Eigenfunction method and mass operator in the quantum electrodynamics of a constant field

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A method is proposed for calculating radiative effects in quantum electrodynamics in an intense constant field; it is based on the use of eigenfunctions of the mass operator and the localization of this operator. A compact expression for an eigenvalue of the mass operator of an electron in an arbitrary constant electric field is found, and the corresponding elastic scattering amplitude is calculated. The imaginary part of the amplitude determines the rate of decay of the various states of the electron in the field, and the real part contains the mass shift, including the anomalous magnetic and electric moments of the electron as functions of the field and the momentum of the electron. Quantities found and studied in detail include the anomalous electric moment which appears in a field for which the pseudoscalar **E**·**H** is not zero, the anomalous magnetic moment in an electric field, which approaches twice the Schwinger value as the field increases, and also the mass shift and the rate of decay of the ground state of the electron in an electric field. For a weak field the mass shift contains the classical term, linear in the absolute value of the field strength, which characterizes the effect of acceleration on the structure of the electron.

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1. INTRODUCTION

The interactions of electrons with an electromagnetic field whose intensity is of the order of the characteristic quantum-electrodynamical value

$$F_0=m^2c^3/e\hbar=4.4\cdot 10^{13}\text{ Oe},$$

is of fundamental interest for quantum electrodynamics owing to the important part played by nonlinear radiative effects. Such fields exist near pulsars, and they can be produced in the laboratory in the rest system of an electron when ultrarelativistic electrons pass through intense laser fields and magnetic fields. The radiative corrections to the motion of electrons and protons in an external field are described by mass and polarization operators and modified Dirac and Maxwell equations. One method for calculating these operators is diagonalizing them by using exact eigenfunctions.

In Sec. 2, based on arguments of a general nature,[1,2] a complete set of operators is found which commute with the exact (in both external and radiation fields) mass operator of an electron in a constant electromagnetic field. This means that a representation is found in which the exact mass operator is diagonal. This radically simplifies the study of radiative effects in an external field and reduces the integrodifferential equations for the wave function and the exact Green's function to algebraic equations. The eigenfunctions of the complete set of operators are derived in explicit form and their properties are studied in detail. By means of them various representations of the propagation function of an electron in a constant field are obtained. A similar program has been carried out for a plane wave and a constant crossed field (see Refs. 1, 2).

In Sec. 3, the eigenfunction method is used to find the eigenvalue of the mass operator of the electron in a constant field. The key point in this is the use of the integral representation (43) of a four-dimensional Gaussian function. This integral transformation plays the same role in the quantum electrodynamics of a constant field as the Fourier transformation does in the quantum electrodynamics of vacuum.

In Sec. 4 the elastic scattering amplitude, or the change Δm of the electron mass, in a constant field is found. This quantity fixes the probability amplitude for an electron to preserve its state as it passes through the field, which is given by $\exp(-i\Delta m\tau)$, where τ is the proper time of the electron's presence in the field. Accordingly, $-2\,\mathrm{Im}\,\Delta m$ is the probability of emission of radiation per unit proper time (more exactly, the rate of decay of the state), and $\mathrm{Re}\,\Delta m$ fixes, for example, the change of velocity of the electron in the field. The two terms of Δm depend linearly on the polarization of the electron and determine the anomalous magnetic and electric moments (AMM and AEM) of the electron as functions of the field intensity and the momentum of the electron. The electron's AEM in nonzero only in a field for which the pseudoscalar $\mathbf{E}\cdot\mathbf{H}\neq 0$. A detailed investigation of the AEM is made in Sec. 5.

The AMM which is found is studied in the case of an electric field in Sec. 6. It is shown that in the ground state (and in not too highly excited states, with the electron's transverse momentum smaller than or of the order of its mass) the AMM first decreases with increase of the field and then increases, approaching twice the Schwinger value. This behavior is radically different from those found previously for the AMM in crossed or magnetic fields. However, in states with large transverse momenta, and also in weak fields, the behavior of the AMM is the same as in the case of crossed fields.

In. Sec. 7 the mass shift and damping of the ground state of an electron in an electric field are found. The dependence of the mass shift on the field strength is like that found in the case of a magnetic field; at first a decrease, linear in the field strength, of order of magnitude αm, and then, at fields larger than F_0, a monotonic increase. The mass shift term linear in the field gives a new structure constant of the electron; unlike the effect of the AMM, this is an inherent effect for a spinless

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particle. It is also interesting that there is no stable state for an electron in an electric field and that the decay rate of the ground state depends strongly on the field, owing to radiation and to the transition of the electron to excited states. The necessity of change of state of the electron agrees with the general rule that a constant field cannot cause a process whose only effect is the production of photons.

The important part of the electron mass shift which is linear in the electric field and does not involve Planck's constant, as found in Sec. 7, is studied in detail in Sec. 8 and is there derived by a purely classical method. It is caused by a restructuring of the proper field of the charge, which is proportional to the acceleration and has the sign opposite to that of the velocity relative to the acceleration, so that in the final state it is negative.

2. THE EIGENFUNCTIONS OF THE MASS OPERATOR

In general a constant field is either crossed or parallel, i.e., such that in some Lorentz coordinate system the electric and magnetic fields are parallel. Here, beginning with Eq. (5), we are concerned with the latter case. We recall that in a parallel field, $\mathbf{E} \parallel \mathbf{H}$, a classical electron moves in a helical path of constant radius and decreasing pitch, losing its longitudinal velocity, until this velocity component becomes zero, after which it takes on longitudinal velocity in the opposite direction, moving along a helical path of the same radius and increasing pitch, and rotating in the same direction (conservation of angular momentum!) (see Ref. 3).

The quantum motion of an electron in an external field, with radiative corrections taken into account, is described by a Green's function which satisfies the Dirac equation proposed by Schwinger[4]:

$$(i\gamma\Pi+m)G(x,y)+\int M(x,x')G(x',y)d^4x'=-i\delta(x-y),\ \Pi_\alpha=-i\partial_\alpha-eA_\alpha. \quad (1)$$

The function $M(x,x')$ which describes the self-energy effects can be regarded as the matrix element $M(x,x')=(x|M|x')$ of a mass operator M in the coordinate representation. The operator M is a scalar γ-matrix function of the operator Π_α and the field $F_{\alpha\beta}$. For a constant field there are only four independent scalars[2]:

$$\gamma\Pi,\ \sigma F,\ (F\Pi)^2,\ \gamma_5 FF^*, \quad (2)$$

on which the operator M can depend. All other scalars can be formed from these four, for example:

$$i\gamma F\Pi={}^1/_2[\gamma\Pi,\sigma F],\quad \gamma_5\gamma F^*\Pi={}^1/_4\{\gamma\Pi,\sigma F\},$$
$$i\gamma FF\Pi={}^1/_4[\gamma F\Pi,\sigma F],\quad F^2={}^1/_2(\sigma F)^2-i\gamma_5 FF^*,$$

and so on. It is not hard to see that the operators (2), and along with them the mass operator also, commute with the squared Dirac operator $(\gamma\Pi)^2=\Pi^2-\frac{1}{2}e\sigma F$, and therefore the operator M is diagonal in the representation of the eigenfunctions $E_p(x)$ of the operator $(\gamma\Pi)^2$:

$$(\gamma\Pi)^2E_p=p^2E_p. \quad (3)$$

The eigenvalue p^2 of the operator $(\gamma\Pi)^2$ can be any real number. It is obvious that $E_p(x)$ is also an eigenfunction of three differential operators that commute with $\gamma\Pi$ and whose eigenvalues number the solutions $\psi(x)$ of the usual Dirac equation

$$(i\gamma\Pi+m)\psi=0. \quad (4)$$

Accordingly, for a general constant field for which the vectors $\mathbf{H}$ and $\mathbf{E}$ are parallel in a suitable coordinate system and directed along the axis 3, and will be denoted by η and ε, and the potential is taken in the form $A_\mu=(0,\eta x_1,-\varepsilon t,0)$, there are four operators:

$$(\gamma\Pi)^2,\ -i\partial_2,\ -i\partial_3,\ \Pi_1^2+\Pi_2^2-e\eta\Sigma_3, \quad (5)$$

which commute with each other and the mass operator and form, along with it, a complete set. We shall denote a set of eigenvalues

$$p^2,\ p_2,\ p_3,\ 2|e\eta|k,\ k=0,1,2,\ldots, \quad (6)$$

of the operators (5) by the letter p. The operators (5) commute with Σ_3 and γ_5 [or with $\frac{1}{2}\sigma F=(\eta+i\varepsilon\gamma_5)\Sigma_3$], and therefore their eigenfunctions can also be distinguished by the eigenvalues $\sigma=\pm1$, $\gamma=\pm1$ of the operators Σ_3 and γ_5.

In the so called spinor representation (see Ref. 5, Sec. 17–22; Ref. 6, Sec. 8), in which γ_5 and Σ_3 are diagonal, the eigenfunctions $E_{p\sigma\gamma}(x)$ are of the form

$$E_{p\sigma\gamma}(x)=\frac{e^{i\pi\lambda/4}\Gamma(-\lambda)e^{i(p_2x_2+p_3x_3)}}{(2\pi|\varepsilon/\eta|)^{1/4}(n!)^{1/2}}D_n(\rho)D_\lambda(\tau)w_{\sigma\gamma}, \quad (7)$$

where $w_{\sigma\gamma}$ are the eigen bispinors of the matrices Σ_3 and and γ_5; in the representation considered the four bispinors $w_{1-1}, w_{-1-1}, w_{11}, w_{-11}$ form the columns 1, 2, 3, 4 of the unit matrix in the γ-matrix space, i.e., $w_{\sigma\gamma}(\alpha)=I_{\alpha\beta}$, if the set of eigenvalues $\sigma\gamma$ is assigned column number β; $D_\nu(z)$ are the parabolic cylinder functions with indices

$$n=k+\frac{\sigma e\eta}{2|e\eta|}-\frac{1}{2},\quad \lambda=-i\frac{2|e\eta|k-p^2}{2|e\varepsilon|}-\frac{\gamma\sigma e\varepsilon}{2|e\varepsilon|}-\frac{1}{2} \quad (8)$$

and arguments

$$\rho=(2|e\eta|)^{1/2}(x_1-p_2/e\eta),\quad \tau=e^{i\pi/4}(2|e\varepsilon|)^{1/2}(t+p_3/e\varepsilon). \quad (9)$$

Accordingly, for a given p the four bispinors (7) form a diagonal four-rowed matrix $E_p(z)$ made up of two diagonal two-rowed matrixes a_p and b_p:

$$E_p=\begin{pmatrix} a_p & 0\\ 0 & b_p\end{pmatrix},$$

$$[\Pi^2-p^2-\sigma e(\eta-i\varepsilon)]a_{p\sigma}=[\Pi^2-p^2-\sigma e(\eta+i\varepsilon)]b_{p\sigma}=0, \quad (10)$$

whose columns $a_{p\sigma}$, $b_{p\sigma}$, $\sigma=\pm1$, are the two-component eigenspinors of the Pauli matrix σ_3 and the solutions of the complex-conjugate equations (10). The spinors $a_{p\sigma}$ and $b_{p\sigma}$ transform independently according to conjugate representations of the proper Lorentz group and are interchanged on inversion (so-called 4-spinors, see Refs. 5, 6). The functions (7) are positive-frequency functions in the sense that when the electric field is turned off (with $\varepsilon\to0$) they go over into functions with positive frequency over a wide range of the time t: $|t+p_3/e\varepsilon|\lesssim(2|e\varepsilon|)^{-1/2}$. Therefore we supply them with a first subscript plus before the p: $E_p=E_{+p}$. The corresponding functions E_{-p} are obtained from them by setting $\tau\to-\tau$. These assertions follow from the asymptotic formula

$$D_\lambda(\pm\tau)=\frac{1}{2^{1/2}}\exp\left[\frac{\lambda}{2}\ln(-\lambda)-\frac{\lambda}{2}\mp(-\lambda)^{1/2}\tau\right],$$
$$\mp(-\lambda)^{1/2}\tau\approx\mp i(2|e\eta|k-p^2)^{1/2}\left(t+\frac{p_3}{e\varepsilon}\right), \quad (11)$$

which holds for $|\lambda|\to\infty$, bounded τ, and $\arg(-\lambda)\leq\pi/2$.

