the liquid metal in the irradiation zone. On increase in the absorbed intensity up to $I_0 \approx 10\ \mathrm{MW/cm^2}$, the temperature of the overheated liquid lead in the irradiation zone approaches the spinodal, where the state of the metastable phase becomes absolutely unstable. Measurements of the recoil pressure under such conditions can give new information on the behavior of the metastable state of the overheated metal in the near-critical region. Information on the rate of "decay" of the metastable phase is essential, particularly in the interpretation of experiments on the evaporation of shock-compressed metals.[17] When laser radiation acts on condensed matter, metastable states may only appear when the recoil pressure in the irradiation zone does not exceed a certain critical value p_c. This can be used to determine the critical pressure of metals.

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Structure of tails produced under the action of perturbations on solitons

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The asymptotic structure of "tails" produced by perturbations acting on solitons and also the evolution of the solitons themselves are studied by a perturbation theory based on the inverse scattering method. A detailed analytic description of the structure of the tail during the first stage of its formation (short times) is presented. A qualitative picture of the tail structure is given for sufficiently long times after switching on the perturbation. In particular, it is shown that the part of the tail adjoining the soliton has the shape of a plateau at both short and long times. The condition that the perturbation operator must satisfy if tail formation is not to occur is derived. Soliton deformation induced by a perturbation is studied in first order.

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1. INTRODUCTION

It has been shown previously[1] that one of the results of the action of a continuously operating perturbation (for example, dissipation) on a soliton is the formation of a "tail"—a wave packet of small amplitude—following the soliton; the length of the tail increases with time. The detailed description of the structure and dynamics of formation of such tails is generally rather complicated. The method developed in Ref. 1 contained an element of averaging at some stage, as a result of which an "averaged" tail was obtained. The contribution of such an averaged tail to the momentum, energy, and other quantities that are conserved in the absence of the perturbation is equal to the contribution of the real tail, as has been pointed out.[1] Thus, from the point

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of view of all the infinite set of conservation laws, which are modified to account for the presence of a perturbation,[2] the averaged and real tails are equivalent. Along with this, there is interest in the study of the asymptotic structure of the real tail formed under the action of the perturbation, for comparison with numerical and laboratory experiments.

The evolution of the tail and the soliton itself are intimately connected with one another. Therefore, by obtaining information on the dynamics of tail formation, we can draw certain conclusions on the character of the deformation of solitons under the action of the perturbation, on their lifetimes and so on. We shall consider this circle of questions using as examples the three most important types of solitons, described by the perturbed Korteweg–de Vries equation (KdV), the modified KdV equation (MKdV), and the nonlinear Schrödinger equation (NSE):

$$u_t-6uu_x+u_{xxx}=\varepsilon R[u], \tag{1.1}$$
$$u_t+6u^2u_x+u_{xxx}=\varepsilon R[u], \tag{1.2}$$
$$iu_t+|u|^2u+{}^1/_2u_{xx}=i\varepsilon R[u]. \tag{1.3}$$

Here R is the perturbation operator, and ε is a small parameter.[1)]

The dynamics of processes in the presence of the perturbation is determined by the relation between the two times t_s and t_p. Here t_s is the so-called soliton time, equal, for example to the time of passage of the soliton over a distance of the order of its length, and t_p is a characteristic time connected with the perturbation, for example, the time of significant evolution of the soliton under the action of the perturbation. Here $t_s/t_p\sim\varepsilon$ (in what follows, these times will be determined exactly in each specific case). The formation of the tail is an important fact of the evolution under the action of the perturbation even at comparatively small t; for example, in the case of KdV solitons at $t_s\ll t\ll(t_s/t_p)^{1/2}t_p$. Section 2 is devoted to the study of the action of the perturbation in this initial period. The general picture of what is obtained in this case is shown in the figure.

The wave pulse, which is assumed to be a "pure" soliton at the initial instant, is changed into a formation consisting of a weakly deformed soliton (soliton core) and the tail following it. In the region adjoining the core, the tail has approximately the form of a plateau with an amplitude of the order ε of the amplitude of the soliton; on the left part of the tail, oscillations appear with decreasing wavelength as one goes away from the soliton. In perturbations of a definite type [condition (2.43)], the tail is not formed. Upon increase of t, the length of the tail increases, but the order of magnitude of its amplitude remains the same. The tail is also gradually modulated. However, the part of the tail adjoining the core keeps the shape of a plateau over distances amounting to many soliton lengths. The shape of the soliton core is here close to that of the soliton; its amplitude and velocity change slowly according to the so-called adiabatic equation.[1,3] Such a picture is preserved, at least up to times of the order of t_p (Sec. 3).

In a number of cases, for example, if the perturbation is a dissipation, the soliton core is generally damped within a time of the order of t_p, so that the results obtained in this research describe the evolution of the soliton throughout all the entire time that it is meaningful to speak of solitons.

In Sec. 4, similar results are obtained also for the MKdV solitons. So far as the NSE solitons are concerned, no tail is formed in this case, and the result of the action of the perturbation is a self-similar change of the soliton with a small deformation (at least if the condition given in note 5 is fulfilled[11]).

2. STRUCTURE OF THE TAIL OF A PERTURBED KdV SOLITON

We consider the solution of Eq. (1.1), which describes the perturbed soliton, writing it in the form

$$\begin{gathered}u(x,t)=u_s(z,\varkappa)+\delta u(z,t),\quad z=\varkappa(t)[x-\xi(t)],\\ u_s(z,\varkappa)=-2\varkappa^2(t)\operatorname{sech}^2 z,\quad \delta u(z,t)=-2\varkappa^2(t)w(z,t),\\ \xi(0)=0,\quad w(z,0)=0.\end{gathered} \tag{2.1}$$

Here the first term of u_s has the form of a soliton pulse with slowly changing $\varkappa(t)$ and $d\xi/dt$. The second term determines the perturbation of the soliton $w\sim\varepsilon$. At $z<0$ and $|z|\gg1$, this term describes the tail.[1]

