Theory of Vavilov-Cerenkov emission in cholesteric liquid crystals

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A development is given of the theory of Vavilov-Cerenkov emission in cholesteric liquid crystals (CLC) under the conditions of diffraction scattering (in higher orders) by the periodic structure of the CLC. Analytic expressions are obtained for the angular, frequency, and polarization characteristics of the radiation in the case of finite CLC specimens. The corresponding expression for radiation losses is also established. It is shown that under the conditions of diffraction scattering, the differential (with respect to angle and frequency) characteristics of the Vavilov-Cerenkov radiation in CLC are very different from the corresponding characteristics in a homogeneous medium although the integrated characteristics, for example, radiation losses, are the same as the corresponding quantities for a homogeneous medium whose permittivity is identical with the permittivity of the CLC.

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INTRODUCTION

It is well known that the unusual optical properties of cholesteric liquid crystals (CLC) are due to their complicated spatial periodic structure.[1,2] Naturally, the optical properties related to the structure of CLC are also seen in the coherent radiation emitted by fast charged particles. The coherent radiation emitted by uniformly moving charged particles in CLC has been considered theoretically by a number of workers.[3–5] It has been shown that, generally speaking, the character of the coherent radiation emitted by fast particles in CLC is similar to that due to particles in media with a simple type of periodicity,[6,7] for example, layered media or media whose permittivity varies periodically in space and is a point function of position. In particular, in addition to the Vavilov-Cerenkov radiation, the condition for which is that the velocity v of the particle must be greater than the phase velocity c_{ph} of light in the medium, cholesteric liquid crystals emit another coherent radiation which is due to the spatial periodicity of the crystal[4] and is occasionally referred to as the structural Vavilov-Cerenkov radiation, the quasi-Cerenkov or the parametric Cerenkov radiation. The Vavilov–Cerenkov radiation emitted in a CLC is diffracted by its periodic structure, so that it emerges not only along the generator of the well-known Cerenkov cone, whose axis lies along the velocity v and has an aperture angle ψ given by $\cos\psi = c_{ph}/v$, but also along the generator of the so-called diffraction cone, the axis of which is not parallel to the particle velocity.

In addition, the emission of Vavilov–Cerenkov radiation in the case of the CLC exhibits a number of specific features due to the complicated (helicoidal) spatial structure of the medium. These features are seen, above all, in the polarization properties of the radiation and in some of the details of its angular distribution. Kats[3] was the first to analyze the loss of energy by a particle through radiation in the CLC. The polarization of the Vavilov–Cerenkov radiation in the CLC is, in general, elliptic, and the emission cones (Cerenkov and diffraction) have a fine structure, so that there are several slightly different cone angles, each of which has its own polarization. These properties of the Vavilov–Cerenkov radiation in the CLC have been established on the basis of a qualitative analysis of the corresponding equations for first-order diffraction scattering of the radiation.[2,5] An analytic description of the Vavilov–Cerenkov emission was obtained for this case only for one direction, namely, the direction of the optic axis of the CLC.

For a more detailed and qualitative analysis of the question, it is therefore interesting to examine the emission of Vavilov–Cerenkov radiation in a case for which an analytic solution can be obtained. This problem is tackled in the present paper in which a theoretical analysis is given of the emission of Vavilov–Cerenkov radiation in a CLC under conditions corresponding to higher-order diffraction scattering of the radiation emitted by the particle. Analytic expressions are obtained for the angular polarization and frequency characteristics of the Vavilov–Cerenkov radiation in the CLC under these conditions. The physical reason why relatively simple analytic expressions are obtained is that the anisotropy in the dielectric properties of the CLC is small and typical values of δ (see below) are of the order of 0.1–0.01.

The Vavilov–Cerenkov emission problem is solved for an infinite CLC and for a specimen in the form of a plane-parallel plate of finite thickness with the optic axis perpendicular to the faces but arbitrary direction of the particle velocity. A detailed analysis is given of the emission of this radiation when the particle moves in a direction parallel to the CLC optic axis.

The character of the Vavilov–Cerenkov radiation under the conditions of high-order diffraction scattering is such that the total intensity of the Vavilov–Cerenkov radiation turns out to be the same to within δ as the intensity in the homogeneous medium with permittivity equal to the mean permittivity of the CLC. The presence of periodicity in the CLC manifests itself in the angular and frequency redistribution of the

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Vavilov–Cerenkov radiation intensity and it its polarization properties which will be considered below.

EMISSION IN AN INFINITE CLC

Basic set of equations. Consider the radiation emitted by a charged particle moving in an unbounded cholesteric crystal with constant velocity exceeding the phase velocity c_{ph} of light in the CLC.

The Maxwell equations for the electric field $\mathbf{E}(\mathbf{r}, t)$ in the CLC, due to a particle moving through it, yield the following equation:

$$-\operatorname{rot}\operatorname{rot}\mathbf{E}=\frac{1}{c^2}\hat{\varepsilon}(\mathbf{r})\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t},$$

where c is the velocity of light; $j=e\mathbf{v}\delta(vt-z)$ is the particle current, e and $\mathbf{v}$ are the charge and velocity of the particle, z is the particle coordinate at time t (the z axis lies along the particle velocity), and $\varepsilon(\mathbf{r})$ is the permittivity of the CLC, which is given by[1]

$$\hat{\varepsilon}(\mathbf{r})=\frac{\varepsilon_1+\varepsilon_2}{2}\begin{bmatrix}1+\delta\cos 2\varphi' & \delta\sin 2\varphi' & 0\\ \delta\sin 2\varphi' & 1-\delta\cos 2\varphi' & 0\\ 0 & 0 & \dfrac{2\varepsilon_3}{\varepsilon_1+\varepsilon_2}\end{bmatrix}, \tag{1}$$

where $\varphi'=2\pi z'/p$, p is the pitch of the cholesteric spiral, the z' axis lies along the optic axis of the CLC, ε_1, $\varepsilon_2=\varepsilon_3$ are the principal values of the permittivity tensor, and $\delta=(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)$.