In the presence of an electric field, because of pair production, each of the solutions $E_{\pm p}$ contains both positive and negative frequencies. The solution E_{+p} contains only the positive-frequency semiclassical wave e^{iS+} for $t\to+\infty$, and the solution E_{-p} contains only the negative wave e^{iS-} for $t\to-\infty$; here $S_\pm$ are the quasiclassical action functions

$$S_\pm=\mp\nu(\operatorname{Arsh}\xi+\xi(1+\xi^2)^{1/2})+p_2x_2+p_3x_3,$$
$$\nu=\frac{2|e\eta|k-p^2}{2|e\varepsilon|},\quad \xi=\left(\frac{|e\varepsilon|}{2\nu}\right)^{1/2}\left(t+\frac{p_3}{e\varepsilon}\right). \tag{12}$$

We note that for the classical time dependence it is essential that the action be large, i.e., that $\max(\nu|\xi|,\nu\xi^2)\gg1$. This condition is equivalent to the requirement that the time be large compared with characteristic quantum times

$$\left|t+\frac{p_3}{e\varepsilon}\right|\gg\max\left[\frac{1}{(2|e\eta|k-p^2)^{1/2}},\frac{1}{|e\varepsilon|^{1/2}}\right]$$
$$\sim\frac{\hbar}{mc}\max\left[\frac{1}{(1+|\eta|/F_0)^{1/2}},\left(\frac{F_0}{|\varepsilon|}\right)^{1/2}\right];$$

the last relation is written on the assumption that $-p^2\sim m^2$.

The matrix functions $E_p(x)$ are orthogonal and are normalized by the condition ($\bar{E}_p\equiv\gamma_4E_p^+\gamma_4$, and ω is the sign of the frequency)

$$\int\bar{E}_{\omega'p'}(x)E_{\omega p}(x)d^4x=(2\pi)^4\delta(p'^2-p^2)\delta(p_2'-p_2)\delta(p_3'-p_3)\delta_{k'k}\delta_{\omega'\omega}. \tag{13}$$

They satisfy the completeness condition

$$\sum_{\pm,k}\int E_{\pm p}(x)\bar{E}_{\pm p}(y)\frac{dp^2dp_2dp_3}{(2\pi)^4}=\delta(x-y). \tag{14}$$

If we apply the operator $\gamma\Pi$ to the matrix $E_{\omega p}$ we obtain the valuable relation

$$\gamma\Pi E_p=E_p\gamma\bar{p}, \tag{15}$$

in which the four-vector $\bar{p}_\mu$ has the components

$$\bar{p}_1=0,\ \bar{p}_2=-\operatorname{sign}(e\eta)(2|e\eta|k)^{1/2},$$
$$\bar{p}_0-\operatorname{sign}(e\varepsilon)\bar{p}_3=\omega(2|e\varepsilon|)^{1/2},\ 2\bar{p}_+\bar{p}_-=2|e\eta|k-p^2,$$
$$\bar{p}_-=\bar{p}_0-\bar{p}_3,\ \bar{p}_+={}^1/_2(\bar{p}_0+\bar{p}_3), \tag{16}$$

and its square is $\bar{p}^2=p^2$. Thus this vector depends only on the dynamical quantum numbers p^2, k, ω and does not depend on the numbers p_2, p_3 which fix only the origin relative to which the coordinates x_i and the time t are measured. As will be seen from Eqs. (48) and (49), $\bar{p}$ is the quantum analog of the kinetic momentum of the electron at the point of natural symmetry of the motion.

It follows in particular from Eq. (15) that the solution of the Dirac equation (4) is obtained by applying the operator $m-i\gamma\Pi$ to the matrix E_p, taken for $p^2=-m^2$:

$$\psi_{\omega p}=(m-i\gamma\Pi)E_{\omega p}=E_{\omega p}(m-i\gamma\bar{p}) \tag{17}$$

and accordingly is of the form $\psi_{\omega pr}(x)=E_{\omega p}(x)u_{pr}$, where u_{pr} is a spinor which satisfies the "free" Dirac equation $(m+i\gamma p)u_{pr}=0$, and the matrix $E_{\omega p}$ and the vector $\bar{p}_\mu$ are taken at the point $p^2=-m^2$ (cf. the plane-wave case, Refs. 1, 2). This solution is the same as that of Nikishov.[7]

In the $E_{\omega p}$ representation the mass operator is diagonal, and the propagation function of the electron in the field is of the form $-i(m+i\gamma p)^{-1}$. In the coordinate representation it can be written in any of the following forms ($\int d^4p\equiv\sum_{k=0}^\infty dp^2dp_2dp_3$, and summation over the sign ω is to be understood throughout):

$$S(x,x')=\int\frac{d^4p}{(2\pi)^4}E_{\omega p}(x)\frac{-i}{m+i\gamma\bar{p}}\bar{E}_{\omega p}(x')$$
$$=-i\int\frac{d^4p}{(2\pi)^4}\frac{E_{\omega p}(x)(m-i\gamma\bar{p})\bar{E}_{\omega p}(x')}{m^2+p^2}=(m-i\gamma\Pi)(-i)$$
$$\times\int\frac{d^4p}{(2\pi)^4}\frac{E_{\omega p}(x)\bar{E}_{\omega p}(x')}{m^2+p^2}=(m-i\gamma\Pi)\int_0^\infty ds\,e^{-im^2s}\int\frac{d^4p}{(2\pi)^4}e^{-ip^2s}$$
$$\times E_{\omega p}(x)\bar{E}_{\omega p}(x'). \tag{18}$$

The path around the pole $p^2=-m^2$ has been specified only in the last (causal) expression, where it is assumed that $m^2\to m^2-i\delta$. The last integral, which is a diagonal γ matrix, can be expressed in terms of elementary functions. To do so we represent its matrix elements in the form

$$\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}e^{ip_2(x-x')_2}\sum_{k=0}^{\infty}D_k\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}e^{ip_3(x-x')_3}\int_{-\infty}^{\infty}\frac{dp^2}{(2\pi)^2}e^{-ip^2s}D_E, \tag{19}$$

where

$$D_k=\left(\frac{|e\eta|}{\pi}\right)^{1/2}\frac{D_n(\rho)D_n(\rho')}{n!},\quad D_E=\frac{\exp({}^1/_4i\pi(\lambda-\lambda'^*))}{(2|e\varepsilon|)^{1/2}}\Gamma(-\lambda)\Gamma(-\lambda'^*)$$
$$\times\sum_\pm D_\lambda(\pm\tau)D_{\lambda'^*}(\pm\tau'^*), \tag{20}$$

noting that in the γ matrix $E\bar{E}$ there are products both of spinors with the same sign $\sigma=\sigma'=\pm1$ and of those with opposite signs $\gamma=-\gamma'=\pm1$. Therefore we have the respective values

$$n=n'=\begin{cases}k\\k-1\end{cases}\qquad\frac{\sigma e\eta}{|e\eta|}=\pm1,$$
$$\lambda=\begin{cases}-i\nu\\-i\nu-1\end{cases},\quad\lambda'^*=\begin{cases}i\nu-1\\i\nu\end{cases}\qquad\frac{\sigma\gamma e\varepsilon}{|e\varepsilon|}=\mp1, \tag{8'}$$

[cf. Eq. (8)].

To calculate the integral over p^2 we use the representation[1)]

$$e^{i\pi\lambda/4}\Gamma(-\lambda)D_\lambda(\pm\tau)=e^{-it^2/4}\int_0^\infty dy\,y^{-\lambda-1}e^{\mp ity-iy^2/2},\qquad\tau=e^{i\pi/4}t, \tag{21}$$

where $t=(2|e\varepsilon|)^{1/2}(x_0+p_3/e\varepsilon)$ is a real variable, perform the integration of the Gaussian integral over p^2, and then that over the variables y, y' of the representation (21). We then get

$$\int_{-\infty}^{\infty}\frac{dp^2}{(2\pi)^2}e^{-ip^2s}D_E=\left(\frac{e\varepsilon}{2\pi\operatorname{sh}2e\varepsilon s}\right)^{1/2}$$
$$\times\exp\left(\frac{i\pi}{4}-\frac{i(t+t')^2}{8\operatorname{cth}|e\varepsilon|s}-\sigma\gamma e\varepsilon s-\frac{e\varepsilon z_0^2}{4\operatorname{th}e\varepsilon s}-i2|e\eta|ks\right), \tag{22}$$

where

$$t+t'=(2|e\varepsilon|)^{1/2}(x_0+x_0'+2p_3/e\varepsilon),\ z_\alpha=(x-x')_\alpha.$$

The next integration, that over p_3, gives

$$\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}e^{ip_3z_3}\int_{-\infty}^{\infty}\frac{dp^2}{(2\pi)^2}e^{-ip^2s}D_E=\frac{e\varepsilon}{4\pi\operatorname{sh}e\varepsilon s}$$
$$\times\exp\left(-ie\varepsilon z_3\frac{(x+x')_0}{2}-\sigma\gamma e\varepsilon s+\frac{ie\varepsilon(z_3^2-z_0^2)}{4\operatorname{th}e\varepsilon s}-i2|e\eta|ks\right). \tag{23}$$

Similarly, we use the representation

$$D_n(\rho)=\left(\frac{2}{\pi}\right)^{1/2}e^{\rho^2/4}\int_0^\infty dy\,y^n e^{-y^2/2}\cos\left(\rho y-\frac{\pi n}{2}\right) \quad (24)$$

to carry out the summation over k and the integration over p_2:

$$\sum_{k=0}^{\infty}D_{\varkappa}e^{-i2|e\eta|ks}=\left(\frac{e\eta}{2\pi\sin 2e\eta s}\right)^{1/2}\times\exp\left(-\frac{i\pi}{4}-\frac{i(\rho+\rho')^2}{8\,\mathrm{ctg}\,|e\eta|s}+i\sigma e\eta s+\frac{ie\eta z_1^2}{4\,\mathrm{tg}\,e\eta s}\right); \quad (25)$$

here

$$\rho+\rho'=(2|e\eta|)^{1/2}(x_1+x_1'-2p_2/e\eta).$$

Thereafter,

$$\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}e^{ip_2z_2}\sum_{k=0}^{\infty}D_{\varkappa}e^{-i2|e\eta|ks}=\frac{-ie\eta}{4\pi\sin e\eta s}\exp\left(ie\eta z_2\frac{(x+x')_1}{2}+i\sigma e\eta s+\frac{ie\eta(z_1^2+z_2^2)}{4\,\mathrm{tg}\,e\eta s}\right). \quad (26)$$