The equations determining the change of $\varkappa$, ξ, and δu, can be obtained with the help of the formalism of the inverse problem of scattering theory. If we write down the Schrödinger equation corresponding to the potential $u(x,t)$,

$$-\psi''(x)+u(x,t)\psi(x)=k^2\psi(x), \tag{2.2}$$

and find the linearly independent solutions of this equation $f(x,k;t)$ and $g(x,k;t)$, which satisfy the conditions

$$f(x,k)\to e^{ikx}(x\to\infty),\ g(x,k)\to e^{-ikx}(x\to-\infty),$$

then $f(x,k;t)$ and $g(x,k;t)$ (the so-called Jost functions) are connected with one another through the coefficients $a(k,t)$, $b(k,t)$:

$$g=af^*+bf,\quad |a|^2-|b|^2=1$$

for $k^2>0$. These coefficients determine the reflection coefficient

$$r(k,t)=b(k,t)/a(k,t). \tag{2.3}$$

The discrete spectrum of Eq. (2.2) $k_r=i\varkappa_r$ ($\varkappa_r<0$, $r=1,2,\ldots N$) is determined by the zeros of the function $a(k)$, analytically continued in the upper half-plane of k. Here

$$\left.\frac{\partial a(k)}{\partial k}\right|_{k=k_r}\equiv a'(i\varkappa_r)\neq0,\quad g(x,k_r;t)=\rho_r(t)f(x,k_r;t).$$

By knowing Jost coefficients $a(k,t)$, $b(k,t)$, $\rho_r(t)$ and discrete spectrum of the Schrödinger equation, the "potential" $u(x,t)$ can be replaced by the formula

$$u(x,t)=-2\frac{\partial}{\partial x}K(x,x;t), \tag{2.4}$$

where $K(x,y)$ is the solution of the Gel'fand-Levitan equation:

$$K(x,y;t)+F(x+y;t)+\int_x^\infty K(x,y';t)F(y+y';t)dy'=0, \tag{2.5}$$

$$F(x;t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} r(k,t)e^{ikx}dk+\sum_{r=1}^{N}\frac{\rho_r(t)}{ia'(i\varkappa_r(t))}\exp[-\varkappa_r(t)x]. \tag{2.6}$$

At $u=u_s$, i.e., for a potential having soliton shape, we find from (2.2)

$$b=0,\ a=a_s=(k-i\varkappa)(k+i\varkappa)^{-1},\ \rho=e^{2\varkappa\xi},$$
$$f(x,k)=f_s(x,k)=e^{ikx}\{k+i\varkappa\,\mathrm{th}[\varkappa(x-\xi)]\}(k+i\varkappa)^{-1}. \tag{2.7}$$

Thus, at $u=u_s$, the quantity $k^2=-\varkappa^2$ is the only eigenvalue of the discrete spectrum.

If $u=u_s+\delta u$, then $b=\delta b$.

$$a(k,t)=a_s(k,\varkappa)+\delta a(k,t). \tag{2.8}$$

In particular, in first order in ε,

$$\delta a=\int_{-\infty}^{\infty}\left[\frac{\delta a}{\delta u(x)}\right]_s\delta u(x)dx=\frac{\varkappa}{ik(k+i\varkappa)^2}\int_{-\infty}^{\infty}(k^2+\varkappa^2\,\mathrm{th}^2 z)w(z)dz, \tag{2.9}$$

where

$$\left[\frac{\delta a}{\delta u(x)}\right]_s=\frac{ia_s}{2k}|f_s(x,k)|^2$$

is the variational derivative of $a(k)$ with respect to $u(x)$ at $u=u_s$. A similar expression can be written down also for $\delta b=b$.

It is useful to note that the change in the eigenvalue λ of the discrete spectrum, due to δu, is equal to zero (in first-order approximation in ε), i.e., $\lambda=-\varkappa^2+O(\varepsilon^2)$, in spite of the fact that $w\sim\varepsilon$. This is connected with the fact that

$$\left[\frac{\delta\lambda}{\delta u(x)}\right]_s=\frac{\varkappa}{2}\mathrm{sech}^2 z,$$

while (see Ref. 1),

$$\int_{-\infty}^{\infty}w(z)\mathrm{sech}^2 z\,dz=0. \tag{2.10}$$

Substituting (2.9) in (2.8) and taking (2.10) into account, we get

$$a(k,t)=\frac{k-i\varkappa}{k+i\varkappa}\left(1-\frac{i\varkappa_0}{k}\right),\quad \frac{\varkappa_0}{\varkappa}=\int_{-\infty}^{\infty}w(z)dz. \tag{2.11}$$

Thus, if the perturbation is such that $\varkappa_0>0$, then a small additional level $k_0=i\varkappa$ due to w appears.

Analysis shows (this will also be seen from the results obtained below) that the expression (2.11) for $a(k,t)$ is valid up to the value $t\sim(t_s/t_p)^{1/2}t_p$. Here $t_s=1/8\varkappa^3$ is the soliton time, $t_p=|\varepsilon q\varkappa^3|^{-1}$ is the characteristic time of the perturbation,[1] where

$$q=\frac{1}{4\varkappa^3}\int_{-\infty}^{\infty}R[u_s(z)]\mathrm{th}^2 z\,dz. \tag{2.12}$$

We first investigate the process of tail formation and soliton deformation at comparatively short times

$$t\ll(t_s/t_p)^{1/2}t_p. \tag{2.13}$$

However, it is assumed that the soliton manages in this case to travel a distance that is much greater than its characteristic dimensions, i.e., we assume $t\gg t_s$, since only in this case will the tail play a significant role in the physics of the phenomenon. On the basis of the results obtained here, a sufficiently complete picture of the evolution of the soliton will be described below also for much greater times $t\sim t_p$.

Taking into account what has been said above, we represent (2.6) in the following form in the case of solution of Eq. (2.5):

$$F(x)=F_s(x)+\delta F(x);$$
$$F_s(x)=2\varkappa e^{\varkappa(2\xi-x)},\quad \xi=\frac{1}{2\varkappa}\ln\left[\frac{\rho}{2i\varkappa a'(i\varkappa)}\right], \tag{2.14}$$

$$\delta F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}r(k)e^{ikx}dk+\frac{\theta(\varkappa_0)\rho_0}{ia'(i\varkappa_0)}\exp(-\varkappa_0 x);$$
$$\theta(\varkappa_0)=1,\ \varkappa_0>0;\ \theta(\varkappa_0)=0,\ \varkappa_0<0. \tag{2.14a}$$

Here $r(k)$ is defined in (2.3).