We shall seek the solutions of the above equations in the form of a superposition of Bloch waves:

$$\mathbf{E}(\mathbf{r},t)=\int e^{-i(\omega t-\mathbf{kr})}d\mathbf{k}d\omega\sum_{s=-\infty}^{\infty}\mathbf{E}_s(\mathbf{k}_s)\exp(i\boldsymbol{\tau}_s\mathbf{r}), \tag{2}$$

where the reciprocal-lattice vectors $\boldsymbol{\tau}_s$ are defined by $\boldsymbol{\tau}_s=s\mathbf{z}'\times 4\pi/p$, and s is an integer. Equations (1) and (2) lead to the following infinite set of linear equations for $\mathbf{E}_s$:

$$-\mathbf{k}_s^2\mathbf{E}_s+\frac{\omega^2}{c^2}\sum_t\hat{\varepsilon}_{s-t}\mathbf{E}_t+\mathbf{k}_s(\mathbf{k}_s\mathbf{E}_s)=-\frac{ie\omega\mathbf{v}}{2\pi^2c^2}\delta(\omega-\mathbf{k}_0\mathbf{v}), \tag{3}$$

where $\mathbf{k}_s=\mathbf{k}_{s-1}+\boldsymbol{\tau}$, and ε_s are the Fourier components of the permittivity tensor of the CLC.

It is well known that analysis of (3) will show that, when δ is small, one or more of the amplitudes $\mathbf{E}_s$ in (2) are greater than all the others by a factor of at least δ^{-1}. Apart from the wave amplitude set by the source, which we shall denote by $\mathbf{E}_0$, for which (3) shows that

$$\omega-\mathbf{k}_0\mathbf{v}=0, \tag{4}$$

the only other large amplitudes will be the amplitudes $\mathbf{E}_s$ for which the following Bragg condition is satisfied (see Fig. 1):

$$\mathbf{k}_s=\mathbf{k}_0+s\boldsymbol{\tau},\quad |\mathbf{k}_s|=|\mathbf{k}_0|. \tag{5}$$

It is therefore possible to obtain an approximate solution of (3) by replacing it with a finite set of equations containing only the large amplitudes $\mathbf{E}_s$. This method of solving the Vavilov–Cerenkov problem for the CLC was used previously[5] in the special case where (3) could be replaced by a set of two equations for the

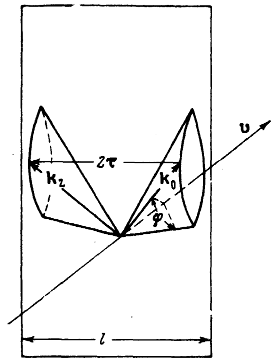

FIG. 1. Angular distribution of Vavilov–Cerenkov radiation in a cholesteric liquid crystal (k_0 and k_2 are wave vectors corresponding to radiation on the Cerenkov and diffraction cones, respectively, and τ defines the orientation of the optic axis of the CLC).

amplitudes of two waves (this is the so-called two-wave approximation). Here, we shall consider the situation in which $\mathbf{E}_0$ and, in general, n further amplitudes must be assumed to be nonzero. To be specific, we shall examine in detail the three-wave case ($n=2$) and then reproduce the results generalized to the case of arbitrary n.

The $n=2$ case enables us to take into account the effect of second-order diffraction scattering in the CLC on Vavilov–Cerenkov emission [$|s|=2$ in (5)]. The form of the permittivity tensor of the CLC given by (1) turns out to be very important for the ensuing analysis. It follows from this expression that the only nonzero harmonics in the Fourier expansion of the permittivity are those corresponding to $s=0, \pm 1$. The fact that the Fourier harmonics are suppressed for $|s|\geq 1$ means that direct diffraction scattering of $\mathbf{E}_0$ into $\mathbf{E}_2$ (and, in general, into other waves with $|s|\geq 2$) is absent, so that the two-wave approximation in which only $\mathbf{E}_0$ and $\mathbf{E}_2$ are nonzero cannot be used. The first approximation in which second-order diffraction scattering can be taken into account, and to which we shall now confine our attention, is the three-wave approximation.

If we retain in (3) only those equations that contain the amplitudes $\mathbf{E}_0$, $\mathbf{E}_1$, and $\mathbf{E}_2$, and then eliminate the amplitude $\mathbf{E}_1$ from the resulting set of equations, as was done in our previous paper,[8] we obtain the following set of equations for $\mathbf{E}_0$ and $\mathbf{E}_2$:

$$\begin{gathered}\left(1-\frac{k_0^2}{\varkappa^2}+\hat{F}_\parallel\right)\mathbf{E}_0+\hat{F}_{02}\mathbf{E}_2=-\frac{ie\delta(\omega-\mathbf{k}_0\mathbf{v})}{2\pi^2\omega\varepsilon}\left[\mathbf{v}-\frac{\mathbf{k}_0(\mathbf{k}_0\mathbf{v})}{\varkappa^2}\right],\\ \hat{F}_{02}^*\mathbf{E}_0+\left(1-\frac{k_2^2}{\varkappa^2}+\hat{F}_\parallel\right)\mathbf{E}_2=0,\\ \varkappa^2=\omega^2\varepsilon/c^2,\quad \varepsilon=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2)\left(1-\tfrac{1}{2}\delta\cos^2\theta\right),\\ \hat{F}_\parallel=\frac{\delta}{2}\begin{pmatrix}\cos^2\theta & 0\\ 0 & -\cos^2\theta\end{pmatrix},\quad \hat{F}_{02}=\frac{\delta^2\operatorname{ctg}^2\theta}{4}\begin{pmatrix}-1 & -i\sin\theta\\ i\sin\theta & -\sin^2\theta\end{pmatrix},\end{gathered} \tag{6}$$

where 2θ is the angle between $\mathbf{k}_0$ and $\mathbf{k}_2$.

