where $\Delta=(\Gamma_1+\Gamma_2)/2$ and

$$s_\infty=-\frac{\sum |c_k|^2(u^4\delta(\Omega-\tilde{\omega}_k)-v^4\delta(\Omega+\tilde{\omega}_k))}{\sum\limits_k |c_k|^2(2N_k+1)(u^4\delta(\Omega-\tilde{\omega}_k)+v^4\delta(\Omega+\tilde{\omega}_k))}. \tag{24}$$

Rewriting (8) in terms of the Fourier components $\mathbf{S}(\chi)$

$$\mathbf{s}(t)=\mathbf{s}_0+\frac{i}{2\pi}\int e^{-i\chi t}\frac{\mathbf{S}(\chi)}{\chi+i\delta}d\chi, \tag{25}$$

we obtain ultimately

$$\begin{aligned} s^{(x)}(t)&=\frac{s_0\omega_1\Delta\omega}{\Omega^2}(e^{-\Gamma t}-e^{-\Delta t}\cos\Omega t)+\frac{\omega_1}{\Omega}s_\infty(1-e^{-\Gamma t}),\\ s^{(z)}(t)&=s_0e^{-\Gamma t}-\frac{\omega_1^2}{\Omega^2}(e^{-\Gamma t}-e^{-\Delta t}\cos\Omega t)+\frac{\Delta\omega}{\Omega}s_\infty(1-e^{-\Gamma t}),\end{aligned} \tag{26}$$

which goes over into the Weisskopf-Wigner result[1] in the limit as $\omega_1\to 0$ and $T\to 0$. As seen from (26), the stationary distribution, in the limit as $t\to\infty$, is equal to $s_\infty$ and is directed along the "effective field."

We can now answer the questions raised at the beginning of the article. First, there are no memory effects in the stationary distribution, i.e., it does not depend on the initial distribution of the two-level system; second, there is a critical frequency

$$\omega_c={}^1/_2\omega_0[1+(\omega_1/\omega_0)^2], \tag{27}$$

such that at $\omega\leqslant\omega_c$ the second delta-function vanishes and we have

$$s_\infty=-\operatorname{cth}\frac{\omega+\Omega}{2T}, \tag{28}$$

which corresponds to a Boltzmann distribution with quasienergy $\omega+\Omega$. At $\omega>\omega_c$ the terms with $\delta(\Omega+\tilde{\omega}_k)$ begin to play an essential role and we find that at these frequencies the stationary distribution is determined by the matrix elements of the operator of the interaction with the thermostat.

Turning to a practical application of (26), we note that it can have a bearing on the theory of quantum amplifiers. It is known that the relaxation processes due to interaction with thermal radiation are the cause of the noise in quantum amplifiers. The corresponding relaxation constants

$$\begin{aligned}\Gamma&=2\pi\sum_k |c_k|^2(2N_k+1)(u^4\delta(\Omega-\tilde{\omega}_k)+v^4\delta(\Omega+\tilde{\omega}_k)),\\ \Delta&=2\pi\sum_k |c_k|^2(2N_k+1)({}^1/_2u^4\delta(\Omega-\tilde{\omega}_k)+{}^1/_2v^4\delta(\Omega+\tilde{\omega}_k)+(uv)^2\delta(\tilde{\omega}_k)),\end{aligned} \tag{29}$$

assume in the resonance case $\Delta\omega\ll\omega_1\ll\omega_0$ the values

$$\Gamma_p=\Delta_p=\Gamma_0/2, \tag{30}$$

where $\Gamma_0$ is the reciprocal decay time in the absence of a signal (the Weisskopf-Wigner constant in the optical region), so that without a signal the noise in the amplifier is double the noise in the resonant case.


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# The fluctuation-dissipation relation in nonlinear electrodynamics


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The linear and nonlinear responses to an external perturbing field in a plasma are considered. It is shown that, apart from the usual fluctuation-dissipation relation connecting the binary correlation function for the charge-density-fluctuations with the linear electric susceptibility, there also exist a number of additional relations connecting correlation functions of higher order with the nonlinear susceptibilities. A number of integral relations between the linear and nonlinear susceptibilities in a plasma are established.



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## 1. INTRODUCTION

As is well known, in linear electrodynamics, for systems in thermodynamic equilibrium, a fluctuation-dissipation relation establishes a general connection between the dissipative properties of the system and the fluctuations of various quantities. Since the dissipative properties of an electro-dynamic system are determined by the macroscopic coefficients in the linear relationship between the induced charges or currents and the fields, specifying these coefficients determines completely the spectral distributions of the fluctuations of the electrodynamic quantities.[1-4] For an equilibrium plasma the spectral distribution of the electromagnetic fluctuations is determined by specifying the permittivity tensor. Conversely, knowing the spectrum



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of the electromagnetic fluctuations, by inverting the fluctuation-dissipation relation we can find the permittivity of the medium.[5]

The fluctuation-dissipation relation can be generalized to the case of a nonequilibrium (albeit in a stable steady state) plasma. In fact, in the derivation of the fluctuation-dissipation relation for equilibrium systems, the relationship between the correlation function of the current fluctuations and the average energy absorbed by the system as a consequence of dissipation is used.[3] An analogous relationship also holds in the absence of thermodynamic equilibrium, and this makes it possible to establish a generalized fluctuation-dissipation relation describing the fluctuations in nonequilibrium systems.[5]

The fluctuation-dissipation relation connecting the linear electric susceptibility and the binary correlation functions for the fluctuations is valid for both linear and nonlinear systems.[1)] When treating the fluctuations in nonlinear electrodynamic systems, besides the usual fluctuation-dissipation relation it is possible to establish a number of additional relations connecting the nonlinear electric susceptibilities with correlation functions of higher than binary order.[8] In the case of a plasma, it is convenient to write the analogous extra relations in the form of a generalized fluctuation-dissipation relation connecting the discontinuities of the nonlinear electric susceptibilities of the plasma across the cuts in the complex frequency planes with the spectral correlation functions for the charge-density fluctuations.