Taking into account the smallness of $\varkappa_0/\varkappa$, we find from (2.11) that $a'(i\varkappa_0)\approx i/\varkappa_0$. Substituting (2.14), (2.14a) in (2.5) and assuming the value of δF to be small ($\sim\varepsilon$) we find the solution of Eq. (2.5) in the first-order approximation in ε (just as was done in Ref. 1), and then, by use of (2.4), the solution of (1.1) in the form (2.1), where

$$w(z,t)=-\frac{1}{2\pi\varkappa}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}r(k)\left(\frac{k+i\varkappa\,\mathrm{th}\,z}{k+i\varkappa}\right)^2e^{2ik(\xi+z/\varkappa)}dk$$
$$+2\theta(\varkappa_0)\rho_0\frac{\varkappa_0}{\varkappa}\frac{\mathrm{th}\,z}{\mathrm{ch}^2 z}. \tag{2.15}$$

We now write down in first order in the equations determining the time dependence of the quantities $\varkappa(t)$, $\rho(t)$ and $b(k,t)$ in the adiabatic approximation[1,3,4]:

$$\frac{d\varkappa}{dt}=-\frac{\varepsilon}{4\varkappa}\int_{-\infty}^{\infty}R[u_s(z)]\mathrm{sech}^2 z\,dz, \tag{2.16}$$

$$\frac{\partial b}{\partial t}=8ik^3b-\frac{i\varepsilon e^{-2ik\xi}}{2k\varkappa(k^2+\varkappa^2)}$$
$$\times\int_{-\infty}^{\infty}R[u_s(z)](k-i\varkappa\,\mathrm{th}\,z)^2e^{-2ikz/\varkappa}dz, \tag{2.17}$$

$$\frac{d\rho}{dt}=8\varkappa^3\rho-\frac{\varepsilon\rho}{2\varkappa^2}\int_{-\infty}^{\infty}R[u_s(z)]\left(\mathrm{th}\,z+\frac{z+\varkappa\xi}{\mathrm{ch}^2 z}\right)dz. \tag{2.18}$$

Using elementary perturbation theory for the Schrödinger equation, it is not difficult to verify that

$$\rho_0=-1+O(\varepsilon). \tag{2.19}$$

We now proceed to the calculation of the reflection coefficient $r(k,t)$. From (2.17), we find

$$b(k,t)=\frac{\varepsilon\exp(8ik^3t)}{2ik}\int_0^t\frac{A(k,\varkappa(t'))}{\varkappa(t')[k^2+\varkappa^2(t')]}\exp[-8ik^3t'-2ik\xi(t')]dt', \tag{2.20}$$

where

$$A(k,\varkappa)=\int_{-\infty}^{\infty}R[u_s(z)](k-i\varkappa\,\mathrm{th}\,z)^2e^{-2ikz/\varkappa}dz. \tag{2.21}$$

Here we have assumed the initial condition $b(k,0)=0$, which corresponds to $w(z,0)=0$.

Integrating by parts in (2.20) and taking it into account that

$$\frac{d\xi}{dt}=4\varkappa^2+O(\varepsilon)$$

by virtue of (2.14), (2.11), and (2.18), we get from (2.3) and (2.11)

$$r(k,t)=\frac{\varepsilon e^{-2ik\xi}A(k,\varkappa)}{16\varkappa(k-i\varkappa)^2(k^2+\varkappa^2)(k-i\varkappa_0)}\frac{1-\exp[8ik^3t+2ik\xi]}{k}+\varepsilon O\left(\frac{t}{t_p}\right). \tag{2.22}$$

where $\varkappa = \varkappa(t)$, $\xi = \xi(t)$, and we have assumed smallness of t in comparison with t_p, in accord with what is shown above.[2)] Substituting (2.22) in (2.15) and introducing the dimensionless quantities

$$p=k/\varkappa,\quad \tau=8\varkappa^3 t=t/t_*,$$

we obtain, with account of (2.19),

$$w(z,t)=-\frac{\varepsilon}{32\pi\varkappa^5}\frac{\partial v}{\partial z}-2\theta(\varkappa_0)\frac{\varkappa_0}{\varkappa}\frac{\operatorname{th} z}{\operatorname{ch}^2 z}, \tag{2.23}$$

where

$$v(z,t)=\int_{-\infty}^{\infty}\frac{A(p)\,(p+i\operatorname{th} z)^2}{(p^2+1)^3(p-i\varkappa_0/\varkappa)}\,\frac{1-\exp(i\tau p^3+2ip\varkappa\xi)}{p}e^{2ipz}dp, \tag{2.24}$$

$$A(p)=\int_{-\infty}^{\infty}R[u_*(z)]\,(p-i\operatorname{th} z)^2\exp(-2ipz)\,dz. \tag{2.25}$$

From (2.23)–(2.25), we obtain

$$\int_{-\infty}^{\infty}w(z,t)\,dz=-\frac{\varepsilon}{32\pi\varkappa^5}[v(\infty,t)-v(-\infty,t]=-\frac{\varepsilon}{2}q\varkappa\xi(t), \tag{2.26}$$

where q is defined in (2.12). On the other hand, from the law of momentum change

$$I_1=\int_{-\infty}^{\infty}u\,dx$$

it follows that[1]

$$\frac{d}{dt}\int_{-\infty}^{\infty}w(z,t)\,dz=-2\varepsilon q(t)\varkappa^3(t). \tag{2.27}$$

Substituting (2.26) here with account of the equality $d\xi/dt=4\varkappa^2+O(\varepsilon)$ we find that (2.27) is fulfilled within the assumed limits of accuracy.

We also note that if (2.11) and (2.26) result in a value $\varkappa_0\neq 0$, then the reflection coefficient (2.22) $r(k,t)\to -1$ as $k\to 0$, which is in agreement with the general properties of the scattering matrix of the one-dimensional Schrödinger equation.[5]

On the basis of the obtained solution (2.23)–(2.25), we now proceed directly to the study of the region of the soliton tail. The latter is located to the left of the soliton ($z<0$) and corresponds to it $|z|\gg 1$. At such z, we can set $\tanh z\approx -1$ in (2.24) and neglect the second term in (2.23). In carrying out the calculations, it is convenient to introduce the functions

$$B(p)=\frac{A(p)}{(p^2+1)(p+i)^2},\quad \varphi(p)=\frac{B(p)}{p-i\delta}-\frac{B(0)}{p-i\delta},\quad \delta=\frac{\varkappa_0}{\varkappa}, \tag{2.28}$$

where, according to (2.25), (2.12), $B(0)=4\varkappa q$. Here $\varphi(p)$ has no singularity at $p=0$.