Separation of polarizations. Next, we perform the separation of polarizations[8] in (6), i.e., we use the fact that, for the π and σ polarizations of $\mathbf{E}_0$ and $\mathbf{E}_2$, the set of equations given by (6) splits into two uncoupled second-order systems in the scalar amplitudes $\mathbf{E}_0^p$ and $\mathbf{E}_0^q$, where p and q are the polarization indices ($p, q=\pi, \sigma$) in which π-polarization is the linear polarization in the plane containing the optic axis of the CLC and the direction of emission, and the σ-polarization is perpendicular to this plane. The corresponding unit vectors will be denoted by π and σ.

This yields the following set of equations

$$
\begin{gathered}
E_0{}^p\left(1-\frac{k_0{}^2}{\varkappa_p{}^2}\right)+F_{pq}E_2{}^q=-\frac{iev_p}{2\pi^2\omega\bar{\varepsilon}}\delta(\omega-k_0v),\\
F_{pq}E_0{}^p+\left(1-\frac{k_2{}^2}{\varkappa_q{}^2}\right)E_2{}^q=0;\\
F_{\pi\pi}={}^1/{}_4\delta^2\cos^2\theta,\quad F_{\sigma\sigma}={}^1/{}_4\delta^2\operatorname{ctg}^2\theta,\quad F_{\pi\sigma}=F_{\sigma\pi}=\delta^2\cos^2\theta/4\sin\theta,\\
\varkappa_{\pi,\sigma}{}^2=\omega^2\varepsilon_{\pi,\sigma}/c^2,\ \varepsilon_\pi=\varepsilon(1-{}^1/{}_2\delta\cos^2\theta),\ \varepsilon_\sigma=\varepsilon(1+{}^1/{}_2\delta\cos^2\theta).
\end{gathered}
\tag{7}
$$

This set of equations describes the following four cases of polarization of $\mathbf{E}_0$ and $\mathbf{E}_2$: 1) π-polarization of $\mathbf{E}_0$ and σ-polarization of $\mathbf{E}_2$; 2) π-polarization of both waves; 3) σ-polarization of both waves, and 4) σ-polarization of $\mathbf{E}_0$ and π-polarization of $\mathbf{E}_2$. The Bragg frequencies ω_B and wave vectors k_0 and k_2 in the neighborhood of which $\mathbf{E}_0$ and $\mathbf{E}_2$ are not small, and the Vavilov–Cerenkov emission is described by (7), must simultaneously satisfy (4) and (5). The corresponding angular and frequency intervals are determined by the coefficients F in (7), i.e., they are of the order of δ^2. This indicates, in particular, that, for a fixed frequency, the change in the wave vector $\mathbf{k}_0$ due to diffraction radiation is of the order of $\delta^2 k_0$ as compared with its value at the same frequency in a homogeneous medium with permittivity $\bar{\varepsilon}$. This change turns out to be smaller than the difference due to birefringence between waves of different polarization, which is of the order of δk_0. Hence, it follows that the intrinsic polarizations of the waves $\mathbf{E}_0$ (which are the π- and σ-polarizations in the case of propagation at an angle to the optic axis of the CLC) are not mixed by the diffraction scattering process we are considering. This is the physical reason for the separation of polarizations in (7).

If the dependence of $\bar{\varepsilon}$ on polarization is explicitly taken into account in (4) and (5), i.e., if we take into account the fact that the aperture angle of the Cerenkov cone and the Bragg frequencies ω_B (or the angles at a fixed frequency) are different for the π- and σ-polarized radiations, it turns out that the above four combinations of polarizations correspond to four somewhat different angles of the diffraction cone (Fig. 2).

The corresponding Bragg frequencies ω_B and their dependence on $\mathbf{k}_0$ on the Cerenkov cone (if we neglect

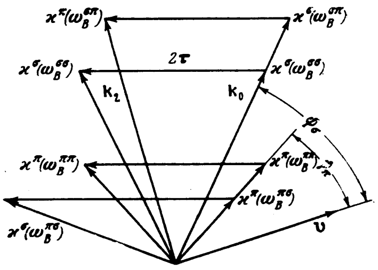

FIG. 2. Fine structure (polarization splitting) of Cerenkov and diffraction cones.

the frequency dispersion of $\hat{\varepsilon}$) are given by the following formulas:

$$
\begin{gathered}
\omega_B{}^{pq}=\omega_B+\Delta\omega_B{}^{pq},\quad \omega_B=\tau c/\varepsilon^{1/2}\sin\theta,\\
\Delta\omega_B{}^{\sigma\sigma}=0,\quad \Delta\omega_B{}^{\sigma\pi}={}^1/{}_8\,\delta\operatorname{ctg}^2\theta\,\omega_B,\\
\Delta\omega_B{}^{\pi\pi}=\frac{\delta\cos^2\theta}{2\sin\theta}\frac{\sin\theta-\cos\beta\cos\psi_0}{\sin^2\psi_0}\omega_B,\\
\Delta\omega_B{}^{\pi\sigma}=-\frac{\delta\operatorname{ctg}^2\theta}{8}\left[1+\frac{2\sin\theta(\cos\beta\cos\psi_0-\sin\theta)}{\sin^2\psi_0}\right]\omega_B,\quad \cos\psi_0=c/v\varepsilon^{1/2},
\end{gathered}
\tag{8}
$$

where β is the angle between $\mathbf{v}$ and τ; the first superscript of ω_B represents the polarization of the wave on the Cerenkov cone and the second on the diffraction cone.