In this way we get for the integral as a whole the invariant expression

$$\int\frac{d^4p}{(2\pi)^4}e^{-ip^2s}E_{\omega p}(x)\bar{E}_{\omega p}(x')=\frac{-ie^2\eta\varepsilon}{(4\pi)^2\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}\times\exp\left(ie\int_{x'}^{x}dy\,A(y)+\frac{i}{2}e\sigma Fs+\frac{i}{4}zeF\,\mathrm{cth}\,eFsz\right), \quad (27)$$

in which all of the nodiagonality in x, x' and all of the dependence on the gauge of the potential $A(y)$ are concentrated in the first term of the exponent, and all of the γ-matrix structure is in the term σF. In writing the third term in the exponent we have used the diagonal representation of the field tensor

$$F=\mathrm{diag}(i\eta,\,-i\eta,\,\varepsilon,\,-\varepsilon) \quad (28)$$

and functions of it. The coefficient of the exponential function can also be written in terms of the field tensor:

$$-i(4\pi)^{-2}[\det(eF/\mathrm{sh}\,eFs)]^{-1/2},$$

but the form shown in Eq. (27) is more intuitive and convenient. On substitution of Eq. (17) in Eq. (18) we obtain the proper-time[2)] representation as found by Fock[8] and by Schwinger[9]:

$$S^c(x,x')=\frac{-ie^{i\varphi}}{(4\pi)^2}\int_0^\infty\frac{ds\,e^2\eta\varepsilon}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}\left[m-\frac{i}{2}\gamma(\beta+eF)z\right]\times\exp\left(-im^2s+\frac{iz\beta z}{4}+\frac{ie\sigma Fs}{2}\right), \quad (29)$$

$$\beta_{\alpha\beta}=(eF\,\mathrm{cth}\,eFs)_{\alpha\beta},\quad \varphi=e\int_{x'}^{x}dy\,A(y).$$

The invariant formulation of the integral (27) also contains the case of a crossed field; for this we must take A_μ and $F_{\mu\nu}$ to be the corresponding potential and field, and set the field invarients η, ε equal to zero. Then the right hand side of Eq. (27) becomes

$$-\frac{i}{(4\pi)^2s^2}\exp\left(i\varphi+\frac{i}{2}e\sigma Fs+\frac{iz^2}{4s}-\frac{is(eFz)^2}{12}\right), \quad (27')$$

and the Green's function takes the form

$$S^c(x,x')=\frac{-ie^{i\varphi}}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\left[m-\frac{i}{2}\gamma\left(\frac{1}{s}+eF+\frac{1}{3}e^2FFs\right)z\right]\times\exp\left(-im^2s+\frac{iz^2}{4s}-\frac{is(eFz)^2}{12}+\frac{ie\sigma Fs}{2}\right). \quad (29')$$

Equations (22)–(27) are essential for the use of the eigenfunction method in more complicated problems. Another representation for the causal propagation function can be derived by closing the path of integration with respect to p^2 in the second or third representation in Eq. (18). To do this we consider the integral

$$\int_{-\infty}^{\infty}\frac{dp^2}{(2\pi)^2}\frac{D_E}{p^2+m^2-i\delta}=-(2\pi)^{-2}\int_{-\infty}^{\infty}\frac{d\nu\,D_E}{\nu-\nu_0+i\delta},\quad \nu=\frac{2|e\eta|k-p^2}{2|e\varepsilon|},\quad \nu_0=\frac{2|e\eta|k+m^2}{2|e\varepsilon|}, \quad (30)$$

and the analytic properties of the function D_E in the complex plane of p^2 or ν. Recalling our footnote[1)] and Eq. (8′), we see that in the complex ν plane the function D_E has simple poles at the integer points on the imaginary axis, moved slightly upward for the function $\Gamma(-\lambda)$, to $\nu=i(n+\delta)$, $n=1,2,\ldots$, and downward for the function $\Gamma(-\lambda'^*)$, $\nu=-i(n+\delta)$, $n=1,2,\ldots$, because of the small bias δ discussed in that footnote.[1)] The presence of poles along the entire imaginary axis of ν in each of the frequency terms of the function D_E is inconvenient for closing the contour, and therefore we introduce two further representations for the function D_E:

$$D_E=\left(\frac{\pi}{|e\varepsilon|}\right)^{1/2}[e^{i\pi/4}\Gamma(-\lambda)D_\lambda(\pm\tau)D_{-\lambda'^*-1}(\mp i\tau'^*)+e^{-i\pi/4}\Gamma(-\lambda'^*)D_{-\lambda-1}(\mp i\tau)D_{\lambda'^*}(\mp\tau'^*)], \quad (31)$$

which follow from Eq. (20) and formulas 8.2(7) and 8.2(8) of the Bateman collection.[10] The first terms of these representations have poles only on the semiaxis $\mathrm{Im}\nu>0$, and have in the half-plane $\mathrm{Im}\nu\leq 0$ the respective asymptotic forms

$$\pi(2|e\varepsilon|)^{-1/2}\exp[{}^1/_4 i\pi\mp i(2|e\varepsilon|\nu)^{1/2}(t-t')]$$

for $|\nu|\to\infty$. The second terms have poles only on the semiaxis $\mathrm{Im}\nu<0$, and in the half-plane $\mathrm{Im}\nu\geq 0$ as $|\nu|\to\infty$ they become

$$\pi(2|e\varepsilon|)^{-1/2}\exp[-{}^1/_4 i\pi\pm i(2|e\varepsilon|\nu)^{1/2}(t-t')].$$

Choosing for D_E the representation (31+) for $t-t'>0$ and the representation (31−) for $t-t'<0$, and closing the path of integration in the ν plane below in the integral of the first term and above for the second, we get a nonvanishing result from the first term only:

$$\int_{-\infty}^{\infty}\frac{dp^2}{(2\pi)^2}\frac{D_e}{p^2+m^2-i\delta}=\frac{i}{2\pi}\left(\frac{\pi}{|e\varepsilon|}\right)^{1/2}e^{i\pi/4}\Gamma(-\lambda)D_\lambda(\pm\tau)D_{-\lambda'^*-1}(\mp i\tau'^*),\quad t-t'\gtrless 0, \quad (32)$$

where in the right member $\nu=\nu_0$ or $p^2=-m^2$.

In this way we arrive at the well known representation of Nikishov[7]:

$$S^c(x,x')=\sum_{k=0}^{\infty}\int\frac{dp_2\,dp_3}{(2\pi)^3}E_{\omega p}(x)(m-i\gamma\bar{p})\bar{E}_{p\omega}(x')(1-e^{-2\pi\nu})^{1/2}=\sum_{k,r}\int\frac{dp_2\,dp_3}{(2\pi)^3}\frac{\omega\psi_{\omega pr}(x)\bar{\psi}_{p\omega r}(x')}{(1-e^{-2\pi\nu})^{1/2}},\quad \omega=\pm\ \text{for}\ t-t'\gtrless 0. \quad (33)$$

The matrices $E_{p\pm}$ are obtained from $E_{\pm p}$ by the replacement

$$e^{i\pi\lambda/4}\Gamma(-\lambda)D_\lambda(\pm\tau)\to\frac{\sqrt{2\pi}\,e^{1/4\,i\pi(-\lambda-2)}}{(1-e^{-2\pi\nu})^{1/2}}D_{-\lambda-1}(\pm i\tau) \quad (34)$$

and have ± frequency behavior as $t \to \mp\infty$. The solutions

$$\psi_{\omega pr} = (1-e^{-2\pi\nu})^{1/2} E_{\omega p} u_{\bar{p}r}, \quad \psi_{p\omega r} = (1-e^{-2\pi\nu})^{1/2} E_{p\omega} u_{\bar{p}r},$$
$$m - i\gamma\bar{p} = \sum_{r=1,2} \omega u_{\bar{p}r} \bar{u}_{\bar{p}r}, \tag{35}$$

of the Dirac equation are normalized in the following way:

$$\int d^3x \begin{pmatrix} \psi^{+}_{\omega' p' r'} \psi_{\omega pr} & \psi^{+}_{\omega' p' r'} \psi_{p\omega r} \\ \psi^{+}_{p'\omega' r'} \psi_{\omega pr} & \psi^{+}_{p'\omega' r'} \psi_{p\omega r} \end{pmatrix}$$
$$= (2\pi)^3 \delta(p_2'-p_2)\delta(p_3'-p_3)\delta_{k'k}\delta_{r'r} \begin{pmatrix} e^{1/2\pi\nu(\omega'\omega-1)} & \delta_{\omega'\omega}(1-e^{-2\pi\nu})^{1/2} \\ \delta_{\omega'\omega}(1-e^{-2\pi\nu})^{1/2} & \omega'\omega e^{1/2\pi\nu(\omega\omega'-1)} \end{pmatrix}. \tag{36}$$

The representation (33) is remarkable for its explicit causality; the state of the electron in the field is affected only by the positive-frequency state in the past and the negative-frequency state in the future. The factor $(1-e^{-2\pi\nu})^{1/2}$ is the probability amplitude for the state with quantum numbers k, p_2, p_3 to be free. The method given here is based on papers by the writer[11] and by Nikishov[7] (see also Refs. 1, 2).

3. THE MASS OPERATOR AND ITS EIGENVALUE

We shall use the eigenfunctions E_p to calculate the mass operator of the electron in a constant field. In second order in the radiation field the mass operator in the x representation is given by the expression

$$M(x, x') = ie^2 \gamma_\mu S^c(x, x') \gamma_\nu D_{\mu\nu}^c(x-x'), \tag{37}$$

where S^c is the propagation function of the electron in the external field, and D^c is the photon propagation function. Using for S^c the representation (29) and for D^c the analogous representation

$$D^c(z) = \frac{-i}{(4\pi)^2} \int_0^\infty \frac{dt}{t^2} \exp\left(\frac{iz^2}{4t}\right), \tag{38}$$

we get for the E_p transform of the mass operator the expression

$$M(p, p') = \frac{-ie^2}{(4\pi)^4} \iint_0^\infty \frac{ds\, dt}{t^2} \frac{e^2 \eta\varepsilon e^{-im^2 s}}{\sin e\eta s \operatorname{sh} e\varepsilon s} J. \tag{39}$$

Here J is an integral over the coordinates, which is a γ matrix:

$$J = \int d^4x\, d^4x' \bar{E}_{\omega p}(x) \gamma_\mu \left(m - \frac{i}{2}\gamma Bz\right) e^{1/2 ie\sigma F s} \gamma_\mu E_{\omega' p'}(x') e^{i\varphi + izAz/4}, \tag{40}$$

where $z = x - x'$, φ is the nondiagonal phase of the Green's function, and $A = \beta + t^{-1}$ and $B = \beta + eF$ are 4×4 matrices, cf. Eq. (29). The γ matrix appearing between the E_p matrices can be written in the form

$$\Gamma = 4m(S + i\gamma_5 P) + ie^{-1/2 ie\sigma F s} \gamma Bz,$$
$$S = \cos e\eta s \operatorname{ch} e\varepsilon s, \quad P = \sin e\eta s \operatorname{sh} e\varepsilon s. \tag{41}$$

Since the E_p matrices commute with the matrices γ_5 and σF, to find J it is necessary to know only the integrals J_0, J_1:

$$J_0, J_1 = \int d^4x\, d^4x' \bar{E}_{\omega p}(x) (1, \gamma Bz) E_{\omega' p'}(x') e^{i\varphi + izAz/4}. \tag{42}$$