In sum, after differentiating with respect to z in (2.23), we get for the region of the tail ($-z\gg 1$),

$$w(z,t)=w_1+w_2+w_3; \tag{2.29}$$

$$w_1(z,t)=-\frac{i\varepsilon q}{4\pi}\int_{-\infty}^{\infty}[1-\exp(i\tau p^3+2i\varkappa\xi p)]e^{2ipz}\frac{dp}{p-i\delta}, \tag{2.30}$$

$$w_2(z,t)=\frac{i\varepsilon}{16\pi\varkappa^5}\int_{-\infty}^{\infty}\varphi(p)\exp[i\tau p^3+2ip(z+\varkappa\xi)]\,dp, \tag{2.31}$$

$$w_3(z)=-\frac{i\varepsilon}{16\pi\varkappa^5}\int_{-\infty}^{\infty}\varphi(p)\,e^{2ipz}dp. \tag{2.32}$$

Calculating the integral in (2.30), we obtain

$$w_1(z,t)=-\frac{\varepsilon q}{2}\int_{-\infty}^{y}\mathrm{Ai}(y')\,dy', \tag{2.33}$$

where $Ai(y)$ is the Airy function,

$$y=2(z+\varkappa\xi)/(3\tau)^{1/3}=2\varkappa x/(3\tau)^{1/3}. \tag{2.34}$$

(x is the coordinate in the laboratory system of coordinates). In obtaining (2.33), we can, for example, make use of the representation

$$\frac{1}{p-i\delta}=i\operatorname{sign}\delta\int_0^{\infty}\exp(-ipz'\operatorname{sign}\delta)\,dz',\quad |\delta|\ll 1$$

and carry out integration first over p and then over z'. In the analysis of the expression (2.33), it is useful to keep it in mind that

$$\int_{-\infty}^{\infty}\mathrm{Ai}(z)\,dz=1.$$

We now turn to the function w_2, rewriting it in the form

$$w_2=\frac{i\varepsilon}{16\pi\varkappa^5(3\tau)^{1/3}}\int_{-\infty}^{\infty}\varphi\left(\frac{p}{(3\tau)^{1/3}}\right)\exp\left[i\left(\frac{p^3}{3}+py\right)\right]dp. \tag{2.35}$$

We are interested in the behavior of this function at $\tau\gg 1$. Investigation shows that if $|y|^{1/2}\ll(3\tau)^{(1/3)}$, then the basic contribution to the integral is made at $|p|\ll(3\tau)^{1/3}$. Because of this, we can set $\varphi(p/(3\tau)^{1/3})\approx\varphi(0)$ (here it is essential that $|\varphi(0)|<\infty$), after which we obtain the Airy integral. Thus, at $\tau\gg 1$ and $|y|^{1/2}\ll(3\tau)^{1/3}$ we have

$$w_2=\frac{i\varepsilon\varphi(0)}{8\varkappa^5(3\tau)^{1/3}}\mathrm{Ai}(y), \tag{2.36}$$

where, in accord with (2.28) and (2.25),

$$\varphi(0)=2i\int_{-\infty}^{\infty}R[u_*(z)]\,(\operatorname{th} z-z\operatorname{th}^2 z+\operatorname{th}^2 z)\,dz. \tag{2.37}$$

At $|y|^{1/2}\gtrsim(3\tau)^{1/3}$ and $y<0$ we calculate (2.35) by the method of stationary phase. As a result,

$$w_2=-\frac{\varepsilon}{16\pi^{1/2}\varkappa^5}\varphi\left(\frac{|y|^{1/2}}{(3\tau)^{1/3}}\right)\frac{\exp[-i(^2/_3|y|^{3/2}+{}^1/_4\pi)]}{(3\tau)^{1/3}|y|^{1/4}}+\text{c.c.} \tag{2.38}$$

Obviously, this formula is valid in the wider interval $|y|\gg 1, y<0$. Here, if $1\ll|y|^{1/2}\ll(3\tau)^{1/3}$, (2.28) is identical with (2.36). The region $y^{1/2}\gtrsim(3\tau)^{1/3}, y>0$ does not concern the tail of the soliton; it is discussed in Sec. 3. At $y\gg 1$, the function w_2 vanishes rapidly.

Finally, the function $w_3(z)$ at $z<0$, $|z|\gg 1$ has the form

$$w_3\approx\frac{\varepsilon}{8\varkappa^5}z^2e^{2z}\int_{-\infty}^{\infty}R[u_*(z')]\operatorname{sech}^2 z'\,dz'+\varepsilon O(ze^{2z}). \tag{2.39}$$

In contrast with w_1 and w_2, this function describes that part of the perturbation which propagates along with the soliton. As the distance $|z|$ from the center of the soliton increases, w_3 falls off more rapidly than w_1 and w_2. Thus, we can assume that w_3 describes the perturbation of the soliton "core", while the tail is described by the sum

$$w^{(t)}=w_1+w_2. \tag{2.40}$$

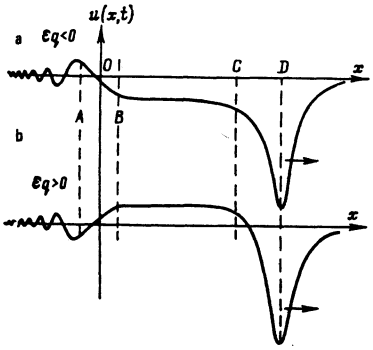

FIG. 1. Typical form of the KdV soliton deformation under action of the perturbation; $AB \sim (3t)^{1/3}$, $OC \sim OD \approx \xi(t)$.