Within the frequency (and angle) intervals $\Delta\omega/\omega_B \sim \delta^2$ around the Bragg frequencies ω_B^{pq} given by (8) and separated from one another by $\delta\omega_B$, a description of the Vavilov–Cerenkov emission in the CLC must be obtained by solving (7) with the corresponding coefficients F_{pq}. The amplitude $\mathbf{E}_2$ can be neglected outside these regions in comparison with $\mathbf{E}_0$, i.e., the Vavilov–Cerenkov radiation in the CLC is described by analogy with the case of homogeneous birefringent media.[9]

Angular and frequency characteristics of the radiation. To be specific, we shall pursue our analysis further by taking the example of the π-polarization of $\mathbf{E}_0$ and $\mathbf{E}_2$ but will omit the polarization indices on the various symbols. Other cases can be examined in an analogous fashion, and the corresponding expressions can be obtained by substituting parameters with the appropriate polarization indices into the final formulas.

If we let $\eta_0=1-k_0^2/\varkappa^2$, $\eta_2=1-k_2^2/\varkappa^2$, $D=\eta_0\eta_2-F^2$, we find that the solution of (7) can be written in the form

$$
E_0=-\frac{ie\pi\mathbf{v}}{2\pi^2\omega\varepsilon}\eta_2D^{-1}\delta(\omega-\mathbf{k}_0\mathbf{v}),\quad E_2=-\frac{ie\pi\mathbf{v}}{2\pi^2\omega\varepsilon}FD^{-1}\delta(\omega-\mathbf{k}_0\mathbf{v}).
\tag{9}
$$

To find the solutions in the form of explicit functions of frequency and k_0, let us express η_2 in terms of η_c. This can conveniently be done by writing the frequency in the form

$$
\omega=\omega_B(1+\Delta\omega/\omega_B),
$$

where $\Delta\omega/\omega_B$ is a small quantity, ω_B is defined by (8), and we use the fact that $\mathbf{k}_0$ and $\mathbf{k}_2$ are related by the Bragg condition (5), and their projections onto the velocity $\mathbf{v}$ of the particle are determined by (4) and $\mathbf{k}_2\cdot\mathbf{v}=\omega+2\tau\cdot\mathbf{v}$, respectively.

In view of the foregoing, we obtain

$$
\eta_2=b\eta_0+\alpha,\quad \alpha=4\sin^2\theta\Delta\omega/\omega_B,\quad b=\cos\widehat{\mathbf{k}_2\mathbf{s}}/\cos\widehat{\mathbf{k}_0\mathbf{s}},
\tag{10}
$$

where s is the normal to $\mathbf{v}$ in the $\mathbf{k}_0, \mathbf{v}$ plane. We now use (10) and write D in the form

$$
D=b(\eta_0-\eta_+)(\eta_0-\eta_-),
$$

where the roots of the equation $D(\eta_0)=0$ are given by

$$
\eta_\pm=b^{-1}[-{}^1/{}_2\alpha\pm({}^1/{}_4\alpha^2+bF^2)^{1/2}].
\tag{11}
$$

Substituting the expression for D obtained in this way in (10), we obtain

$$E_0=-\frac{ie\pi\mathbf{v}}{2\pi^2\omega\varepsilon}\frac{b\eta_0+\alpha}{b(\eta_0-\eta_+)(\eta_0-\eta_-)}\delta(\omega-\mathbf{k}_0\mathbf{v}),$$
$$E_2=-\frac{ie\pi\mathbf{v}}{2\pi^2\omega\varepsilon}\frac{F}{b(\eta_0-\eta_+)(\eta_0-\eta_-)}\delta(\omega-\mathbf{k}_0\mathbf{v}). \tag{12}$$

From this, we can easily show that, for any azimuth of $\mathbf{k}_0$ that is fixed relative to $\mathbf{v}$, the amplitude of the wave radiated into the diffraction cone in the frequency region $\Delta\omega/\omega_B\sim\delta^2$ is of the order of the amplitude of the wave emitted into the Cerenkov cone. When $\Delta\omega/\omega_B\gg\delta^2$, the amplitude $\mathbf{E}_2$ decreases in proportion to $(\Delta\omega/\omega_B)^{-1}$, and $\mathbf{E}_0$ tends to the usual expression for the Vavilov–Cerenkov amplitude.

Next, let us consider the spectral density of the energy lost by the particle by radiation in the CLC. As usual,[10] this can be done by calculating the force due to the reaction exerted by the radiation field on the particle:

$$\mathscr{F}=\int \mathbf{v}e(E_0\exp\{i\mathbf{k}_0\mathbf{r}\}+E_2\exp\{i\mathbf{k}_2\mathbf{r}\})e^{-i\omega t}\,d\mathbf{k}_0\,d\omega,$$

which is equal to the total radiative loss per unit length of the particle trajectory. Similarly, using (2) and (12), we find that the rate of radiative loss in the direction with given azimuth φ (Fig. 1) is given by

$$\frac{\partial^2\mathscr{F}}{\partial\varphi\partial\omega}=\sum_{\pm,\omega_1}\frac{i\xi}{(2\pi)^2}\frac{dI_c}{dz}\int_{-\infty}^{\infty}\frac{(b\eta_0+\alpha)\,d\eta_0}{b(\eta_0-\eta_+)(\eta_0-\eta_-)},\quad \frac{dI_c}{dz}=\frac{e^2}{c^2}\left(1-\frac{c^2}{v^2\varepsilon}\right)\omega, \tag{13}$$

where dI_c/dz is the spectral density of the energy loss through the Vavilov–Cerenkov emission per unit length of the particle path in a homogeneous medium and ξ is the polarization factor. For the above case of π-polarization of the radiation on the Cerenkov cone,

$$\xi_\pi=\frac{(\pi\hat{\mathbf{v}})^2}{1-c^2/v^2\varepsilon},$$

and for the σ-polarization

$$\xi_\sigma=\frac{(\sigma\hat{\mathbf{v}})^2}{1-c^2/v^2\varepsilon},$$

where $\hat{\mathbf{v}}$ is a unit vector.

Evaluation of the integrals in (13) shows that, when $b>0$, the radiative loss at given frequency (and given azimuth) is equal to within $\sim\delta$ to the corresponding loss by radiation in a homogeneous medium with permittivity $\bar{\varepsilon}$.

