Bound states of magnons in a spin chain with "easy-plane" anisotropy

G. I. Georgiev, Yu. A. Kosevich, and V. M. Tsukernik

A. M. Gor'kii Khar'kov State University
and Physicotechnical Institute for Low Temperatures, Academy of Sciences of the Ukrainian SSR
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The bound stationary states of an arbitrary number of inverted spins (spin complexes) in a chain with uniaxially anisotropic Heisenberg interaction are investigated when the anisotropy parameter (the ratio of the longitudinal to the transverse exchange constant) is less than unity. It is shown that, for a given number of inverted spins, there exists a finite set of nonoverlapping ranges of anisotropy-parameter values for which such a bound state is possible. It is shown that, in a particular interval of values of the anisotropy parameter, a spin complex with a specified number of inverted spins is energetically the most favored.

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The one-dimensional spin system that we are considering is described by a Hamiltonian with an anisotropic Heisenberg interaction:

$$\mathscr{H}=-J\sum_m(s_m^x s_{m+1}^x+s_m^y s_{m+1}^y+\gamma s_m^z s_{m+1}^z)-2\mu H\sum_m s_m. \tag{1}$$

Here, s_m^j is the operator of the j-th spin projection at the m-th site ($s=\frac{1}{2}$), J is the positive nearest-neighbor exchange interaction constant, μ is the Bohr magneton, H is the external uniform magnetic field, and γ is the positive anisotropy parameter.

If $\gamma\gg 1$ the ground state of the system is ferromagnetic for any value of the field H, and for $\gamma>1$ the z axis coincides with the easy axis. In papers by Ovchinnikov[1] and Gochev[2] it was shown that bound states (spin complexes) of arbitrary length and with arbitrary quasimomentum exist in such a chain.[1)]

In the present paper, the bound states in a chain with the Hamiltonian (1) with $0\leqslant\gamma\leqslant 1$ are investigated. It is found that, in this case, a spin complex of given length n (with n inverted spins) does not exist, generally speaking, for all values of γ in the interval $(0,1)$.[2)] For a given n the region of existence of the complex consists of a set of intervals of values of γ, the lengths and number of which are determined by the number n.

1. The ground state of the system that we are considering is ferromagnetic only when $2\mu H\geqslant(1-\gamma)J$. Henceforth we shall assume that the external field satisfies this condition.[7] Inasmuch as, in the system,

$$\sum_m s_m{}^z=s^z$$

is conserved, the stationary states can be classified according to the number of inverted spins. The vector of a stationery state with n inverted spins can be expanded in the complete set of the corresponding site vectors:

$$|n\rangle=\sum_{m_1<m_2<\ldots<m_n}B_{m_1m_2\ldots m_n}s_{m_1}^- s_{m_2}^-\ldots s_{m_n}^-|0\rangle, \tag{2}$$

where $s_{mj}^-=s_{mj}^x-is_{mj}^y$, and $|0\rangle$ is the vector of the ground ferromagnetic state.

For a bound state of n inverted spins the wavefunction $B_{m_1m_2\ldots m_n}$ can be represented in the form[2]

$$B_{m_1m_2\ldots m_n}=\exp\left\{i\frac{k}{n}(m_1+m_2+\ldots+m_n)\right\}\prod_{\nu=1}^{n-1}r_\nu^{m_{\nu+1}-m_\nu}, \tag{3}$$

where k is the quasimomentum of the complex. Here the first factor describes the motion of the complex as a whole, and the second describes the relative motion of the inverted spins. From the Schrödinger equation for the stationary states with n inverted spins we obtain[2] a system of equations for the parameters r_ν:

$$2\gamma r_\nu=e^{-ik/n}r_{\nu+1}+e^{ik/n}r_{\nu-1},\quad \nu=1,\ 2,\ldots,\ n-1 \tag{4}$$

with the additional condition $r_0=r_n=1$.