Substituting (2.33) and (2.38) in (2.40), we obtain the following relations, which determine the asymptote of the tail ($z<0$, $|z| \gg 1$) at $1 \ll \tau \ll (t_p/t_s)^{1/2}$:

$$w^{(1)} \approx w_1 = -\frac{\varepsilon q}{2}\int_{-\infty}^{y} \mathrm{Ai}(y')\,dy', \quad |y|^{1/2} \ll (3\tau)^{1/3} \tag{2.41}$$

(in this region $w_1 \gg w_2$), and

$$w^{(1)} \approx -\frac{\varepsilon A(s)\exp[-i(2\tau s^3+\pi/4)]}{16\varkappa^5(3\pi\tau)^{1/2}s^{1/2}(s^2+1)(s+i)^2} + \text{c.c.} \tag{2.42}$$

at

$$s = \frac{|y|^{1/2}}{(3\tau)^{1/3}} = \left(\frac{2\varkappa|x|}{3\tau}\right)^{1/2} \gg 1, \quad x<0.$$

The general form of the tail is shown in the figure. At $y>0$, $1 \ll y^{1/2} \ll (3\tau)^{1/3}$, the expression (2.41) describes a line close to the horizontal line $w=-\varepsilon q/2$. Upon decrease in y, we fall into the transitional region $2\varkappa|x| < (3\tau)^{1/3}$, the width of which increases with time like $(3\tau)^{1/3}$. At large negative y, oscillations appear which are described by the expression (2.42). At $s \gg 1$, these oscillations are rapidly damped, since $A(s)$ falls off rapidly as $s \to \infty$ (see the figure). This picture of the tail is naturally more detailed than that which was described by the averaged expression of Ref. 1; however, it preserves its two basic characteristics, which follow from the averaged expression, namely the length of the tail $\sim 4\varkappa^2 t$ and its area $\sim 4\varepsilon q\varkappa^4 t$.

We note that the perturbation for which $q=0$, i.e.,

$$\int_{-\infty}^{\infty} R[u_s(z)]\,\mathrm{th}^2 z\,dz = 0, \tag{2.43}$$

does not lead to the formation of an increasing tail. In this case, the main result of the perturbation is the self-similar change of the soliton, described by $u_s[z, \varkappa(t)]$, on which small distortions are superimposed, propagating along with the soliton and localized in the region $|z| \lesssim 1$. Actually, it follows from (2.33) that $w_1=0$ at $q=0$, while the function w_2 spreads out with passing of time. Its appearance in the considered case is obviously connected with the initial conditions assumed here, which correspond to an instantaneous turning on of the perturbation at $t=0$.

It follows from what has been set forth above that the additional discrete level $k=i\varkappa_0$ is due to a long but shallow potential well, connected with the tail at $\varepsilon q<0$ (see the Figure, a). The width of this well is $\sim\xi(t)$, and the depth is $\delta u \approx \varepsilon q\varkappa^2$. The value of the additional level increases with time. According to (2.11) and (2.26) at the small times t considered here, we have

$$\begin{aligned} \varkappa_0 &= -\frac{1}{2}\varepsilon q\varkappa^2\xi, \\ \frac{d\varkappa_0}{dt} &\approx -\frac{\varepsilon q\varkappa^2}{2}\frac{d\xi}{dt} \\ &\approx -2\varepsilon q\varkappa^4. \end{aligned} \tag{2.44}$$

At $\varepsilon q>0$ (Figure, b), this level is absent (although the increment δa remains). The condition (2.13), as is not difficult to show, is equivalent to a situation in which the value of the level $\varkappa_0^2$ is much less than the depth of the potential well δu. At $\varepsilon q<0$ and $t \gtrsim (t_s/t_p)^{1/2} t_p$, new levels appear which must be taken successively into consideration. However, as will be seen from the next section, the most important qualitative results obtained above remain in force, at least up to $t \sim t_p$.

Finally, we note still one more interesting effect connected with the tail. It turns out that the growth of the latter leads to a change of velocity of the soliton $d\xi/dt$ by an amount of the order of ε in comparison with the case in which the tail is not formed. Actually, it is seen from (2.11) and (2.26) that the difference of the coefficient a from a_s is brought about by the tail. Substituting (2.11) in the second formula of (2.14) and differentiating it with respect to time, we get, with accuracy to terms of order ε^2,

$$\frac{d\xi}{dt} = \frac{d}{dt}\left(\frac{\ln\rho}{2\varkappa}\right) + \frac{1}{2\varkappa^2}\frac{d\varkappa_0}{dt}. \tag{2.45}$$

The second term on the right side determines the reaction of the tail to the velocity of the soliton (the "recoil effect").

Now, using (2.18), (2.16), and (2.44), we obtain

$$\frac{d\xi}{dt} = 4\varkappa^2 - \frac{\varepsilon}{4\varkappa^3}\int_{-\infty}^{\infty} R[u_s(z)](z\,\mathrm{sech}^2 z + \mathrm{th}\,z + \mathrm{th}^2 z)\,dz. \tag{2.46}$$

Equation (2.46) differs from the corresponding adiabatic-approximation equation[1,3,4] by the presence of the last term in the round parentheses in the integral, which stems from the second term in (2.45) and thus describes the reaction of the tail. If the latter is not formed, then this term will not make a contribution by virtue of (2.43).

3. EVOLUTION OF THE SOLITON CORE UNDER THE ACTION OF THE PERTURBATION

The formation of a tail under the action of a perturbation takes place simultaneously with the deformation of the soliton and is closely connected with the latter. We investigate the asymptotic form of the deformed soliton. Under the condition (2.13), this deformation is determined by the formulas (2.23)–(2.25) in the region $|z| \ll \varkappa\xi$. The integral in (2.24) is conveniently divided into two, corresponding to the two terms in the numerator of the next to the last factor. Here we can set $p \to p \pm i0$ in its denominator. We then substitute (2.25) and interchange the order of integration over p and z. Then the first integral can easily be calculated with the help of the residue theorem. In the first integral, as is easily established, the principal contribution is made by the region $p \lesssim (\varkappa\xi)^{-1}$, so that we can set $\exp(i\tau p^3) \approx 1$ in the case $\varkappa\xi \gg 1$. After this, the second integral is also calculated. We finally obtain

$$v(z;\xi)=\int_{-\infty}^{\infty} R[u_s(z')]I(z,z')dz'+4\pi\int_z R[u_s(z')][\operatorname{th} z' \operatorname{th}^2 z$$
$$-\operatorname{th} z \operatorname{th}^2 z'+(z-z')\operatorname{th}^2 z \operatorname{th}^2 z']dz'+16\pi\theta(\varkappa_0)\varkappa^6 q\xi \operatorname{th}^2 z, \tag{3.1}$$

where

$$I(z,z')=\begin{cases} J(z,z'), & z>z' \\ J(z',z), & z<z', \end{cases}$$

$$J(z,z')=-\frac{\pi}{8\operatorname{ch}^2 z \operatorname{ch}^2 z'}[2(e^{-2z}\operatorname{ch}^2 z+4e^{z'-z}\operatorname{ch} z \operatorname{ch} z'+e^{2z'}\operatorname{ch}^2 z')$$
$$-14(e^{-z}\operatorname{ch} z+e^{z'}\operatorname{ch} z')\mathscr{P}_1(z-z')+15\mathscr{P}_2(z-z')],$$
$$\mathscr{P}_1(z)=1+{}^4/_7 z,\quad \mathscr{P}_2(z)=1+{}^{14}/_{15}z+{}^4/_{15}z^2.$$