The way these integrals are calculated is given in the Appendix; the result is the foundation of the E_p-function method:

$$J_0, J_1 = (2\pi)^4 \delta(p-p') \delta_{\omega\omega'} \frac{4ie^{-1/2 ie\sigma F w - i\bar{p}w\bar{p}}}{(\det(A-eF))^{1/2}} (1, 2\gamma B(A-eF)^{-1}\bar{p}). \tag{43}$$

Here w is a symmetric matrix function of the field matrix F:

$$w = \frac{1}{eF} \operatorname{Arcth} \frac{A}{eF} = \frac{1}{2eF} \ln \frac{A+eF}{A-eF}, \tag{44}$$

and $(2\pi)^4 \delta(p - p')\delta_{\omega\omega'}$ denotes the right hand side of Eq. (13). We remark that Eq. (43) is valid for arbitary 4×4 matrices A and B, under the condition that A is a symmetric function of F. In our case $A = eF \coth eFs + t^{-1}$ and the matrix w has two twofold positive eigenvalues,

$$w_1 = \frac{1}{e\eta} \operatorname{arcctg}\left(\operatorname{ctg} e\eta s + \frac{1}{e\eta t}\right), \quad w_2 = \frac{1}{e\varepsilon} \operatorname{Arcth}\left(\operatorname{cth} e\varepsilon s + \frac{1}{e\varepsilon t}\right), \tag{45}$$

which play the role of magnetic and electric characteristic times of an electron which interacts not only with the external field but also with the radiation field. Owing to the latter, $w_{1,2}$ lag s. But w_1 is always in the same period as s: $n\pi/|e\eta| < w_1 < s < (n+1)\pi/|e\eta|$; and it even coincides with s when it passes through a boundary between periods, i.e., a singularity of $\cot e\varepsilon s$. At the same time, w_2 for $s \to \infty$ and finite t goes further and further from s, approaching $(2e\varepsilon)^{-1}\ln(1+2e\varepsilon t)$. For $t \to \infty$ both functions $w_{1,2} \to s$, and for $t \to 0$, $w_1 \to n\pi/|e\varepsilon|$, the beginning of the period that contains s, and $w_2 \to 0$.

Finally, we give the expansions

$$w_1 = \omega + {}^1/_3(e\eta\omega)^2(s-\omega) + \ldots, \quad w_2 = \omega - {}^1/_3(e\varepsilon\omega)^2(s-\omega) + \ldots, \tag{46}$$

which are valid for sufficiently small fields η, ε. We shall call w the matrix of radiative characteristic time.

Equation (43) may usefully be compared with the analogous formula in electrodynamics without an external field ($F=0$), in which plane waves e^{ipx} replace the functions E_p, p being the ordinary four-momentum p_α, and in which we have the ordinary conservation laws

$$J_0, J_1 = (2\pi)^4 \delta(p-p') \frac{4ie^{-ipA^{-1}p}}{(\det A)^{1/2}} (1, 2\gamma BA^{-1}p). \tag{47}$$

It follows from Eq. (43) that the effective (mean) value of the relative coordinate z_α is given by

$$z_{\alpha\,\mathrm{eff}} = 2(A-eF)^{-1}_{\alpha\beta} \bar{p}_\beta = \left(\frac{e^{2eFw}-1}{eF}\right)_{\alpha\beta} \bar{p}_\beta. \tag{48}$$

This formula acquires a simple physical meaning if we recall the law of motion of a classical particle in a constant field[9]:

$$x_\alpha(s) - x_\alpha(0) = \left(\frac{e^{2eFs}-1}{eF}\right)_{\alpha\beta} \Pi_\beta(0). \tag{49}$$

Thus the effective relative coordinate varies like the coordinate of a classical particle, except for two differences: It is not s that plays the part of a characteristic time, but w (owing to the additional interaction with the radiation field), and the quantum vector $\bar{p}$ plays the role of the initial[3)] kinetic momentum.

In what follows we shall need the expansion of the matrix w in terms of two independent matrices:

$$w_{\alpha\beta} = \frac{\varepsilon^2 w_1 + \eta^2 w_2}{\eta^2+\varepsilon^2} \delta_{\alpha\beta} - \frac{w_1 - w_2}{\eta^2+\varepsilon^2} (FF)_{\alpha\beta}, \tag{50}$$

[compare with (56)] which in the important special case of a crossed field reduces to

$$w_{\alpha\beta}=\omega\delta_{\alpha\beta}-{}^{1}/_{3}\omega^{2}(s-\omega)(e^{2}FF)_{\alpha\beta},\quad \omega^{-1}=s^{-1}+t^{-1}. \tag{51}$$

In the general case this last formula is approximate, with accuracy up to terms of order $(eF)^4$. It can be seen from the representation (50) that the vector $\bar{p}$ appears in the exponent in Eq. (43) only in the form of the invariants p^2 and $(eF\bar{p})^2$.

By the use of Eqs. (39), (40), and (43) the mass operator reduces to the diagonal form

$$M(p,p')=(2\pi)^4\delta(p-p')\delta_{\omega\omega'}M_R(\bar{p},F),$$

where

$$M_R(\bar{p},F)=\frac{\alpha}{2\pi}\int\!\!\int_0^\infty\frac{ds\,dt}{t^2}\left\{\frac{\sin e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}\exp\left(-im^2s-i\bar{p}w\bar{p}-\frac{i}{2}e\sigma Fw\right)\right.$$
$$\times\left[2m(S+i\gamma_5P)+i\exp\left(-\frac{i}{2}e\sigma Fs\right)\gamma\exp(eF(w+s))\frac{\mathrm{sh}\,eFw}{\mathrm{sh}\,eFs}\bar{p}\right]$$
$$\left.-\frac{\omega^2}{s^2}\exp(-im^2s-ip^2\omega)\left(2m+i\gamma\bar{p}\frac{\omega}{s}\right)\right\}+M_R^0(\bar{p}). \tag{52}$$

In this expression a renormalization has been carried out[4)] (see Refs. 2, 12), and use has been made of the relation

$$(\det(\beta+t^{-1}-eF))^{1/2}=\frac{e^2\eta\varepsilon}{\sin e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2}. \tag{53}$$

Because the motion in the electric field is infinite, an infrared divergence for $t\to\infty$ arises, in general, in M; this divergence can be removed by introducing a factor $e^{-i\mu^2 t}$, i.e., a small photon mass μ.

It would be more correct to call the γ-matrix function $M(\bar{p},F)$ an eigenvalue of the mass operator, in analogy with the fact that $\gamma\bar{p}$ is an eigenvalue of the operator $\gamma\Pi$. Like every eigenvalue, $M(\bar{p},F)$ is an invariant; it does not depend on the choice of representation. In a different representation, characterized by a different complete set (5), different eigenvalues (6), and different eigenfunctions (7), $M(\bar{p},F)$ would be the same function of $\bar{p}$ and F, and only the dependence of $\bar{p}$ on the new eigenvalues would be different.

In the special case of a crossed field the mass operator (52) is identical with that found previously by the present writer,[1,2] and in the special case of a magnetic field it is identical with the result of Tsai and Yildiz.[13] In the general case of electric and magnetic fields a mass operator has been found by Baĭer, Katkov, and Strakhovenko.[14] However, the operator they give is cumbersome, and a comparison with Eq. (52) is difficult.

4. THE ELASTIC SCATTERING AMPLITUDE

On the mass shell $p^2=-m^2$ the matrix element of the mass operator between "free" spinors gives the characteristic amplitude for elastic scattering of the electron in an intense constant field:

$$T(\bar{p},F)=-\bar{u}_{\bar{p}\zeta}M(\bar{p},F)u_{\bar{p}\zeta}=-\mathrm{tr}[M(\bar{p},F)u_{\bar{p}\zeta}\bar{u}_{\bar{p}\zeta}]. \tag{54}$$

Here $u_{\bar{p}\zeta}$ is the spinor corresponding to a definite polarization of the electron, so that $i\gamma_5\gamma\bar{s}u_{\bar{p}\zeta}=u_{\bar{p}\zeta}$. As for the free electron, the polarization four-vector $\bar{s}$ satisfies the conditions $\bar{s}^2=1$, $\bar{s}\bar{p}=0$. Accordingly

$$u_{\bar{p}\zeta}\bar{u}_{\bar{p}\zeta}=\frac{c}{4m}(m-i\gamma\bar{p})(1+i\gamma_5\gamma\bar{s}),\quad c=\mathrm{tr}(u\bar{u}), \tag{55}$$

is the corresponding polarization density matrix.

Matrix functions of the field matrix F appear in the calculation of the trace. In the final stage of the calculations it is convenient to represent these functions as expansions in terms of four independent matrices with invariant coefficients

$$f(F)=\frac{1}{\mathrm{tr}(\bar{F}F)}[\mathrm{tr}(F^*F^*\bar{f})I+\mathrm{tr}(\bar{f})FF+\mathrm{tr}(F\bar{f})F+\mathrm{tr}(F^*\bar{f})F^*]. \tag{56}$$

Here a bar over a matrix means that it is multipled by the involution matrix diag$(-1,-1,1,1)$. As usual, $F^*_{\alpha\beta}=(i/2)\varepsilon_{\alpha\beta\gamma\delta}F_{\gamma\delta}$ is the tensor dual to $F_{\alpha\beta}$, and its form in the diagonal representation is diag$(i\varepsilon,-i\varepsilon,-\eta,\eta)$ [cf. Eq. (28)]. We have already used the representation (56) for the symmetric matrix w [see Eq. (50)].

The result is the following expression for the amplitude:

$$T(\bar{p},F)=-cm\frac{\alpha}{2\pi}\int\!\!\int_0^\infty\frac{ds\,dt}{t^2}\left\{\frac{\sin e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}e^{-im^2s-i\bar{p}w\bar{p}}\left[a+\chi^2b\right.\right.$$
$$\left.\left.+i\frac{e\bar{s}F^*\bar{p}}{m^3}c_1+i\frac{e\bar{s}F\bar{p}}{m^3}c_2\right]-\frac{\omega^2}{s^2}\left(2-\frac{\omega}{s}\right)e^{-im^2(s-\omega)}\right\}. \tag{57}$$

Here a, b, c_1, c_2 are invariant functions of the parameters η, ε and the characteristic times:

$$a=2\cos e\eta s\,\mathrm{ch}\,e\varepsilon s\cos e\eta w_1\,\mathrm{ch}\,e\varepsilon w_2+2\sin e\eta s\,\mathrm{sh}\,e\varepsilon s\sin e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2$$
$$-\frac{\varepsilon^2}{\eta^2+\varepsilon^2}\mathrm{ch}\,e\varepsilon(w_2+s)\frac{\sin e\eta w_1}{\sin e\eta s}-\frac{\eta^2}{\eta^2+\varepsilon^2}\cos e\eta(w_1+s)\frac{\mathrm{sh}\,e\varepsilon w_2}{\mathrm{sh}\,e\varepsilon s}, \tag{58}$$

$$b=\frac{m^4}{e^2(\eta^2+\varepsilon^2)}\left[\mathrm{ch}\,e\varepsilon(w_2+s)\frac{\sin e\eta w_1}{\sin e\eta s}-\cos e\eta(w_1+s)\frac{\mathrm{sh}\,e\varepsilon w_2}{\mathrm{sh}\,e\varepsilon s}\right], \tag{59}$$

$$\binom{c_1}{c_2}=\frac{m^2}{e(\eta^2+\varepsilon^2)}\left[\binom{\eta}{-\varepsilon}\left(2\cos e\eta s\,\mathrm{ch}\,e\varepsilon s\sin e\eta w_1\,\mathrm{ch}\,e\varepsilon w_2\right.\right.$$
$$\left.-2\sin e\eta s\,\mathrm{sh}\,e\varepsilon s\cos e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2-\sin e\eta(w_1+s)\frac{\mathrm{sh}\,e\varepsilon w_2}{\mathrm{sh}\,e\varepsilon s}\right)$$
$$+\binom{\varepsilon}{\eta}\left(2\cos e\eta s\,\mathrm{ch}\,e\varepsilon s\cos e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2+2\sin e\eta s\,\mathrm{sh}\,e\varepsilon s\sin e\eta w_1\,\mathrm{ch}\,e\varepsilon w_2\right.$$
$$\left.\left.-\mathrm{sh}\,e\varepsilon(w_2+s)\frac{\sin e\eta w_1}{\sin e\eta s}\right)\right]. \tag{60}$$