The solution of this system has the form

$$r_\nu=\sin^{-1}n\varkappa\,[e^{ik\nu/n}\sin(n-\nu)\varkappa+e^{-ik(n-\nu)/n}\sin\nu\varkappa], \tag{5}$$

where we have introduced the notation

$$\gamma=\cos\varkappa,\quad 0\leqslant\varkappa\leqslant\pi/2. \tag{6}$$

For the energy of the complex, reckoned from the ferromagnetic ground state, we obtain the following expression:

$$\varepsilon_n(k)=2n\mu H+J\frac{\sin\varkappa}{\sin n\varkappa}(\cos n\varkappa-\cos k). \tag{7}$$

2. From the requirement that the wavefunction (3) be bounded in all the arguments $m_{j+1}-m_j$ ($j=1,2,\ldots,n$) follow the conditions $|r_\nu|\leqslant 1$, and for the bound states we must have

$$|r_\nu|<1,\quad \nu=1,\ 2,\ldots,\ n-1. \tag{8}$$

From (5) and (8) for $\sin n\varkappa\neq 0$ we obtain the following system of inequalities:

$$\sin\nu\varkappa\sin(n-\nu)\varkappa(\cos n\varkappa-\cos k)>0,\quad \nu=1,\ 2,\ldots,\ n-1. \tag{9}$$

For a bound state it is necessary that all the inequalities (9), which we shall regard as condition for the param-

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eters $\varkappa$ and k, be fulfilled simultaneously. To obtain these conditions we note first of all that, for fixed $\varkappa$ and k, in the inequalities (9) the products $\sin\nu\varkappa\sin(n-\nu)\varkappa$ should have the same sign for all $\nu=1,2,\ldots,n-1$. This imposes restrictions on the allowed values of $\varkappa$.

If we put

$$\varkappa=\varkappa_l=l\pi/n, \quad l=1,\ 2,\ldots,\ [n/2], \tag{10}$$

then

$$\sin\nu\varkappa_l\sin(n-\nu)\varkappa_l=(-1)^{l+1}\sin^2\nu\varkappa_l \tag{11}$$

and the sign of the product is the same for all ν ($[a]$ denotes the integer part of the number a). The number l can be regarded as the label of the allowed interval for $\varkappa$. The values of ν at which each of the factors in (11) changes sign are determined by the inequalities

$$\nu\varkappa_l<m\pi, \quad (\nu+1)\varkappa_l>m\pi, \quad m=1,\ 2,\ldots,\ l-1.$$

Hence, $\nu=\nu_m=[mn/l]$, and the l-th allowed interval of values of $\varkappa$ is determined by the inequalities

$$\max_{(m)}\frac{m\pi}{[mn/l]+1}<\varkappa<\min_{(m)}\frac{m\pi}{[mn/l]}, \tag{12}$$

where $m=1,2,\ldots,l-1$, and none of the numbers mn/l should be an integer. Otherwise, there is no interval with the corresponding l. Thus, the only allowed values of l are numbers that are coprime with n.

To indicate the algorithm for finding the upper limit of the interval (12) we represent the ratio n/l in the form

$$n/l=a_0+r_0/l, \quad r_0<l, \quad a_0=[n/l].$$

Then,

$$[mn/l]=ma_0+[mr_0/l].$$

As a result,

$$\frac{m}{[mn/l]}=\frac{1}{a_0+m^{-1}[mr_0/l]}.$$

Hence,

$$\min_{(m)}\frac{m}{[mn/l]}=1\Big/\left\{a_0+\max_{(m)}\frac{1}{m}\left[\frac{mr_0}{l}\right]\right\}.$$

For a given value of $[mr_0/l]=m_1$ the maximum of the fraction $m^{-1}[mr_0/l]$ is reached when $m=[m_1l/r_0]+1$. Hence,

$$\max_{(m)}\frac{1}{m}\left[\frac{mr_0}{l}\right]=\max_{(m_1)}\frac{m_1}{[m_1l/r_0]+1}=1\Big/\left\{a_1+\min_{(m_1)}\frac{1}{m_1}\left(1+\left[\frac{m_1r_1}{r_0}\right]\right)\right\},$$

where $l/r_0=a_1+r_1/r_0$, and $m_1=1,2,\ldots,r_0-1$.