Substitution of (3.1) in (2.23) leads to the expression w, which depends only on $z, \varkappa$, since the latter terms on the right sides of (3.1) and (2.23) (which are proportional to ξ) cancel one another. Here the asymptote of $w(z)$ at $1 \ll -z \ll \varkappa\xi$, as follows from (3.1), has the form

$$w(z)\approx-\frac{\varepsilon}{8\varkappa^5}\operatorname{th}^2 z\int_z^{\infty} R[u_s(z')]\operatorname{th}^2 z' dz'+w_3(z), \tag{3.2}$$

where $w_3(z)$ is defined in (2.39). It is not difficult to see that the first term in (3.2) goes over into the tail as $z \to -\infty$. This tail was obtained in the previous section. At $q=0$, this term tends to zero. The asymptote of w at $z \gg 1$ is obtained if we set $w_3(z) \to -w_3(-z)$ in (3.2). We note that (3.1) corresponds to what was obtained for the considered (near) region in Ref. 1; however, the second term in (2.23) was absent there.

The fact that the expression found here for the deformation of the soliton does not depend on ξ suggests that its applicability is limited not by the condition (2.13) for which it was obtained, but by a much greater time. In order to verify this, we substitute the expression

$$u(z)=-2\varkappa^2[\operatorname{sech}^2 z+w(z)] \tag{3.3}$$

in Eq. (1.1), where $w(z) \sim \xi$ and, in correspondence with (3.1), we assume that $w(z) \to 0$ as $z \to \infty$. Then, after linearization, we obtain the following equation:

$$\hat{\Lambda}w(z)\equiv\left(\frac{d^2}{dz^2}+12\operatorname{sech}^2 z-4\right)w(z)=\frac{1}{2\varkappa^5}\Big[2\varkappa^3(\xi_t-4\varkappa^2)\operatorname{sech}^2 z$$
$$+2\varkappa\varkappa_t(1-\operatorname{th} z-z\operatorname{sech}^2 z)+\varepsilon\int_z^{\infty} R[u_s(z')]dz'\Big]+O(\varepsilon^2). \tag{3.4}$$

Substituting the integral representation for w from (2.23)–(2.25) in the left side of (3.4)[3)] and making use of the identity

$$\hat{\Lambda}\frac{d}{dz}[(p+i\operatorname{th} z)^2 e^{2ipz}]=-8ip(1+p^2)(p+i\operatorname{th} z)^2 e^{2ipz}, \tag{3.5}$$

and also Eqs. (2.16) and (2.46), it is not difficult to verify that $w(z)$ actually satisfies Eq. (3.4). Here the error associated with the discarded terms can generally increase with time and become of the order of the retained terms at $t \sim t_p(t_p/t_s)$. Therefore $w(z)$, which is determined by the asymptotic expressions (2.23) and (3.1), describes the deformation of the soliton at[4)] $t_s \ll t \ll t_p(t_p/t_s)$. At large t, the figure refers only to the region $\Delta x \ll \varkappa^{-1}(t_p/t_s)^{1/3}$, adjoining the soliton core.

The portions of the tail that are more distant from the nucleus are modulated; at $\varepsilon q<0$, small solitons begin to develop from them, corresponding to the levels of the potential well formed by the tail. The amplitudes of these solitons will be of the order of $\varepsilon q\varkappa^2$.

4. STRUCTURE OF THE SOLITON TAILS FOR PERTURBED MKdV AND NSE EQUATIONS

As was shown in Refs. 1 and 6, the solution of Eqs. (1.2), (1.3), which describe the perturbation of the soliton pulse, has the form $u=u_s+\delta u$,

$$u_s(z,t)=2\nu e^{i\mu z/\nu+i\delta}\operatorname{sech} z,\quad z=2\nu(x-\xi),$$
$$\delta u=2\nu w(z,t)e^{i\mu z/\nu+i\delta}, \tag{4.1}$$

where, in first order in ε,

$$w=\frac{e^{-i(\delta+\mu z/\nu)}}{2\pi i\nu}\int_{-\infty}^{\infty} r(\lambda)\left(\frac{\lambda-\mu+i\nu\operatorname{th} z}{\lambda-\mu+i\nu}\right)^2 e^{i\lambda(z/\nu+2\xi)}d\lambda$$
$$-\frac{\nu e^{i(\delta+\mu z/\nu)}}{2\pi i\operatorname{ch}^2 z}\int_{-\infty}^{\infty} r^*(\lambda)\frac{e^{-i\lambda(z/\nu+2\xi)}}{(\lambda-\mu-i\nu)^2}d\lambda. \tag{4.2}$$

Here $r(\lambda)=b(\lambda)/a(\lambda)$ is the coefficient of reflection, and the change in time of the real parameters $\nu(t)>0$, $\mu(t)$, $\xi(t)$, and $\delta(t)$ is described by the equations of the adiabatic approximation.[7,3,4,6]

$$\frac{\partial a}{\partial t}=-\frac{i\varepsilon}{(\lambda-\mu+i\nu)^2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{\lambda-\mu-i\nu\operatorname{th} z}{\operatorname{ch} z}R[u_s(z)]e^{-i\mu z/\nu-i\delta}dz, \tag{4.3}$$

$$\frac{\partial b}{\partial t}=ih(\lambda)b+\frac{i\varepsilon A(\lambda,\mu,\nu)e^{i\delta-2i\lambda\xi}}{2\nu[(\lambda-\mu)^2+\nu^2]}; \tag{4.4}$$

where

$$A(\lambda,\mu,\nu)=\int_{-\infty}^{\infty} e^{i(\mu-\lambda)z/\nu}\{(\lambda-\mu-i\nu\operatorname{th} z)^2 R[u_s(z)]e^{-i\mu z/\nu-i\delta}$$
$$-\nu^2\operatorname{sech}^2 z R^*[u_s(z)]e^{i\mu z/\nu+i\delta}\}dz, \tag{4.5}$$

and $h(\lambda)=8\lambda^3$ and $h(\lambda)=-2\lambda^2$ for MKdV and NSE, respectively (on the case of a real MKdV equation (1.2) one should set $\mu=\delta=0$ in all the formulas).