When $b<0$, the above differential radiative energy loss in the CLC differs from the corresponding loss in a homogeneous medium with permittivity $\bar{\varepsilon}$. Evaluation of the integrals in (13) now leads to the following expressions:

$$\frac{\partial^2\mathscr{F}}{\partial\varphi\partial\omega}=\frac{\xi}{2\pi}\frac{dI_c}{dz}\frac{|\nu|}{(\nu^2-1)^{1/2}},\quad |\nu|=2\sin^2\theta\frac{|\Delta\omega|}{\omega_B F(|b|)^{1/2}}>1;$$
$$\frac{\partial^2\mathscr{F}}{\partial\varphi\partial\omega}=0,\quad |\nu|\leqslant 1. \tag{14}$$

This means that, for frequencies $|\nu|\leqslant 1$, the particle does not, in general, emit Vavilov–Cerenkov radiation, i.e., there is a forbidden frequency band. Outside this band, the above radiative energy losses are greater than in a uniform medium, whereas, on the boundary of the forbidden band, there is a frequency singularity.[3,4] However, it follows from (14) that the energy loss integrated with respect to frequency (or angle φ) is equal to the energy loss in the uniform medium, as before.

Thus, to within $\sim\delta$, the total radiative energy loss in the CLC is equal to the energy loss in the uniform medium with permittivity $\bar{\varepsilon}$, but there is an angular redistribution of the radiation whilst, for $b<0$, there is also a frequency redistribution of the radiative energy loss.

The above solutions can also be used to obtain an idea about the maximum intensity of the Vavilov–Cerenkov radiation that can escape from the CLC in the direction of the diffraction and Cerenkov cones. To obtain this estimate, we must take into account the presence of absorption in the CLC. In a nonabsorbing CLC, the intensity of the radiation leaving the specimen is equal to the total energy lost by the particle by radiation, and increases in proportion to the path traversed by the particle in the CLC, so that it becomes infinite for an infinite specimen.

If we now take into account the imaginary part of $\bar{\varepsilon}$, and use the corresponding modifications of the above formulas, we can write down the expressions for the radiative energy flux through a plane cutting the particle trajectory in the direction of the diffraction and Cerenkov cones:

$$\frac{\partial^2 W_0}{\partial\varphi\partial\omega}=\frac{\partial^2 W_c}{\partial\varphi\partial\omega}\frac{\nu^2+(b+1)/4}{\nu^2+(b+1)/4b},$$
$$\frac{\partial^2 W_2}{\partial\varphi\partial\omega}=\frac{\partial^2 W_c}{\partial\varphi\partial\omega}\frac{[(b+1)/4b]\mathbf{k}_2\mathbf{n}}{[\nu^2+(b+1)/4b]\mathbf{k}_0\mathbf{n}},\quad b>0, \tag{15}$$

$$\frac{\partial^2 W_0}{\partial\varphi\partial\omega}=\frac{\partial^2 W_c}{\partial\varphi\partial\omega}\frac{\nu^2+(|b|-3)/4}{\nu^2+(|b|-1)^2/4|b|}\frac{|\nu|}{(\nu^2-1)^{1/2}},$$
$$\frac{\partial^2 W_2}{\partial\varphi\partial\omega}=\frac{\partial^2 W_c}{\partial\varphi\partial\omega}\frac{(|b|+1)/4|b|}{\nu^2+(|b|-1)^2/4|b|}\frac{|\nu|}{(\nu^2-1)^{1/2}}\frac{\mathbf{k}_2\mathbf{n}}{\mathbf{k}_0\mathbf{n}},\quad b<0,\quad |\nu|>1, \tag{15'}$$

where n is the normal to the plane through which the energy flux is evaluated and $\partial^2 W_c/\partial\varphi\partial\omega$ is the flux of the Vavilov–Cerenkov radiation in the uniform medium:

$$\frac{\partial^2 W_c}{\partial\varphi\partial\omega}=\frac{e^2\varepsilon^{1/2}(\pi\mathbf{v})^2\cos\widehat{\mathbf{k}_0\mathbf{n}}}{2\pi c v^2(\operatorname{Im}\varepsilon)\cos\beta}.$$

The above expressions determine the intensity of radiation leaving the specimen whose linear dimensions are greater than the absorption lengths for light in the CLC and are equal to the corresponding limits of the expressions obtained for finite CLC specimens by solving the boundary value problem (see below).

It is clear from (15) that, when $b>0$, the radiation intensity in the diffraction cone near ω_B is of the same order as in the Cerenkov cone, whilst the intensity in the Cerenkov direction is lower than in the case of the homogeneous medium.

It follows from the foregoing that the polarization of the radiation on the diffraction and Cerenkov cones is either π or σ, depending on the particular case [see (8)]. For the diffraction cone, there is a meaningful expression for the polarization characteristics averaged with respect to frequency (over four cones) for a fixed azimuthal angle φ. The result of this averaging for the diffraction cone corresponds to a partially σ-polarized radiation with polarization

$$P=\frac{(\sigma\mathbf{v})^2F_{\sigma\sigma}-(\sigma\mathbf{v})^2F_{\sigma\pi}+(\pi\mathbf{v})^2F_{\pi\sigma}-(\pi\mathbf{v})^2F_{\pi\pi}}{(\sigma\mathbf{v})^2F_{\sigma\sigma}+(\sigma\mathbf{v})^2F_{\sigma\pi}+(\pi\mathbf{v})^2F_{\pi\sigma}+(\pi\mathbf{v})^2F_{\pi\pi}}$$

SOLUTION OF BOUNDARY VALUE PROBLEM

Field amplitudes. Let us now consider the Vavilov–Cerenkov radiation in a plane-parallel plate of the CLC of thickness l (Fig. 1). The radiation field in the CLC is now the superposition of the above solutions in the case of the inhomogeneous system (7) and the solutions for the homogeneous system obtained from (7) by equating to zero the right-hand sides:

$$\vec{\mathscr{E}}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)+\sum_{i=1}^{4}c_i\vec{\mathscr{E}}_i\quad(\mathbf{r},t). \tag{16}$$