The amplitude depends on the two field parameters $e\eta m^{-2}$, $e\varepsilon m^{-2}$ and on a dynamical parameter χ which includes the entire dependence on the momentum $\bar{p}$:

$$\chi=m^{-3}((eF\bar{p})^2)^{1/2}=m^{-3}[e^2\eta^2\bar{p}_\perp^2+e^2\varepsilon^2(m^2+\bar{p}_\perp^2)]^{1/2}. \tag{61}$$

The parameter χ takes discrete values if the magnetic field is not zero, $\eta\neq 0$, since in this case $\bar{p}_\perp^2=2|e\eta|k$, $k=0,1,2,\ldots$. If $\eta=0$, then $\bar{p}_\perp^2$ and χ take continuous values. Finally, if $\eta=\varepsilon=0$ but $F\neq 0$ (crossed field), χ is continuous, while the right member of Eq. (61) takes the form $m^{-3}|eF|p_-$, where p_- is an eigenvalue of the operator Π_-. As for the invariants $\bar{s}F^*\bar{p}$ and $\bar{s}F\bar{p}$, they are the mean values of the spin matrices $\frac{1}{2}\sigma F$ and $\frac{1}{2}\sigma F^*$ in the state $u_{\bar{p}\zeta}$:

$$-\frac{\bar{s}F^*\bar{p}}{m}=c^{-1}\bar{u}_{\bar{p}\zeta}\frac{1}{2}\sigma Fu_{\bar{p}\zeta},\quad \frac{\bar{s}F\bar{p}}{m}=c^{-1}\bar{u}_{\bar{p}\zeta}\frac{1}{2}\sigma F^*u_{\bar{p}\zeta}. \tag{62}$$

Two mutually orthogonal spinors $u_{\bar{p}\zeta}$ and $u_{\bar{p}-\zeta}$ can be chosen so that for them

$$-\bar{s}F^{*}\bar{p}/m=\pm\eta,\quad \bar{s}F\bar{p}/m=\pm\varepsilon. \qquad (63)$$

One such pair of mutually orthogonal spinors is the first and fourth columns of the matrix $m-i\gamma\bar{p}$; their polarization vectors are $\pm\bar{s}_{\alpha}$, where

$$\bar{s}_1=0,\ \bar{s}_2=-\bar{p}_2/m,\ \bar{s}_-=-\bar{p}_-/m,\ \bar{s}_+=(m^2-\bar{p}_2^2)/2m\bar{p}_-. \qquad (64)$$

Another pair of mutually orthogonal spinors satisfying Eq. (63) is the third and second columns of the same matrix $m-i\gamma\bar{p}$; for them the polarization vectors are $\pm\bar{s}'_{\alpha}$, where

$$\bar{s}_1'=0,\ \bar{s}_2'=\bar{p}_2/m,\ \bar{s}_-'=(-m^2+\bar{p}_2^2)/2m\bar{p}_+,\ \bar{s}_+'=\bar{p}_+/m. \qquad (65)$$

Thus it is clear that the terms in the amplitude proportional to $\bar{s}F^{*}\bar{p}$ and $\bar{s}F\bar{p}$ depend on the spin orientation and are governed by the interactions of the AMM with the magnetic field and of the AEM with the electric field.

The amplitude (57) has not been previously derived. In the case $\eta=\varepsilon=0$ (but $F\neq 0$) it becomes the elastic scattering amplitude for an electron in a crossed field,[12] and in the case $\varepsilon=0$ it agrees with the amplitude for an electron in a magnetic field.[13]

In proceeding with the study of the amplitude (57), we note that it is invariant under the replacements $\eta\rightleftarrows -i\varepsilon$ if the parameter χ and the spin invariants are kept the same. We shall now consider three physical effects: The appearance of an AEM of the electron in an electromagnetic field, the behavior of the AMM in an electric field, and the mass shift and damping of the ground state of an electron in an electric field.

5. THE ANOMALOUS ELECTRIC MOMENT

As is well known, besides its normal magnetic moment $e/2m$ an electron in the vacuum has an AMM determined by its internal structure—its interaction with the radiation field. An intense external field changes the interaction with the radiation field and therefore changes the AMM. In the vacuum the electron has no electric moment (neither normal nor anomalous), because the interaction is invariant under space and time inversions. An applied field, however, by changing the radiative interaction, can produce an electric moment if the field has a pseudoscalar $\mathbf{E}\cdot\mathbf{H}\neq 0$.

The anomalous magnetic and electric moments are linear in the spin, but whereas the AMM is a pseudovector, the AEM is a vector, and is therefore odd with respect to both the electric field and the magnetic field. The interactions of the AMM with the magnetic field and of the AEM with the electric field lead to an electron-mass change that depends on the spin orientation. The change of mass Δm of an electron in an external field is related to the characteristic elastic scattering amplitude T by the simple equation $T=-c\Delta m$. It follows from Eq. (57) that the change of mass Δm_E brought about by the AEM is given by

$$\Delta m_E=\frac{ie\bar{s}F\bar{p}}{m^2}\frac{\alpha}{2\pi}\int\!\!\int_0^\infty\frac{ds\,dt}{t^2}\frac{\sin e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}e^{-im^2s-i\bar{p}w\bar{p}}c_2. \qquad (66)$$

The magnitude $\Delta\nu$ of the AEM is given by the real part of Δm_E, namely $\mathrm{Re}\Delta m_E=-\Delta\nu E=-\Delta\nu\bar{s}F\bar{p}/m$ (compare this with the energy $-\Delta\boldsymbol{\nu}\cdot\mathbf{E}$ of a classical dipole moment). Then in units of the Bohr magneton $\nu_0=e/2m$ we get

$$\frac{\Delta\nu}{\nu_0}=-\frac{\alpha}{\pi}\int\!\!\int_0^\infty\frac{ds\,dt}{t^2}\frac{\sin e\eta w_1\,\mathrm{sh}\,e\varepsilon w_2}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}\sin(m^2s+\bar{p}w\bar{p})c_2. \qquad (67)$$

This expression, like (66), is a complicated function of η, ε, and χ and vanishes if either of the field η or ε is equal to zero [factor c_2, see Eq. (60)]. Because it is odd under inversion of time reversal, $\Delta\nu/\nu_0$ changes sign for $\varepsilon\to-\varepsilon$ or $\eta\to-\eta$.

Let us examine the important special case when the fields η, ε are small in comparison with m^2/e. Then

$$c_2={}^5/_3e^2\eta\varepsilon m^2\omega s(s-\omega)+\ldots,\quad \bar{p}w\bar{p}=-m^2\omega+{}^1/_3m^6\omega^2(s-\omega)\chi^2+\ldots, \qquad (68)$$

see also Eq. (46). We change to the variables

$$u=s/t,\quad x=m^2\omega(s/t)^{1/3}\chi^{2/3}. \qquad (69)$$

Then

$$\Delta m_E=\frac{e\bar{s}F\bar{p}}{m^2}\frac{5\alpha}{6\pi}\frac{e^2\eta\varepsilon}{m^4}\int_0^\infty\frac{du\,z^3f''(z)}{u^2(1+u)},\quad z=\left(\frac{u}{\chi}\right)^{2/3}, \qquad (70)$$

where $f(z)$ is a special function introduced previously[12] by the writer and studied in detail in Ref. 2:

$$f(z)=i\int_0^\infty dx\exp\{-i(xz+x^3/3)\}.$$

This function is characteristic for processes in crossed fields, which is essentially what we have here, since the entire dependence on the actual field parameters has been factored out in the factor $e^2\varepsilon\eta/m^4$. The expression for $\Delta\nu/\nu_0$ is obtained from Eq. (70) by setting $e\bar{s}F\bar{p}/2m^2\to-1, f''(z)\to\Upsilon''(z)$, where $\Upsilon(z)=\mathrm{Re}f(z)$ (cf. Ref. 2).

For small and large values of the parameter χ [see Eq. (61)] we get

$$\Delta m_E=\frac{e\bar{s}F\bar{p}}{m^2}\frac{5\alpha}{6\pi}\frac{e^2\eta\varepsilon}{m^4}\times\begin{cases}-2\left(\ln\frac{3^{1/2}\gamma}{\chi}-\frac{9}{4}+\ldots\right)+i\frac{3^{1/2}\pi}{4\chi}\left(1-\frac{4\chi}{3^{1/2}}+\ldots\right) & \chi\ll 1,\\ -\chi^{-2}\ln\frac{\chi}{3^{1/2}\gamma}+\ldots+\frac{i\pi}{2}\chi^{-2}+\ldots, & \chi\gg 1.\end{cases} \qquad (71)$$

According to Eq. (61) the parameter χ cannot be smaller than $\chi_{\min}=|e\varepsilon|m^{-2}$. Accordingly, as χ increases the AEM $\Delta\nu/\nu_0$ decreases monotonically from the value

$$\frac{10\alpha}{3\pi}\frac{e^2\eta\varepsilon}{m^4}\left(\ln\frac{3^{1/2}\gamma m^2}{|e\varepsilon|}-\frac{9}{4}\right)$$

to zero.

6. THE ANOMALOUS MAGNETIC MOMENT

The formula for the AMM is obtained by replacing c_2 with $-c_1$ in Eq. (67). In particular, for an electron in an electric field the AMM is given by

$$\frac{\Delta\mu}{\mu_0}=\frac{\alpha}{\pi\beta}\int\!\!\int_0^\infty\frac{ds\,dt}{t^2}\frac{\omega\,\mathrm{sh}\,e\varepsilon w_2}{s\,\mathrm{sh}\,e\varepsilon s}\left[2\,\mathrm{ch}\,e\varepsilon s\,\mathrm{sh}\,e\varepsilon w_2-\frac{\omega}{s}\mathrm{sh}\,e\varepsilon(s+w_2)\right]\times\sin m^2\left(s-\omega+\frac{\chi^2}{\beta^2}(\omega-w_2)\right), \qquad (72)$$

where $\beta=e\varepsilon m^{-2}$. This expression thus depends on two independent parameters, β and χ.