Using the equations of the adiabatic approximation for $\nu(t)$ and $\mu(t)$, it is not difficult to establish the fact that Eq. (4.3) is satisfied at

$$a=a_s=(\lambda-\mu-i\nu)(\lambda-\mu+i\nu)^{-1},$$

i.e., $\delta a=a-a_s=0$ (a_s corresponds to u_s). This can also be proved by direct calculation of a with the help of variational derivatives [in analogy with (2.29)]. Correspondingly, the tails formed in this case under the action of the perturbation will not lead to a change in the adiabatic-approximation equations for μ, ν, δ and ξ.

Everywhere in what follows, just as was done in Ref. 1, we shall assume the perturbation to be such that μ, ν and $A(\lambda,\mu,\nu)$ change slowly with the time.[5)] Then, from (4.4), by integration by parts, we obtain (at $t \ll t_p^*$)

$$r(\lambda,t)=-\frac{\varepsilon A(\lambda,\mu,\nu)[1-\exp(ih(\lambda)t+2i\lambda\xi-i\delta)]}{2\nu[h(\lambda)+2\lambda\xi_t-\delta_t](\lambda-\mu-i\nu)^2}e^{i\delta-2i\lambda\xi}$$
$$+\varepsilon O\left(\frac{t}{t_p^*}\right), \tag{4.6}$$

where

$$\frac{1}{t_p^*}=\left|\frac{\varepsilon}{2\nu}\operatorname{Re}\int_{-\infty}^{\infty}\operatorname{sech} z R[u_s(z)]e^{-i\mu z/\nu-i\delta}dz\right|.$$

We now consider a real MKdV equation (1.2), i.e., we set $\mu=\delta=0$, $h(\lambda)=8\lambda^3$. Taking it into account that $r^*(-\lambda)=-r(\lambda)$ in this case, and introducing $p=\lambda/\nu$ and $\tau=8\nu^3t=t/t_s$, we obtain from (4.2) and (4.6),

$$w(z,t)=\frac{i\varepsilon}{32\pi\nu^4}\int_{-\infty}^{\infty}\frac{A(p)[(p+i\operatorname{th}z)^2+\operatorname{sech}^2z]}{p(p^2+1)^3}\times(1-e^{i\tau p^3+2ip\nu\xi})e^{ipz}dp,\tag{4.7}$$

where

$$A(p)=\int_{-\infty}^{\infty}R[u_s(z)](p^2-2ip\operatorname{th}z-1)\exp(-ipz)dz.$$

Integrating (4.7) with respect to z, we obtain

$$\int_{-\infty}^{\infty}w(z,t)dz=\frac{\varepsilon\xi(t)}{8\nu^3}\int_{-\infty}^{\infty}R[u_s(z)]dz.\tag{4.8}$$

At the assumed degree of accuracy, this same result [taking into account the fact that $\xi_t=4\nu^2+O(\varepsilon)$] follows from the law of variation of the first invariant

$$I_1=\int_{-\infty}^{\infty}u\,dx$$

for a real MKdV equation.[1]

The regions of the tail of interest to us correspond to $z<0$, $|z|\gg1$ (the soliton, just as in the case of the KdV equation, moves from left to right). Taking this into account and setting

$$B(p)=\frac{A(p)}{(p^2+1)(p+i)^2},\quad\varphi(p)=\frac{B(p)-B(0)}{p},\tag{4.9}$$

we obtain from (4.7)

$$w(z,t)=w_1+w_2+w_3;\tag{4.10}$$

$$w_1(z,t)=\frac{i\varepsilon B(0)}{32\pi\nu^4}\int_{-\infty}^{\infty}[1-\exp(i\tau p^3+2i\nu p\xi)]e^{ipz}\frac{dp}{p},\tag{4.11}$$

$$w_2(z,t)=-\frac{i\varepsilon}{32\pi\nu^4}\int_{-\infty}^{\infty}\varphi(p)\exp[i\tau p^3+ip(z+2\nu\xi)]dp,\tag{4.12}$$

$$w_3(z)=\frac{i\varepsilon}{32\pi\nu^4}\int_{-\infty}^{\infty}\varphi(p)e^{ipz}dp.\tag{4.13}$$

Reasoning in the estimate of the integral in (4.12) exactly in the same manner as in Sec. 2, and taking into account that $w_3\sim\varepsilon z^2e^z$ describes the distortion of the soliton core, we find the asymptote of the tail from (4.10)–(4.12) at $1\ll\tau\ll t_p^*/t_s$, $z<0$, and $|z|\gg1$:

$$w^{(t)}\approx w_1=\frac{\varepsilon q}{16}\int_{-\infty}^{2\nu x(3\tau)^{-1/3}}\operatorname{Ai}(y)dy,\quad B(0)=\nu^4q\tag{4.14}$$

in the region $|2\nu x|^{1/2}\ll(3\tau)^{1/3}$ and

$$w^{(t)}=w_1+w_2\approx\frac{\varepsilon A(s)\exp[-i(2\tau s^3+\pi/4)]}{32\nu^4(3\pi\tau)^{1/2}s^{1/2}(s^2+1)(s+i)^2}+\text{c.c.},\quad s=\left|\frac{2\nu x}{3\tau}\right|^{1/2}\tag{4.15}$$

in the region $s>1$, $x<0$.

Thus, the qualitative picture of the formation of the tail is the same here as in the case of the KdV equation, and is characterized by the presence in $w(z)$ of the plateau $w\approx\varepsilon q/16$ in the region $(3\tau)^{1/3}\lesssim2\nu x\ll3\tau$, a transition region $2\nu|x|<(3\tau)^{1/3}$, and a region of rapid exponentially decreasing oscillations at $|2\nu x|^{1/2}\gg(3\tau)^{1/2}$, $x<0$. The condition of absence of an increasing tail, similar to (2.43), has the following form in the given case:

$$q=\frac{1}{\nu^4}\int_{-\infty}^{\infty}R[u_s(z)]dz=0.\tag{4.16}$$

Finally, we show that by integrating in (4.7) at $|z|\ll2\nu\xi$, we can obtain a solution of $w(z)$ in explicit form for the region of the soliton core. This solution remains valid at least to $t\sim t_p^*$. Just as in Sec. 3, this is demonstrated by the substitution of $u=u_s+\delta u$ in Eq. (1.2). At $-z\gg1$, this solution transforms into the plateau $w\approx\varepsilon q/16$ mentioned above. At $z\gg1$, its asymptote is identical with that found earlier.[1]