The solutions of the homogeneous system, $\vec{\mathscr{E}}_i$, have been investigated in sufficient detail,[8] and the coefficients c_i in (16) are obtained from boundary conditions for the fields on the specimen surface. The intensity of the Vavilov-Cerenkov radiation escaping from the specimen is determined by the value of the expression given by (16) on the surface of the CLC, i.e., for $z=0$ and $z=l$. If, as before, we express the radiation field in the specimen (solutions of the inhomogeneous and homogeneous systems) in the form of a superposition of the two waves $\vec{\mathscr{E}}_0$ and $\vec{\mathscr{E}}_2$ and ignore radiation on the boundary (transition radiation), we can reduce the boundary conditions to the form

$$\mathbf{E}(z=0,t)+\sum_{i=1}^{4}c_i\vec{\mathscr{E}}_i\quad(z=0,t)=0, \tag{17}$$

if the direct and the diffracted Vavilov–Cerenkov waves escape from the CLC through the same specimen surface, and

$$\begin{aligned}&\mathbf{E}_0(z=0,t)+\sum_{i=1}^{4}c_i\vec{\mathscr{E}}_{0i}\quad(z=0,t)=0,\\&\mathbf{E}_2(z=l,t)+\sum_{i=1}^{4}c_i\vec{\mathscr{E}}_{2i}\quad(z=l,t)=0,\end{aligned} \tag{18}$$

if the direct and diffracted waves leave the CLC specimen through opposite surfaces.

Next, we confine our attention to the case in which the optic axis of the CLC is perpendicular to the specimen surfaces (Fig. 1) and the boundary conditions have the form given by (18). The field amplitudes for radiation escaping from the specimen are then

$$\begin{aligned}E_0=&-\frac{ie\pi\hat{\mathbf{v}}\exp(ia\eta_0)}{2\pi^2\omega\varepsilon(-\eta_0^2+2\gamma\eta_0-F^2)\cos\beta}\{[(-\eta_0+2\gamma)\gamma-F^2]i\sin a(\gamma^2-F^2)^{1/2}\\&+(-\eta_0+2\gamma)(\gamma^2-F^2)^{1/2}[\cos a(\gamma^2-F^2)^{1/2}-\exp(ia(\gamma-\eta_0))]\}\\&\times[(\gamma^2-F^2)^{1/2}\cos a(\gamma^2-F^2)^{1/2}+i\sin a(\gamma^2-F^2)^{1/2}]^{-1},\\E_2=&-\frac{ie\pi\hat{\mathbf{v}}F}{2\pi^2\omega\varepsilon(-\eta_0^2+2\gamma\eta_0-F^2)\cos\beta}\\&\left\{\frac{i(\eta_0-\gamma)\sin a(\gamma^2-F^2)^{1/2}\mp(\gamma^2-F^2)^{1/2}[\cos a(\gamma^2-F^2)^{1/2}-\exp[ia(\eta_0-\gamma)]]}{(\gamma^2-F^2)^{1/2}\cos a(\gamma^2-F^2)^{1/2}+i\sin a(\gamma^2-F^2)^{1/2}}\right\},\end{aligned} \tag{19}$$

$$\eta_0=(-2\sin\theta\sin\psi_0)\Delta\psi/\cos\beta,\ a=\varkappa l/2\sin\theta,\ \Delta\psi=\psi-\psi_\pi,$$
$$\cos\psi_\pi=c/v\varepsilon_\pi^{1/2},\quad\gamma=\frac{2\sin\theta}{\sin\psi_0}(\sin\theta\cos\psi_0-\cos\beta)\Delta\psi+(2\sin^2\theta)\frac{\Delta\omega}{\omega_B}.$$

The angular dependence of the field amplitudes of

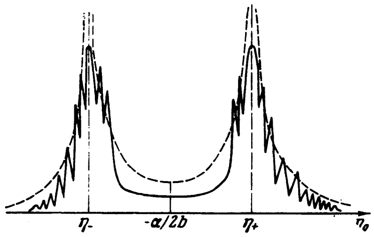

FIG. 3. Angular dependence of the amplitude of the wave emitted into the diffraction cone in an infinite CLC (broken curve) and in a finite specimen (solid curve).

the radiation from the crystal (at fixed frequency; see Fig. 3) is, on the whole, analogous to the corresponding angular dependence in the case of the inhomogeneous solutions. However, finite specimens differ from the infinite medium by the fact that the field amplitudes in the former case exhibit beats as functions of the angle of observation, with periods depending on l, and the field amplitudes remain finite for finite l. Nevertheless, if the crystal is not too thin, the radiation on the Cerenkov cone is emitted in practically two directions, determined by the condition $D(\eta_0)=0$ for each frequency, just as in the case of the infinite crystal. For frequencies in the region $\Delta\omega/\omega_B\sim\delta^2$, the radiation amplitudes on the Cerenkov and diffraction cones may be of the same order. For $\Delta\omega/\omega_B\gg\delta^2$, the radiation field on the diffraction cone falls off as $(\Delta\omega/\omega_B)^{-1}$, whereas, on the Cerenkov cone, it tends to the value corresponding to the Vavilov-Cerenkov radiation in the homogeneous medium (E_c).

