v}\mathbf{j}+\mathbf{v}^{(n)}\mathbf{p}+\boldsymbol{\omega}\boldsymbol{\lambda}+\boldsymbol{\omega}_\alpha^{(l)}\boldsymbol{\lambda}_\alpha^{(l)}+\boldsymbol{\omega}_\alpha^{(n)}\boldsymbol{\lambda}_\alpha^{(n)}-E. \tag{37}$$

The law of energy conservation has the form (29) with the energy flux density

$$\mathbf{Q}=(P+E-sT-\mathbf{v}^{(n)}\mathbf{p})\mathbf{v}+(sT+\mathbf{v}^{(n)}\mathbf{p})\mathbf{v}^{(n)}+(\boldsymbol{\mu}_\alpha^{(L)}+w_{i\alpha}^{(l)}v_i)\boldsymbol{\eta}_\alpha^{(l)}+(\boldsymbol{\mu}_\alpha^{(s)}+w_{i\alpha}^{(n)}v_i)\boldsymbol{\eta}_\alpha^{(n)}-(\lambda_{i\alpha}^{(l)}v_i)\boldsymbol{\omega}_\alpha^{(l)}-(\lambda_{i\alpha}^{(n)}v_i)\boldsymbol{\omega}_\alpha^{(n)}-(\lambda_i v_i)\boldsymbol{\omega}. \tag{38}$$

Thus, the developed Hamiltonian approach enables us to obtain nondissipative equations of hydrodynamics of a quantum fluid, which automatically lead to the laws of conservation energy and momentum as well as of the quantities connected with the invariance to the intrinsic symmetry group.

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