We now consider the NSE (1.3). Setting

$$h(\lambda)=-2\lambda^2,\ \xi_t=2\mu+O(\varepsilon),\ \delta_t=2(\nu^2+\mu^2)+O(\varepsilon)$$

in (4.6), we get from (4.2) the solution in the form $w(z,t)=\bar w+\tilde w$. The quantity $\bar w$, which changes slowly with time, represents the averaged correction to the soliton and which decays as $|z|\to\infty$. It was found previously at $\mu=0$.[1] At arbitrary μ, the quantity $\bar w$ is obtained by formal replacement $\delta\to\delta+\mu z'/\nu$ in the formulas (5.14) from Ref. 1. For the rapidly changing part of the solution $\tilde w$ at

$$|z|\gg1,\ 1\ll\tau=2\nu^2t\ll2\nu^2t_p^*$$

we find, by the method of stationary phase,

$$\tilde w\approx\frac{\varepsilon A(\nu s,\mu,\nu)\exp[i(\tau s^2-\mu z/\nu-\delta+\pi/4)]}{8\nu^3(\pi\tau)^{1/2}[(s-\mu/\nu)^2+1](s-\mu/\nu\mp i)^2}\quad z\to\pm\infty,\tag{4.17}$$

where $s=\nu x/\tau$ and $A(\nu s,\mu,\nu)$ is determined by the formula (4.5). This quantity, being a consequence of the chosen initial conditions, vanishes as $\tau\to\infty$, in correspondence with the results of Ref. 1. Thus, in the given case, the perturbation does not lead to the formation of a tail, and the soliton, slightly deformed, changes in self-similar fashion in correspondence with the formulas of the adiabatic approximation. In the work of Pereira and Stenflo,[9] a similar result was obtained for the case in which $\varepsilon R[u]$ describes the growth or decay of the waves.

[1] In all three cases, it is assumed that $u(x,t)\to0$, $R[u(x,t)]\to0$ as $|x|\to\infty$, and that the operator R does not depend explicitly on the time.

[2] If $q=0$, then $t_p=\infty$, and in place of it we need to take, for example,

$$t_p'=\left|\frac{\varepsilon}{4\varkappa^2}\int_{-\infty}^{\infty}R[u_s(z)]dz\right|^{-1}.$$

[3] Thanks to the conditions $|z|\ll\varkappa\xi$ and $\varkappa\xi\gg1$, we can set $\exp(i\tau p^3)=1$ in (2.24), as was done in the derivation of (3.1).

[4] Strictly speaking, at $t\sim t_p$, we should substitute the quantity $\varkappa$ in $u_s(z,\varkappa)$. This quantity is defined in the approximation that follows from (2.16), since the error $\delta\varkappa\sim\varepsilon$ at $t\sim t_p$. However, this makes no changes in the physical picture set forth above and therefore the corresponding correction is not calculated here. We also note that if $\varepsilon R[u]$ describes the dissipation, then, generally speaking the soliton ceases to exist after at several t_p intervals.

[5] This is true, for example, if $R[ue^{i\delta(t)}]=e^{i\delta(t)}R[u]$, where $\operatorname{Im}\delta(t)=0$.

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Mechanism of nonadiabatic losses in a dipole trap

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The principal mechanisms of the nonadiabatic escape of particles from a trap are considered—stochastic instability and universal Arnold instability. The boundaries of the regions of existence of these instabilities are investigated. It is shown that the results of numerical calculations of the limits of the adiabatic behavior of the particles agree with analytic estimates that follow from stochasticity theory. It is observed in experiment that introduction of azimuthal inhomogeneity of the magnetic field can lead to a decrease of the nonadiabatic losses.

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Processes that control the dynamics of particles in magnetic traps can be connected, besides the global diffusion due to the scattering by the residual gas, with the universal Arnold diffusion and with stochastic instability.[1] In particular, it was suggested[1,2] that the nonadiabatic decrease of the particle lifetime with increasing relative cyclotron radius $\chi = \rho/R_c$ (R_c is the radius of curvature of the force line), which is observed in linear traps with mirrors,[3] is due to the Arnold universal instability. To understand the reason for the nonadiabatic behavior of the particles it is necessary to determine first the limits of the regions where the proposed instability mechanisms operate. The present paper is devoted to a determination of these limits and of the form of the instability responsible for the nonadiabatic departure of a particle from a dipole trap. The questions considered here, which touch upon the general physical problem of the investigation of very subtle and universal interaction mechanisms of resonances that control stochastization processes, are of great interest, for example, in the investigation of singularities of the spatial distribution of the high-energy part of the spectrum of the charged particles in the magnetic traps of the earth and of Jupiter, etc.

1. The change of the adiabatic invariant in multiperiod systems is due to the interaction of nonlinear resonances, for example, resonances between fast Larmor rotation and higher harmonics of slow oscillations between the magnetic mirrors. In the vicinity of the separatrix of each resonance there appears the so-called stochastic layer, which constitutes a certain region of unstable motion.[1,4] In the case of axial symmetry of the field (two-dimensional motion) the thin stochastic layers of different resonances do not intersect on the phase plane, and the instability is therefore localized within the confines of a single stochastic layer, and causes only bounded oscillations of the adiabatic invariant and of the frequencies. If the parameter χ is large enough, the stochastic layers broaden to the dimensions of their resonances, neighboring resonances in phase space overlap, and strong stochastic instability results.

The boundary of the stochastic instability, determined by the criterion of overlap of the nonlinear resonances, can be represented, according to Ref. 4, in the form

$$\Delta\mu/\mu \approx \pi^{1/2}\Omega/2\omega, \tag{1}$$

where μ is the orbital magnetic moment of the particle, ω is the cyclotron frequency, and Ω is the frequency of the oscillations between the reflection points. We recognize that in a dipole magnetic field the period of the oscillation of the leading center between the reflection point is given by[5]

$$\tau_2 = 4R_e T(\alpha)/v, \quad 0 \leqslant \alpha \leqslant \pi/2, \tag{2}$$

where v is the particle velocity, R_e is the radius of the force line in the median plane, $T(\alpha) = 1.3-0.56 \sin\alpha$, and α is the angle between the velocity vector and the

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