Introducing, analogously, $m_2=[m_1r_1/r_0]+1$, we arrive at the result that for a given m_2 the minimum is reached when $m_1=[m_2r_0/r_1]$, and

$$\min_{(m_1)}\frac{1}{m_1}\left(1+\left[\frac{m_1r_1}{r_0}\right]\right)=\min_{(m_2)}\frac{m_2}{[m_2r_0/r_1]},$$

where $m_2=1,2,\ldots,r_1-1$.

Thus, the problem has been reduced to the determination of the minimum of an expression of the same form as the original expression, but for a smaller set of values of the variable m_2. Continuing this procedure, we finally arrive at the relation

$$r_{q-2}=a_qr_{q-1}+1 \quad (r_{q-1}>1)$$

(we recall that n and l are coprime). The corresponding m_q, which realizes the extremum, is equal to $r_{q-1}-1$.

The lower limit of the l-th interval is found analogously.

As a result, the algorithm for determining the limits of the intervals reduces to finding the remainders on successive division of n and l (in accordance with Euclid's algorithm):

$$\begin{gathered}n=a_0l+r_0, \quad l=a_1r_0+r_1,\\ r_0=a_2r_1+r_2,\\ \cdots\cdots\\ r_{q-3}=a_{q-1}r_{q-2}+r_{q-1},\\ r_{q-2}=a_qr_{q-1}+1.\end{gathered}$$

The desired values of m for the upper and lower limits of the interval are found by means of the recurrence relations.

a) for the upper limit of the interval,

$$m_{2k}=\left[m_{2k+1}\frac{r_{2k-1}}{r_{2k}}\right]+1, \quad m_{2k-1}=\left[m_{2k}\frac{r_{2k-2}}{r_{2k-1}}\right].$$

b) for the lower limit,

$$m_{2k}=\left[m_{2k+1}\frac{r_{2k-1}}{r_{2k}}\right], \quad m_{2k-1}=\left[m_{2k}\frac{r_{2k-2}}{r_{2k-1}}\right]+1,$$

where we have put $m_0=m$, $r_{-1}=l$, $r_{-2}=n$.

In addition to the intervals considered, for any n the interval

$$0<\varkappa<\pi/(n-1), \tag{13}$$

in which $\sin\nu\varkappa$ and $\sin(n-\nu)\varkappa$ are positive for all ν, is an allowed interval. Formally, this interval can be regarded as corresponding to $l=1$.

As $\varkappa$ tends to one of the limits of the intervals (12) the parameters r_ν and $r_{n-\nu}$, with index $\nu=[mn/l]$ corresponding to a limit value of $\varkappa$, tend in modulus to unity.[3)] This means that the wavefunction (3) of the spin complex does not vanish in the limit $m_{\nu+1}-m_\nu\to\infty$ or $m_{n-\nu+1}-m_{n-\nu}\to\infty$ for this ν. In other words, "disintegration" of the spin complex occurs, i.e., the complex of n inverted spins ceases to exist as a single entity. As $\varkappa$ tends to the right-hand limit of the interval (13), $|r_1|\to1$ and $|r_{n-1}|\to1$, i.e., the complex disintegrates at its ends. With increase of the label l of the interval the points of disintegration approach the middle of the complex.

As follows from (9) and (11), for intervals with even l,

$$\cos n\varkappa<\cos k, \tag{14}$$

and for intervals with odd l,

$$\cos n\varkappa>\cos k. \tag{15}$$

Consequently, in the former case long-wavelength (in the limit, stationary) complexes exist, and in the latter case short-wavelength complexes exist.

A quasiclassical description[3-5] of a magnetic system with the Hamiltonian (1) with $\gamma\geqslant1$ has revealed the existence of localized oscillations of the magnetization vector and has shown that the results obtained by a consistently quantum-mechanical procedure go over into the classical results when $n \gg 1$. The analogous analysis in our case ($\gamma < 1$) results in absence of localized conditions (the classical analog of a spin complex). On the other hand, as $n \to \infty$ the lengths of the intervals (12) and (13) that we have found tend to zero. Thus, for $\gamma < 1$ the classical description corresponds to the absence of ranges of values of γ for which spin complexes exist.