Let us examine the important case in which the electric field is weak and the transverse momentum of the electron is large compared with the mass. Then $\beta \ll 1$, χ, while χ can be either small or large compared with unity. Expanding the integrand in Eq. (72) in powers of the field and introducing the variables (69) instead of s, t, we get

$$\frac{\Delta\mu}{\mu_0}=\frac{\alpha}{\pi}\int_0^\infty \frac{du\, z\Upsilon(z)}{(1+u)^3}, \quad z=\left(\frac{u}{\chi}\right)^{2/3}. \tag{73}$$

The function $\Upsilon(z)$ was already mentioned in Sec. 5. This expression is identical with the AMM in a crossed field, if we take χ to mean the expression corresponding to a crossed field, $|eF|p_-m^{-3}$ [(see Eq. (12)]. We call attention to the fact that for small values of the parameter χ the AMM is close to the Schwinger value $\alpha/2\pi$, and for large χ it decreases as $\chi^{-3/2}$:

$$\frac{\Delta\mu}{\mu_0}=\begin{cases}\frac{\alpha}{2\pi}\left[1-\chi^2\left(12\ln\frac{3^{1/2}\gamma}{\chi}-37\right)+\ldots\right], & \beta\ll\chi\ll1\\ \frac{\Gamma(^1/_3)\alpha}{9\cdot3^{1/3}(3\chi)^{2/3}}+\ldots, & \chi\gg1\end{cases}. \tag{74}$$

Equation (73) ceases to be valid when χ approaches β, i.e., when the transverse momentum of the electron is comparable with or smaller than the mass of the electron.

Let us, therefore, consider a different case, that in which $\bar{p}_\perp=0$, so that χ takes its smallest value, $\chi=\beta$. In this case Eq. (72) becomes a function of the single parameter β. For small values of β we get

$$\frac{\Delta\mu}{\mu_0}=\frac{\alpha}{2\pi}\left[1-\beta^2\left(\frac{8}{3}\ln\frac{\gamma}{2\beta}-\frac{46}{9}\right)+\ldots\right], \quad \beta=\chi\ll1. \tag{75}$$

This dependence on small $\beta=\chi$ is like the dependence on small χ for $\beta \ll \chi \ll 1$, Eq. (47). However, the coefficient on the main, logarithmic term in Eq. (75) is much smaller than that in Eq. (74). This means that the parameter β not taken into account in Eq. (74) increases the AMM, at the same time that the parameter χ diminishes it. With increasing β its effect in increasing the AMM becomes stronger, as can be seen by investigating the limiting case $\beta=\chi\gg1$:

$$\frac{\Delta\mu}{\mu_0}=\frac{\alpha}{\pi}\left[1-\frac{\pi}{4\beta}\ln\frac{2\beta}{\gamma}+\ldots\right], \quad \beta=\chi\gg1. \tag{76}$$

Accordingly, as the parameter $\beta=\chi$ increases, the AMM first falls below the Schwinger value $\alpha/2\pi$; it reaches a minimum value for $\beta=\chi\sim1$, and then increases, approaching the limit α/π, which is twice the Schwinger value.

7. THE SHIFT AND DAMPING OF THE GROUND STATE IN AN ELECTRIC FIELD

Setting $\eta=0$ in Eq. (75), we get the amplitude for the change of mass of the electron in an electric field:

$$\Delta m=m\frac{\alpha}{2\pi}\int_0^\infty\!\!\int\frac{dx\,du}{x(1+u)}\left\{\frac{\operatorname{sh}y}{\operatorname{sh}x}\left[2\operatorname{ch}x\operatorname{ch}y-\frac{\operatorname{ch}(x+y)}{1+u}+\frac{\chi^2}{\beta^2}\left(\frac{\operatorname{ch}(x+y)}{1+u}-\frac{\operatorname{sh}y}{\operatorname{sh}x}\right)\right]\exp\left[\frac{-i}{\beta}\left(\frac{xu}{1+u}+\frac{\chi^2}{\beta^2}\left(\frac{x}{1+u}-y\right)\right)\right]-\frac{1+2u}{(1+u)^2}\exp\left[-\frac{ixu}{\beta(1+u)}\right]\right\}e^{-i\lambda x/u}. \tag{77}$$

Here instead of s, t we have used the variables $x=|e\varepsilon|s$, $u=s/t$ and have introduced the notation $y=|e\varepsilon|w_2$. The function (77) depends on three dimensionless parameters χ, $\beta=|e\varepsilon|m^{-2}$ and $\lambda=\mu^2/|e\varepsilon|$. The last one, the infrared parameter, will always be regarded as the smallest, and we set it equal to zero wherever possible.

In an electric field Δm does not depend on the polarization of the electron; the interaction with the AMM is turned off because there is no magnetic field, and the interaction with the AEM vanishes because the AEM is zero. Let us examine in more detail the Δm for the ground state of the electron, i.e., the state with $\bar{p}_\perp^2=0$ (so-called hyperbolic motion). In this case $\chi=\beta$, and we get

$$\Delta m=m\frac{\alpha}{2\pi}\int_0^\infty\!\!\int\frac{dx\,du}{x(1+u)}\left\{\frac{\operatorname{sh}y}{\operatorname{sh}x}\left(2\operatorname{ch}x\operatorname{ch}y-\frac{\operatorname{sh}y}{\operatorname{sh}x}\right)\exp\left[-\frac{i(x-y)}{\beta}\right]-\frac{1+2u}{(1+u)^2}\exp\left[-\frac{ixu}{\beta(1+u)}\right]\right\}e^{-i\lambda x/u}. \tag{78}$$

Using the definition (45), we can express the functions of y appearing here in terms of x and u:

$$y=\frac{1}{2}\ln\frac{\operatorname{cth}x+ux^{-1}+1}{\operatorname{cth}x+ux^{-1}-1},$$
$$\frac{\operatorname{sh}y}{\operatorname{sh}x}\left(2\operatorname{ch}x\operatorname{ch}y-\frac{\operatorname{sh}y}{\operatorname{sh}x}\right)=\frac{\operatorname{cth}^2x+2ux^{-1}\operatorname{cth}x+1}{(\operatorname{cth}x+ux^{-1}-1)(\operatorname{cth}x+ux^{-1}+1)}. \tag{79}$$

The singularities of the integrand in the sector $-\pi/2<\arg x<0$ of the x plane, at the points were $\coth x+ux^{-1}-1=0$, do not allow us to turn the path of integration over x onto the negative imaginary axis, on which the integrand is real. However, a proper consideration of these singularities leads to a different representation for Δm.

Let us consider two limiting cases: $\beta\ll1$ and $\beta\gg1$. In the case $\beta\ll1$ the essential contributions to the integral (78) come from two ranges of integration which we shall call the quantum region and the classical region. In the classical region $x_{\text{eff}}\sim1$, $u_{\text{eff}}\sim\beta\ll1$, and in the quantum region $x_{\text{eff}}\sim\beta\ll1$ and $u_{\text{eff}}\sim1$. The contribution from each region can be calculated by dividing the range of integration over x at a point x_0, which satisfies the condition $\beta\ll x_0\ll1$, and expanding the integrand in the quantum region $(0<x<x_0)$ in terms of x, and in the classical region $(x_0<x<\infty)$ in terms of u. The total contribution does not depend on x_0, and the result is

$$\operatorname{Re}\Delta m=m\frac{\alpha}{2\pi}\left[-\beta\pi+\beta^2\left(\frac{4}{3}\ln\frac{\gamma}{2\beta}+\frac{4}{9}\right)+\ldots\right],$$
$$\operatorname{Im}\Delta m=-m\frac{\alpha}{2\pi}\left[\beta\left(2\ln\frac{2\beta m}{\gamma\mu}-1\right)-\beta^2\frac{2\pi}{3}+\ldots\right]. \tag{80}$$

The first terms in these expansion are purely classical: they do not depend on $\hbar$, when we take into account, the fact that in classical theory one must use instead of the photon mass a minimum wave number $k_{\perp0}=\mu c/\hbar$. The classical term in the probability $-2\operatorname{Im}\Delta m$ of emission of radiation is identical with the integral of the classical radiation spectrum found by Nikishov and the writer.[15]

The term in $\operatorname{Re}\Delta m$ which is linear in the magnitude of the field is larger by a factor 2π than the term linear in the field in the case of a magnetic field, in which it is due to the AMM found by Schwinger. In the present case this term represents the work which the electron does in the rearrangement of its self-field on being accelerated in the electric field. Therefore Δm_{cl} is determined as the difference of the interactions in the field and in vacuum (indices F and 0):

$$\Delta W_{\rm cl}=-\Delta m_{\rm cl}\tau=\frac{i}{2}\int d^4x d^4x' j_\alpha(x) j_\alpha(x')\Delta^c(x-x',\mu)\Big|_0^F$$

$$=\frac{ie^2}{2}\int_{-\infty}^{\infty}\int ds ds' \dot{x}_\alpha(s)\dot{x}_\alpha(s')\Delta^c(x(s)-x(s'),\mu)\Big|_0^F$$

$$=-\tau m\frac{\alpha}{2\pi}\beta\int_0^\infty dz \exp\left(\frac{-i\lambda}{2z}\right)\left[e^{iz}K_1(iz)-\left(\frac{\pi}{2iz}\right)^{1/2}\right],\quad \lambda=\left(\frac{\mu m}{e\varepsilon}\right)^2\to 0, \tag{81}$$

where $j\alpha(x)$ and $x\alpha(s)$ are the classical current and coordinate of a charge executing myperbolic motion in the electric field. Equation (81) reproduces the classical terms of the expansions (80).

As the field is made larger, the motion of the electron ceases to be classical and the change of its mass becomes an essentially nonlinear function of the field.

In the case $\beta \gg 1$ the main contribution to the integral (78) is from the region $x_{\rm eff} \sim \beta \gg 1$, $(\mu/m)^2 \lesssim u_{\rm eff} \lesssim 1$. We find

$$\begin{aligned}\operatorname{Re}\Delta m&=m\frac{\alpha}{2\pi}\left[\frac{1}{4}\left(\ln\frac{2\beta}{\gamma}\right)^2+\ldots\right],\\ \operatorname{Im}\Delta m&=-m\frac{\alpha}{2\pi}\left[2\beta\ln\frac{m}{\mu}+\frac{\pi}{4}\ln\frac{2\beta}{\gamma}+\ldots\right].\end{aligned} \tag{82}$$

Accordingly, as the field increases, the mass of the electron in an electric field at first decreases, reaches a minimum at a field of the order of the critical field ($\beta \sim 1$), and then increases as the square of the logarithm of the field. This dependence of $\operatorname{Re}\Delta m$ on the electric field is similar to the dependence of the mass shift on a magnetic field, as found by Demeur and considerd in papers[16] by Jancovici[17] and Newton.[18] In both cases the largest diminution of the mass is small in comparison with m, of the order of αm, but for the electric field it is larger by about a factor 2π than in the case of the magnetic field.

A qualitative feature of the motion of an electron in an electric field is that there is no stable state; even in the ground state ($\bar{p}_\perp^2=0$) the electron radiates. The electron thus goes into an excited state with transverse momentum $\bar{p}_\perp' \neq 0$, and a photon carries of the transverse momentum $\mathbf{k}_\perp = -\bar{\mathbf{p}}_\perp'$. The necessary energy for such a transition arises because of the work the electric field does on the charge. Thus the instability of the ground state is a manifestation of the ability of the field to do work on the charge, and the radiation is a consequence of a change of state of the electron. Nikishov and the writer have called attention to the fact that a system in a constant electromagnetic field cannot emit photons without there being a change of state of electrons; for example, a field in vacuum can emit photons only as a result of pair production in the field.

As can be seen from Eqs. (80) and (82), the probability of radiation emission, $w_{\rm rad}=-2\operatorname{Im}\Delta m$ is given by the classical expression over a wide range of the parameter β. Quantum corrections become important for $\beta \gtrsim 1$ and slightly weaken the increase of the probability of radiation with increase of the field. For $\beta \sim \alpha^{-1}=137$ the imaginary part of Δm becomes of the order of m, and we may suppose that the interaction with the radiation field becomes strong. It is, however, not out of the question that the next radiative correction may be of the order of $\alpha^2\beta\ln\beta$, so that the parameter of the perturbation treatment would be $\alpha\ln\beta$.