Spectral and polarization properties. The frequency dependence of the Vavilov–Cerenkov radiation can be obtained by integrating with respect to $\Delta\psi$ in (19) (see Appendix), and the total emitted radiation (for fixed azimuth φ) can be obtained by subsequently integrating with respect to ν. For the diffraction cone, the total radiation intensity is, of course, proportion to δ^2 ($L=aF$):

$$W_2=\int_0^{2\pi}\frac{e^2(\pi\hat{\mathbf{v}})^2\omega_BL\,\mathrm{th}\,L}{4\varepsilon^{1/2}c\cos\beta\sin\theta}d\varphi. \tag{20}$$

Direct evaluation will show that, with the adopted precision,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|\mathbf{E}_0|^2+|\mathbf{E}_2|^2-|\mathbf{E}_c|^2)d(\Delta\psi)d\nu=0,$$

i.e., the total intensity emitted into the Cerenkov and diffraction cones is equal to the intensity of the Vavilov–Cerenkov radiation in the homogeneous medium.

For an arbitrary direction of motion of the particle in the specimen, the expressions for the intensity as a function of frequency are very unwieldy and are usually written in the form of series. We shall therefore reproduce the radiation intensity I_0 on the Cerenkov cone and I_2 on the diffraction cone only in the case where the particle moves along the optic axis of the CLC, i.e., at right-angles to the specimen surface:

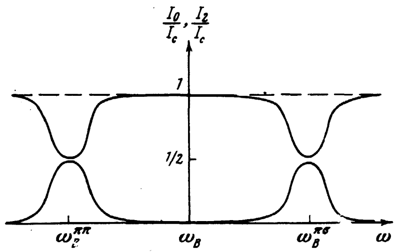

FIG. 4. Frequency distribution of the density of Vavilov-Cerenkov radiation escaping from a thick CLC in the direction of the Cerenkov (upper curve) and diffraction (lower curve) cones, I_0 and I_2, respectively, when the particle travels along the optic axis of the CLC: $\omega_B^{\pi\pi} = \omega_B(1 - 1/8\delta\, \mathrm{tg}^2\psi_0)$, $\omega_B^{\pi\sigma} = \omega_B(1 + 1/8\delta\, \mathrm{tg}^2\psi_0)$.

$$I_0 = I_c\left\{\frac{(1-\nu^2)\,\mathrm{sh}\,L + 2\nu\sin L\nu}{2L\,\mathrm{ch}\,L(1+\nu^2)^2} + \frac{\nu^2+{}^1/_2}{\nu^2+1}\right\},$$
$$I_2 = I_c\left\{\frac{(\nu^2-1)\,\mathrm{sh}\,L - 2\nu\sin L\nu}{2L\,\mathrm{ch}\,L(1+\nu^2)^2} + \frac{{}^1/_2}{\nu^2+1}\right\}, \quad I_c = \frac{e^2}{c^2}\left(1 - \frac{c^2}{v^2\bar{\varepsilon}}\right) l\omega. \tag{21}$$

These intensity distributions are shown in Fig. 4 for the case of thick crystals. The intensity distribution on the diffraction cone in the case of thin crystals ($L \ll 1$) is $I_2 = I_c L^2/6$, i.e., it is proportional to the cube of the specimen thickness and is much smaller than the intensity on the Cerenkov cone. For thick crystals ($L \gg 1$), the radiation intensity on the diffraction cone is

$$I_2 = I_c[2(\nu^2+1)]^{-1},$$

i.e., it is proportional to the specimen thickness and is comparable with the radiation intensity on the Cerenkov cone.

To be specific, we have confined our attention to π-polarization on the Cerenkov and diffraction cones (Fig. 2) and have omitted the polarization indices on all the quantities that were polarization-dependent. Since the expressions for other polarizations are completely analogous, the required expressions can be obtained by simply substituting the appropriate polarization indices. As an example, we demonstrate this for the polarization characteristics. As already noted, the Cerenkov and diffraction cones in the CLC are separated, and the polarization properties of the radiation are completely definite in the immediate neighborhood (in angle) of each of the resulting cones. These are, in fact, the π- and σ-polarizations (see Fig. 2).

However, since the angular separation between these cones is small, the polarization characteristics are usually averaged over angles and frequencies (for fixed azimuth φ) under experimental conditions. The result of this averaging in the case of the diffraction cone corresponds to partially σ-polarized radiation with polarization given by

$$P = \frac{(\sigma\nu)^2 L_{\sigma\sigma}\,\mathrm{th}\,L_{\sigma\sigma} - (\sigma\nu)^2 L_{\sigma\pi}\,\mathrm{th}\,L_{\sigma\pi} + (\pi\nu)^2 L_{\pi\sigma}\,\mathrm{th}\,L_{\pi\sigma} - (\pi\nu)^2 L_{\pi\pi}\,\mathrm{th}\,L_{\pi\pi}}{(\sigma\nu)^2 L_{\sigma\sigma}\,\mathrm{th}\,L_{\sigma\sigma} + (\sigma\nu)^2 L_{\sigma\pi}\,\mathrm{th}\,L_{\sigma\pi} + (\pi\nu)^2 L_{\pi\sigma}\,\mathrm{th}\,L_{\pi\sigma} + (\pi\nu)^2 L_{\pi\pi}\,\mathrm{th}\,L_{\pi\pi}}. \tag{22}$$

In particular, when the particle travels along the optic axis of the CLC, the radiation is partially σ-polarized whatever the dependence on φ, and the polarization is

$$P = \frac{L_{\pi\sigma}\,\mathrm{th}\,L_{\pi\sigma} - L_{\pi\pi}\,\mathrm{th}\,L_{\pi\pi}}{L_{\pi\sigma}\,\mathrm{th}\,L_{\pi\sigma} + L_{\pi\pi}\,\mathrm{th}\,L_{\pi\pi}} \tag{23}$$

It follows from this expression that the degree of polarization decreases monotonically with crystal thickness. In particular, for an ultrarelativistic particle, the polarization is determined only by the anisotropy of the permittivity of the CLC, and is given by $(\bar{\varepsilon}^{1/2} - 1)/(\bar{\varepsilon}^{1/2} + 1)$ for thick crystals and by $(\bar{\varepsilon} - 1)/(\bar{\varepsilon} + 1)$ for thin crystals.