As pointed out above, the conditions for the existence of spin complexes have reduced to the inequalities (9) with $\sin n\varkappa \neq 0$, i.e., with $\varkappa \neq l\pi/n$ (l is an integer). It can be seen from (12) and (13) that $\varkappa = l\pi/n$ ($l = 1, 2, \ldots, [n/2]$) fall in the allowed intervals. The conditions (14) and (15) show that the intervals of allowed values of k collapse to the points $k=0$ and $k=\pm\pi$, respectively. The limit values r_ν are obtained from (5) on substituting these values of k and then taking the limit $\varkappa \to l\pi/n$. As a result we obtain

$$|r_\nu| = \begin{cases} |\cos \nu l\pi/n| & \text{for even } l, \\ |\sin \nu l\pi/n| & \text{for odd } l. \end{cases}$$

3. In a paper by Gochev[2] it was shown that for $\gamma > 1$ the energy of a complex of length n is always lower than the energy of any other state with n inverted spins. In our case ($\gamma < 1$) we cannot make such a statement. The difference between the total energy of two complexes of lengths p and $n-p$ with momenta k_1 and $k-k_1$ and the energy of a complex of length n with momentum k can be represented in the form

$$\varepsilon_p(k_1)+\varepsilon_{n-p}(k-k_1)-\varepsilon_n(k)=J\frac{\sin\varkappa\sin n\varkappa}{2\sin p\varkappa\sin(n-p)\varkappa}\Big[1+|r_p|^2 -2|r_p|\cos(k_1-\varphi)\,\mathrm{sign}\frac{\sin n\varkappa}{\sin p\varkappa\sin(n-p)\varkappa}\Big], \qquad (16)$$

where r_p is determined by the expression (5), and

$$\operatorname{tg}\varphi=\frac{\sin p\varkappa\sin k}{\sin(n-p)\varkappa+\sin p\varkappa\cos k}.$$

As can be seen from (16), the sign of this difference is determined by the sign of the ratio $\sin n\varkappa/\sin p\varkappa \sin(n-p)\varkappa$, since the expression in square brackets is always positive. Each of the allowed intervals (12) and (13) is divided into two intervals, in one of which this ratio is positive while in the other it is negative. The difference under consideration vanishes at the points $\varkappa_1 = l\pi/n$. In the intervals in which the difference is positive, a complex of n magnons is energetically more favorable than two individual complexes with the same total length and total momentum. In the case of a negative difference we must, in addition, ascertain whether it is possible for two such complexes to exist in the corresponding interval of $\varkappa$. In particular, in the first interval

$$\frac{\pi}{n}<\varkappa<\frac{\pi}{n-1} \qquad (17)$$

this possibility is not realized, since the inequalities

$$\cos n\varkappa>\cos k, \quad \cos p\varkappa>\cos k_1, \quad \cos(n-p)\varkappa>\cos(k-k_1)$$

are found to be incompatible. The impossibility of division into two complexes implies at the same time that division into a larger number is also impossible, inasmuch as the interval (13) is the fundamental interval for any complex of length less than n. Thus, in the first, fundamental interval (13), a complex of length n, as in the case $\gamma > 1$, is energetically more favorable than any other stationary state with n inverted spins.

For the second interval, which exists for odd n:

$$\frac{2\pi}{n+1}<\varkappa<\frac{2\pi}{n}, \qquad (18)$$

division into complexes with lengths $(n-1)/2$ and $(n+1)/2$, for which the conditions of existence are compatible, is possible. For any other division the compatibility is destroyed. For each of the complexes the interval (18) is part of the fundamental interval, i.e., further division is now impossible.

To investigate the question of all the possible divisions of a complex for values of $\varkappa$ from the interval (12) with arbitrary label l requires the analysis of quantum states with many spin complexes for a given total number of inverted spins and not just of states with one complex, and goes beyond the scope of the present article.

1) In the case $\gamma \geq 1$ a classical investigation of the localized magnetization oscillations has been carried out by solving the nonlinear Landau-Lifshitz equation.[3-5]

2) The case $n=2$ for $0<\gamma<1$ has been considered earlier.[6]

3) Since more than one value of m, and, consequently, of ν, can correspond to a limit value of $\varkappa$, the number of pairs of parameters r_ν and $r_{n-\nu}$ tending in modulus to unity can also be greater than one.

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