We note that the probability of quantum radiation of an electron in an electromagnetic field was first found by Nikishov,[19] in quite a different form than the imaginary part of (77) and (78), namely in the form of a double or single (for $\bar{p}_\perp=0$) integral of the square of a hypergeometric function.

The dependence of Δm on β which we have found holds qualitatively also for excited states with $\bar{p}_\perp^2 \lesssim m^2$. If $\bar{p}_\perp^2 \gg m^2$, the important parameter is χ, and the dependence of Δm on χ will be the same as in the case of a crossed field.[12]

8. CONCERNING THE CLASSICAL PART OF THE MASS SHIFT

Because of the importance of the classical mass shift, which is not restricted to electrodynamics, we shall consider it in more detail. According to the quantum theory of scattering the quantity $\Delta W_{\rm cl}$, Eq. (81), determines the probability amplitude $\exp(i\Delta W_{\rm cl})$ that a classical electron with a quantized proper field does not change its state, i.e., is scattered without emission of photons. Since the electron executes an infinite motion and interacts with an arbitrarily large number of soft photons, this amplitude has no meaning unless the proper time τ of the motion in the field is finite (otherwise $\Delta W_{\rm cl}=-\Delta m_{\rm cl}\tau\to\infty$) and unless, in the sense of Bloch and Nordsieck, there is a parameter (in our case μ) by means of which one distinguishes between elastic and inelastic scattering.

Following this doctrine, we consider $\Delta m_{\rm cl}$ as the limit of the value in theory in which the photon has a small mass, $\mu\to 0$, and the propagation function is $\Delta^c(z,\mu)$. The integral (81), which gives $\Delta m_{\rm cl}$, essentially depends on a region in which the timelike interval $[-(x-x')^2]^{1/2}$ between the points of emission and absorption of a virtual photon satisfies the condition

$$m/e\varepsilon \lesssim (-(x-x')^2)^{1/2} \lesssim \mu^{-1}. \tag{83}$$

Then, if $\mu \ll e\varepsilon m^{-1}$, $\operatorname{Re}\Delta m_{\rm cl}$ does not depend on μ, and $\operatorname{Im}\Delta m_{\rm cl}$ depends logarithmically on μ. If, on the other hand, we set $\mu=0$ before integrating over the interval $[-(x-x')^2]^{1/2}$, i.e., replace $\Delta^c(z,\mu)$ with $D^c(z)$, then $\Delta m_{\rm cl}$ loses its meaning; $\operatorname{Im}\Delta m_{\rm cl}$ goes to $-\infty$, and $\operatorname{Re}\Delta m_{\rm cl}$ becomes indeterminate $(0\cdot\infty)$. On the other hand, the use of different propagation functions for the calculation of the real and imaginary parts violates the causal connection between them.

The fact that for $\mu \ll e\varepsilon m^{-1}$ the value of $\operatorname{Re}\Delta m_{\rm cl}$ does not depend on μ and $\hbar$ means that there is a completely classical way to calculate this quantity. We shall not present this method.

The propagator (29) is essentially an integral over s of the function e^{iW}, where

$$W(x,x',s)=-m^2s+{}^1/_4 zeF\,\mathrm{cth}\,eFsz+e\int_{x'}^{x} dy_\alpha A_\alpha(y),\quad z_\alpha=x_\alpha-x_\alpha', \tag{84}$$

is the action of a classical charge moving in a constant field from the point x' to the point x during the proper time s. This action is defined as the integral of the Lagrangian,

$$W=\int_0^s ds L,\quad L=-m^2+\frac{1}{4}\left(\frac{dx_\alpha}{ds}\right)^2+eA_\alpha\frac{dx_\alpha}{ds}, \tag{85}$$

taken along the actual trajectory, i.e., that which satisfies the Lagrangian equations of motion. According to the equations of motion [see Eq. (49)], the partial derivative

$$\left.\frac{\partial W(x,x',s)}{\partial s}\right|_{x,x'}=-m^2-\Pi_\alpha^2,\quad \Pi_\alpha=\frac{\partial W}{\partial x_\alpha}-eA_\alpha, \tag{86}$$

is a constant of the motion and is the difference between the square of the mass of the particle, equal to $-\Pi^2$, and the parameter m^2. Thus the condition $\partial W/\partial s=0$ determines m as the mass of the particle.

Inclusion of the interaction between the particle and its proper field leads to a change of the Lagrange function (58); additional terms appear in it. This change is infinite, but for the electron in vacuum it is equivalent to a renormalization of the mass.

For the electron in an external field the change of the Lagrange function can be represented as the difference between its changes in the field and in the vacuum (for identical positions and velocities of the electron at a given time) plus its change in the vacuum, which can be omitted, using the renormalization of the mass of the free electron. Thus if m is the mass of the actual electron in the vacuum, then the change of the action on account of the change of the interaction of the electron with its proper field because of the external field is:

$$\Delta W=\int_0^s ds\Delta L=\int_{t'}^{t} dt(1-v^2)^{1/2}\frac{\Delta L}{2m},$$
$$(1-v^2)^{1/2}\frac{\Delta L}{2m}=\left[\int dVA_\alpha j_\alpha+\int dV\frac{E^2-H^2}{2}\right]_0^F, \tag{87}$$

where A_α, $\mathbf{E}$, $\mathbf{H}$ are the potential and intensities of the proper field of the electron, determined by its current j_α, and the indices F, 0 denote the difference of the quantity in brackets in the field and in the vacuum under the specified conditions.

Let us find ΔW for an electron in an electric field, with zero transverse momentum. In this case the motion of the electron is uniformly accelerated,

$$x_3=\frac{m}{e\varepsilon}\operatorname{ch}2e\varepsilon s,\quad x_0=\frac{m}{e\varepsilon}\operatorname{sh}2e\varepsilon s,\quad x_1=x_2=0, \tag{88}$$

with constant acceleration in its rest system equal to $w_0=e\varepsilon m^{-1}=\beta m$. The retarded potential and field of a change executing such a motion have been found by Born and Schott, and are discussed in detail by Fulton and Rohrlich.[20] Using these potentials and fields, we find that the contribution of the first term in Eq. (87) is equal to zero,[5)] and the contribution of the second term is nonzero and gives the result

$$\frac{\Delta L}{2m}=\alpha w_0\frac{1}{4}\left[v+\frac{1}{v}-\frac{1}{2\gamma^4v^2}\ln\frac{1+v}{1-v}\right]=\alpha w_0\frac{d}{2dx}(x\operatorname{cth}x), \tag{89}$$

where v is the velocity of the electron, $\gamma=(1-v^2)^{-1/2}$, $x=2\operatorname{Artanh}v=4e\varepsilon s$.

Accordingly, the change of the action is

$$\Delta W=\alpha\beta(\operatorname{cth}x-1/x)m^2s\to\pm\alpha\beta m^2s\quad\text{at}\quad v\to\pm1. \tag{90}$$

It then follows from the condition $\partial(W+\Delta W)/\partial s=0$ that the mass of the electron in the field is equal to $m-\Delta L/2m$, is a function of its velocity, and approaches the constant values $m\mp\alpha w_0/2$ for $v\to\pm1$. Consequently, the mass shift is proportional to the acceleration in the proper (instantaneous rest) system and is positive when the electron is being slowed down by the field, approaching the turning point, equal to zero at the turning point, and negative when the electron is speeded up as it moves away from the turning point. The region in which the restructuring of the proper field of the electron occurs and its mass changes from its largest to its smallest value is not large; it is of the order of the inverse acceleration $w_0^{-1}=m/e\varepsilon$.

The integral (87) which gives the mass shift comes from a region near the electron with transverse and longitudinal dimensions of the orders of w_0^{-1} and of $v\gamma w_0^{-1}$, respectively. [cf. Eq. (83)]. The quantum calculation of the elastic scattering amplitude predicts a mass shift of the final state (i.e., as $v\to1$), as is assumed in the S-matrix approach. The altered mass will appear in experiments in which the effect on the electron is weak enough to produce no additional mass change and is of sufficiently low frequency.

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APPENDIX: CALCULATION OF THE INTEGRALS J_0 AND J_1

Let us first consider the integral J_0. We change to variables of integration $z=x-x'$ and $X=(x+x')/2$. It is easily seen that only the phase of the integrand depends on X_2 and X_3, and the integration over these variables gives the conservation laws $p_2=p_2'$, $p_3=p_3'$, so that

$$J_0=(2\pi)^2\delta(p_2-p_2')\delta(p_3-p_3')J_0^MJ_0^E, \tag{A.1}$$

where

$$J_0^M=\left(\frac{|e\eta|}{\pi n!n'!}\right)^{1/2}\int dX_1dz_1dz_2D_n(\rho)D_{n'}(\rho')\exp\left[i(e\eta X_1-p_2)z_2+\frac{i}{4}A_1(z_1^2+z_2^2)\right], \tag{A.2}$$

$$J_0^E=\frac{\exp(\frac{1}{4}i\pi(\lambda'-\lambda^*))\Gamma(-\lambda^*)\Gamma(-\lambda')}{(2|e\varepsilon|)^{1/2}}\int dX_0dz_3dz_0D_{\lambda^*}(\omega\tau^*)\times D_{\lambda'}(\omega'\tau')\exp\left[-i(e\varepsilon X_0+p_3)z_3+\frac{i}{4}A_2(z_3^2-z_0^2)\right]. \tag{A.3}$$

Here $A_1=e\eta\operatorname{cot}e\eta s+t^{-1}$ and $A_2=e\varepsilon\operatorname{coth}e\varepsilon s+t^{-1}$; the other notations as are in Sec. 2.