CONCLUSIONS

We have given a detailed discussion of the Vavilov-Cerenkov emission under the conditions corresponding to second-order diffraction scattering. A completely analogous solution can be given in the case of diffraction scattering corresponding to higher orders since the corresponding sets of equations differ from (7) only by the coefficients. Thus, in the n-th order, the diagonal coefficients are the same as in (7) whereas the off-diagonal coefficients contain the additional factor

$$\frac{n^2}{4[(n-1)!]^2}\left(\frac{\delta n^2\,\mathrm{ctg}^2\,\theta}{8}\right)^{n-2}.$$

As a result, the Vavilov-Cerenkov picture remains qualitatively the same as in the case of the second-order scattering, but the angular and frequency intervals within which diffraction is seen are of the order of δ^n. This means that the Cerenkov cone splits into two, the cone angles differing by an amount of the order of δ and the radiation being linearly polarized so that, on one of the cones, the radiation is π-polarized and, on the other, it is σ-polarized. The diffraction cone splits into four with cone angles differing by an amount again of the order δ. On each of these cones, the emitted radiation lies within the frequency band $\Delta\omega/\omega_b \sim \delta^n$ near the corresponding Bragg frequency. On two of these cones, the radiation is polarized in the plane containing the optic axis of the CLC and the direction of emission (π-polarization), and on the other two the polarization is at right-angles to this plane (σ-polarization). The polarization characteristics are obtained by averaging over all the four cones and correspond, in the case of the diffraction cone, to partially σ-polarized radiation.

It is important to note that the above results may be useful for describing coherent radiation emitted by charged particles in media with scalar permittivity that is a periodic function of position. The various effects in the Vavilov-Cerenkov radiation in cholesteric liquid crystals are usually small because δ is small, so that experimental verification is best carried out with cholesteric liquid crystals with maximum possible permittivity.

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APPENDIX

To determine the frequency distribution of the radiation leaving the CLC specimen of finite size we have to evaluate an integral of the form

$$I=\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x^2-\cos^2 L(x^2-1)^{1/2}}=\int_{-\infty}^{\infty}\frac{f(x)\,dx}{\sin^2 L(x^2-1)}\frac{1}{2ix}\left(\frac{1}{m_+(x)}-\frac{1}{m_-(x)}\right), \tag{1a}$$

$$m_\pm(x)=(x^2-1)^{1/2}\operatorname{ctg} L(x^2-1)^{1/2}\mp ix.$$

We shall use the fact that the functions $m_\pm(x)$ have no zeros for $\operatorname{Im} x\geqslant 0$, respectively.[11]

We also take into account the fact that the two-valued function $g(z)=(z^2-1)^{1/2}$ has a regular branch assuming positive values on the positive axis outside the interval $(-1, 1)$. We begin by considering integrals with finite limits $-R$, R along the real axis, including the ε-intervals of the singular points

$$z_0=0,\quad z_k=\frac{k}{|k|}\left[1+\left(\frac{\pi k}{L}\right)^2\right]^{1/2},\quad (k\neq 0).$$

We have

$$I_{R\varepsilon}^{\pm}=\int_{-R}^{R}\frac{f(z)\,dz}{\sin^2 L(z^2-1)^{1/2}\cdot 2iz m_\pm(z)}. \tag{2a}$$

Substituting $R=[1+\pi^2(k+\frac{1}{2})^2/L^2]^{1/2}$, we consider the contour of integration for $I_{R\varepsilon}^{+}$ in the complex plane, which includes the semicircle of radius R for $\operatorname{Im} z\geqslant 0$, semicircles of small radius ε around the first-order poles z_0 and z_k, and the segment $[-R, R]$ along the real axis without the ε-intervals of the singular points z_0 and z_k. If

$$\left|\frac{zf(z)}{\sin^2 L(z^2-1)^{1/2}}\right|\leqslant M,\quad \operatorname{Im} z\geqslant 1, \tag{3a}$$

is satisfied on this contour, where $M=\text{const}$, the evaluation of $I_{R\varepsilon}^{+}$ reduces to the summation of the poles z_0 and z_k subject to the condition that $f(z)$ has no poles in the upper half-plane. Similarly, evaluating $I_{R\varepsilon}^{-}$ for $\operatorname{Im} z\leqslant 0$ and taking $\varepsilon\to 0$, $R\to\infty$, we obtain

$$I=-\frac{\pi f(0)}{\operatorname{sh} L\operatorname{ch} L}+\sum_{k=-\infty}^{\infty}\frac{\pi}{L[1+(\pi k/L)^2]}f\left(\frac{k}{|k|}\left(1+\left(\frac{\pi k}{L}\right)^2\right)^{1/2}\right). \tag{4a}$$

This formula was used in evaluating (21) and (22) with the function f given by (20).

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Translated by S. Chomet

Structure of the field near a singularity arising from self-focusing in a cubically nonlinear medium

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The structure of the field is studied near a singularity that arises in the propagation of intense light beams in a nonlinear (cubic) medium. The structure near the focus is determined by a method of numerical integration of a parabolic equation with the step of the transverse coordinate changed automatically as the singularity is approached. Methods of analyzing this solution, based on extension of the scales and the use of functional relations which are invariant with respect to the exact position of the focus, make it possible to develop an idea of the formation of the field near the singularity by a bell-shaped beam with a Townes profile with adjacent weakly focusing wings, which converges to a point. On the basis of this concept the analytic form of the field near the singularity is described by the function $E\sim[\ln|(z_{sf}-z)|/(z_{sf}-z)]^{1/2}$, which is in complete agreement with the numerical results.

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In the propagation of an intense light beam in a medium with a cubic nonlinearity $[\varepsilon=\varepsilon_0(1+\varepsilon'|E|^2)]$ there can be regions where the amplitude increases without limit—foci.[1,2] The character of the singularity of the field at a focus has been discussed in the approximation of a parabolic equation

$$2i\,\partial E/\partial z=\Delta_\perp E+|E|^2E \tag{1}$$

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