We introduce instead of z_1, z_2 the polar coordinates $z_\perp, \varphi$. Then

$$J_0^M=(2\pi n!n'!)^{-1/2}\int_0^\infty dz_\perp z_\perp\int_0^{2\pi} d\varphi J_{nn'}(r,\varphi)\exp\left(\frac{i}{4}A_1z_\perp^2\right),\quad r=\left(\frac{|e\eta|}{2}\right)^{1/2}z_\perp, \tag{A.4}$$

where $J_{nn'}$ denotes the integral that appears when we integrate over X_1:

$$J_{nn'}(r,\varphi)=\int_{-\infty}^{\infty} d\xi\, D_n(\xi+r\sin\varphi)D_{n'}(\xi-r\sin\varphi)\exp\left(i\xi\frac{e\eta}{|e\eta|}r\cos\varphi\right). \tag{A.5}$$

Using the properties of the parabolic cylinder functions, we readily show that

$$\frac{\partial J_{nn'}}{\partial\varphi}=i(n'-n)\frac{e\eta}{|e\eta|}J_{nn'};$$

and therefore

$$J_{nn'}(r,\varphi)=J_{nn'}(r,0)\exp\left[i(n'-n)\frac{e\eta}{|e\eta|}\varphi\right]. \tag{A.6}$$

Then J_0^M reduces to a known integral of a Laguerre polynomial:

$$J_0^M=\frac{(2\pi)^{1/2}}{n!}\delta_{nn'}\int_0^\infty dz_\perp z_\perp J_{nn}(r,0)\exp\left(\frac{i}{4}A_1z_\perp^2\right)$$

$$=\frac{2\pi\delta_{nn'}}{|e\eta|}\int_0^\infty dx\exp\left[-\frac{1}{2}x\left(1-i\frac{A_1}{|e\eta|}\right)\right]L_n(x)=\frac{4\pi\delta_{nn'}}{|e\eta|-iA_1}\left(\frac{A_1-i|e\eta|}{A_1+i|e\eta|}\right)^n. \tag{A.7}$$

Since the integral appears only in matrix elements that conserve the spin ($\sigma=\sigma'=\pm1$), we have $\delta_{nn'}=\delta_{kk'}$. Moreover, using Eq. (8) for n, we can write the result in the form

$$J_0^M=2\pi\delta_{kk'}\frac{2i}{A_1+i\sigma e\eta}\left(\frac{A_1-i|e\eta|}{A_1+i|e\eta|}\right)^k,\quad k=\frac{\bar p_\perp^2}{2|e\eta|}. \tag{A.8}$$

In the integral J_0^E we use the representation (21) for the D function. Then the integral over X_0 gives a delta function

$$\frac{2\pi}{|e\varepsilon|}\delta\left[z_-+\left(\frac{2}{|e\varepsilon|}\right)^{1/2}(\omega y-\omega'y')\right],\quad z_-=z_0-\frac{e\varepsilon}{|e\varepsilon|}z_3. \tag{A.9}$$

Changing from z_3, z_0 to the variable

$$z_-,z_+=\frac{1}{2}\left(z_0+\frac{e\varepsilon}{|e\varepsilon|}z_3\right)$$

and integrating over z_+, we get another delta function

$$2\pi\left(\frac{2}{|e\varepsilon|}\right)^{1/2}\delta\left[\omega y+\omega'y'+\frac{A_2}{|e\varepsilon|}(\omega y-\omega'y')\right]. \tag{A.10}$$

Integrating over z_- we get

$$J_0^E=\frac{(2\pi)^2}{(e\varepsilon)^2}\iint_0^\infty dy\,dy'y^{-\lambda^*-1}y'^{-\lambda'-1}\delta\left[\omega y+\omega'y'+\frac{A_2}{|e\varepsilon|}(\omega y-\omega'y')\right]. \tag{A.11}$$

Since $A_2>|e\varepsilon|$, the delta function can have zero argument only for $\omega=\omega'$. Accordingly we get

$$J_0^E=\frac{(2\pi)^2}{|e\varepsilon|}\frac{\delta_{\omega\omega'}}{A_2+|e\varepsilon|}\left(\frac{A_2-|e\varepsilon|}{A_2+|e\varepsilon|}\right)^{-\lambda^*-1}\int_0^\infty\frac{dy'}{y'}\exp[-(\lambda^*+\lambda'+1)\ln y']$$

$$=(2\pi)^3\delta(p_\parallel^2-p_\parallel'^2)\delta_{\omega\omega'}\frac{2}{A_2+|e\varepsilon|}\left(\frac{A_2-|e\varepsilon|}{A_2+|e\varepsilon|}\right)^{-\lambda^*-1}. \tag{A.12}$$

In the general case, according to Eq. (8)

$$\lambda^*+\lambda'+1=i\frac{2|e\eta|(k-k')-p^2+p'^2}{2|e\varepsilon|}-\frac{\gamma\sigma+\gamma'\sigma'}{2|e\varepsilon|}e\varepsilon, \tag{A.13}$$

but the integral in question appears only in elements for which $\gamma\sigma+\gamma'\sigma'=0$. Therefore the integral over y gives the delta function as written, with

$$p_\parallel^2=p^2-2|e\eta|k,\quad p_\parallel'^2=p'^2-2|e\eta|k'.$$

Now using Eq. (8) for $-\lambda^*-1$, we can write the result in the form

$$J_0^E=(2\pi)^3\delta(p_\parallel^2-p_\parallel'^2)\delta_{\omega\omega'}\frac{2}{A_2+\gamma\sigma e\varepsilon}\left(\frac{A_2-|e\varepsilon|}{A_2+|e\varepsilon|}\right)^{-i\nu},\ \nu=\frac{-p_\parallel^2}{2|e\varepsilon|}=\frac{\bar p_+\bar p_-}{|e\varepsilon|}. \tag{A.14}$$

By multiplying the matrix element (A.8) and the element (A.14), as indicated in Eq. (A.1), we get for J_0 the matrix shown in Eq. (43).

Going on to the integral J_1, we note that in a representation in which B is diagonal

$$\gamma Bz={}^1/_2(\gamma_2-i\gamma_1)B_{11}(z_2+iz_1)+{}^1/_2(\gamma_2+i\gamma_1)B_{22}(z_2-iz_1)$$
$$+{}^1/_2(i\gamma_4+\gamma_3)B_{33}(z_0+z_3)+{}^1/_2(i\gamma_4-\gamma_3)B_{44}(z_0-z_3). \tag{A.15}$$

Therefore J_1 differs from J_0 by the appearance either of the coordinate $z_2\pm iz_1=z_\perp e^{\pm i\varphi}$ in J^M or of the coordinate $z_0\pm z_3$ in J^E. Denoting the corresponding integrals by $J_\pm^M, J_\pm^E$, we get for $J_\pm^M$ instead of (A.17)

$$J_\pm^M=\left(\frac{2\pi}{n!n'!}\right)^{1/2}\delta_{n\mp e\eta/|e\eta|,n'}\int_0^\infty dz_\perp z_\perp^2J_{nn'}(r,0)\exp\left(\frac{i}{4}A_1z_\perp^2\right)$$

$$=2\pi\delta_{n\mp e\eta/|e\eta|,n'}\frac{i}{e\eta}\left(\frac{2}{(m+1)|e\eta|}\right)^{1/2}\int_0^\infty dx\,x\exp\left(-\frac{x}{2}\left(1-\frac{iA_1}{|e\eta|}\right)\right)L_m^1(x)$$

$$=2\pi\delta_{n\mp e\eta/|e\eta|,n'}\frac{4ie\eta}{(iA_1-|e\eta|)^2}\left(\frac{2(m+1)}{|e\eta|}\right)^{1/2}\left(\frac{A_1-i|e\eta|}{A_1+i|e\eta|}\right)^m,$$
$$m=\min(n,n'). \tag{A.16}$$

Since the $J_\pm^M$ are multiplied by the matrices $\gamma_2\mp i\gamma_1$, whose nonvanishing elements have $\sigma=\pm1, \sigma'=\mp1$, according to Eq. (8) a change of n and a change of σ are equivalent to conservation of k; besides this, $m=k-1$. Accordingly, Eq. (A.16) can be written

$$J_\pm^M=2\pi\delta_{kk'}\frac{4i\bar p_\pm}{(A_1+ie\eta)(A_1-ie\eta)}\left(\frac{A_1-i|e\eta|}{A_1+i|e\eta|}\right)^k. \tag{A.17}$$

Similarly, for $J_\pm^E$ we get instead of Eq. (A11)

$$J_\pm^E=\left(\frac{2\pi}{e\varepsilon}\right)^2\left(\frac{2}{|e\varepsilon|}\right)^{1/2}\iint_0^\infty dy\,dy'y^{-\lambda^*-1}y'^{-\lambda'-1}\left[-i\left(1\pm\frac{e\varepsilon}{|e\varepsilon|}\right)\delta'(u)\right.$$

$$\left.-\frac{1}{2}\left(1\mp\frac{e\varepsilon}{|e\varepsilon|}\right)(\omega y-\omega'y')\delta(u)\right], \tag{A.18}$$

where $\delta(u)$ is the same delta function as that in Eq. (A.11).

Calculating these integrals in the same way as we did those for Eq. (A.11) and using the fact that they are multiplied by matrices $i\gamma_4\pm\gamma_3$ [see Eq. (A.15)], whose nonzero elements have $\gamma\sigma=\gamma'\sigma'=\pm1$, we get instead of (A.14) the expression

$$J_\pm^E=(2\pi)^3\delta(p_\parallel^2-p_\parallel'^2)\delta_{\omega\omega'}\frac{4(\bar p_0\pm\bar p_3)}{(A_2+e\varepsilon)(A_2-e\varepsilon)}\left(\frac{A_2-|e\varepsilon|}{A_2+|e\varepsilon|}\right)^{-i\nu}. \tag{A.19}$$

By multiplying together the matrix elements (A.17), (A.14) and (A.19), (A.7), we get the matrix J_1 given in the text.

Note added in proof (September 26, 1978): The change of the energy of the proper field of the electron, found in analogy with Eq. (87) is given by the expression

$$\int dV\frac{E^2+H^2}{2}\Big|_0^F=\frac{2}{3}\alpha w_0 v\gamma,$$

which is identical with the Schott acceleration energy.[20]

Later Erratum by Author [Zh. Eksp. Teor. Fiz. 76, 383 (1979)]: The interpretation presented here for formula (77) for the shift of the electron mass in an electric field is incorrect. What this formula actually determines is the shift of the "center of gravity" of the mass doublet, since it had been obtained from the general formula not only by causing the magnetic field to vanish, but also by choosing a particular direction of the spin (satisfying the condition (63) and perpendicular to the proper one for the mass shift in the electric field).

In the general case the mass shift is given by formula (57) and takes the form

$$\Delta m(\bar{s}) = \Delta m_0 + \bar{s}_\alpha S_\alpha, \quad S_\alpha = eF_{\alpha\beta}{}^*\bar{p}_\beta m^{-2}C_1 + eF_{\alpha\beta}\bar{p}_\beta m^{-2}C_2,$$

where C_1 and C_2 determine the anomalous magentic and electric moments, depend on the parameters η, ε, and χ, and are cited in (57) as integrals of the functions $(\alpha/2\pi)ic_{1,2}$. The eigenvalues $\Delta m_\pm$ of the mass spectrum are the values of $\Delta m(\bar{s})$ for the spin $\bar{s}_\alpha$ directed parallel or antiparallel to the vector S_α that defines both the proper direction of the spin for the mass shift and the value of the splitting. Thus, $\Delta m_\pm = \Delta m_0 \pm \sqrt{S^2}$. It follows therefore that for a purely electric field $\Delta m_\pm \Delta m_0 \pm e\varepsilon\bar{p}_\perp m^{-2}C_1$, and the spin-independent part Δm_0 is given by the aforementioned formula (77). The part $\pm e\varepsilon\bar{p}_\perp m^{-2}C_1$ that depends on the spin direction is due to the interaction of the anomalous magentic moment with the magnetic field that appears in the proper system because of the transverse momentum $\bar{p}_\perp \neq 0$ and is investigated in Sec. 6.

1) We note that D_λ is an integral function of the index λ, and $\Gamma(-\lambda)$ is analytic everywhere except for poles at the points $\lambda = 0, 1, 2, \ldots$. In order for the solution $E_{p\sigma\gamma}$, Eq. (7), to exist for all real p^2, a small negative imaginary part $-i\delta$ has been added to p^2, so that $\mathrm{Re}\lambda \leqslant -\delta < 0$.

2) We here use the proper time s, which is related to the usual τ by $\tau = 2ms$.

3) The point $s = 0$ is a point of natural symmetry of the trajectory (49), i.e., of symmetry of its curvature and torsion.

4) The renormalization consists of satisfying the boundary condition that for $F \to 0$ the quantity $M(\bar{p}, F)$ must approach $M_R^0(\bar{p})$, the mass operator of the electron in vacuum.

5) When the current is eliminated by means of the Maxwell equations this term must reduce to three terms which do not vanish, but cancel each other.[3]

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