## 2. LINEAR AND NONLINEAR RESPONSES OF THE SYSTEM TO AN EXTERNAL PERTURBATION

To describe the behavior of the system we introduce the microscopic distribution function $D(t)$, the temporal evolution of which is described by the Liouville equation

$$\frac{\partial D(t)}{\partial t}=\{H+V(t),D(t)\}, \tag{1}$$

where $\{\ldots,\ldots\}$ are the classical Poisson brackets, $H$ is the Hamiltonian of the system, and $V(t)$ is the external, time-dependent perturbation. With neglect of the motion of the ions, the Hamiltonian $H$ describing the plasma can be written in the form

$$H=\sum_\alpha\frac{\mathbf{p}_\alpha^2}{2m}+\frac{1}{2}\sum_{\alpha\neq\alpha'}U(\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}), \tag{2}$$

where $\mathbf{p}_\alpha^2/2m$ is the kinetic energy of a single electron and $U(\mathbf{r}_\alpha-\mathbf{r}_{\alpha'})$ is the Coulomb interaction energy of two electrons (the summation in (2) runs over all the electrons). The Liouville equation (1) must be supplemented by the initial condition

$$D(t)|_{t=-\infty}=D_0 \tag{3}$$

($D_0$ is the microscopic distribution at the initial time $t=-\infty$).

Introducing the Liouville operator

$$L\ldots\equiv-i\{H,\ldots\}, \tag{4}$$

we can write the formal solution of Eq. (1) with the initial condition (3) in the form

$$D(t)=D_0+\int_0^\infty dt' e^{it'L}\{V(t-t'),\ D(t-t')\}. \tag{5}$$

Let the external perturbation be due to the action of an electric field characterized by the potential $\Phi(\mathbf{r},t)$. In this case,

$$V(t)=\int d\mathbf{r}\,\Phi(\mathbf{r},t)\rho(\mathbf{r},t), \tag{6}$$

where $\rho(\mathbf{r},t)$ is the microscopic electric-charge density in the system:

$$\rho(\mathbf{r},t)=\sum_\alpha e\delta(\mathbf{r}-\mathbf{r}_\alpha(t)) \tag{7}$$

($r_\alpha(t)$ is the position vector of electron $\alpha$ at time $t$). It is more convenient to rewrite the external perturbation (6) in the form

$$V(t)=\sum_\mathbf{k}\Phi_\mathbf{k}(t)\rho_{-\mathbf{k}}(t), \tag{8}$$

where $\Phi_\mathbf{k}$ and $\rho_\mathbf{k}$ are the spatial Fourier components of the field potential and charge density. We note that the quantity $\rho_\mathbf{k}$, generally speaking, depends on the coordinates and velocities of all the electrons, and its value at a given time is determined by the state of the system.

In the absence of the external perturbation the average value of the total charge density (including the charges of the ions) $\rho_\mathbf{k}$ is equal to zero, since the system is electrically neutral and spatially uniform:

$$\langle\rho_\mathbf{k}\rangle^0\equiv\int d\Gamma\,\rho_\mathbf{k}D_0=0 \tag{9}$$

($d\Gamma$ is an element of volume in the phase space of the whole system). The external perturbation leads to the appearance of a nonzero charge density. We shall define the linear and non-linear responses of the charge density to the external perturbation by the relation

$$\langle\rho_\mathbf{k}(t)\rangle^{(n)}\equiv\int d\Gamma\,\rho_\mathbf{k}(t)D^{(n)}(t),\qquad n=1,2,\ldots, \tag{10}$$

where $D^{(n)}(t)$ is the corresponding interational correction in the external perturbation in the solution (5). For the linear, quadratic, and cubic responses, it is not difficult to obtain the following general expressions:

$$\langle\rho_\mathbf{k}(t)\rangle^{(1)}=\sum_{\mathbf{k}'}\int_0^\infty dt'\,\Phi_{\mathbf{k}'}(t-t')\Psi^{(1)}_{\mathbf{k};-\mathbf{k}'}(t'), \tag{11}$$

$$\langle\rho_\mathbf{k}(t)\rangle^{(2)}=\sum_{\mathbf{k}_1,\mathbf{k}_2}\int_0^\infty dt_1\int_0^\infty dt_2\Phi_{\mathbf{k}_1}(t-t_1)\Phi_{\mathbf{k}_2}(t-t_1-t_2)\Psi^{(2)}_{\mathbf{k};-\mathbf{k}_1,-\mathbf{k}_2}(t_1,t_2), \tag{12}$$

$$\langle\rho_\mathbf{k}(t)\rangle^{(3)}=\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\int_0^\infty dt_1\int_0^\infty dt_2\int_0^\infty dt_3\Phi_{\mathbf{k}_1}(t-t_1)\Phi_{\mathbf{k}_2}(t-t_1-t_2)\times\Phi_{\mathbf{k}_3}(t-t_1-t_2-t_3)\Psi^{(3)}_{\mathbf{k};-\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_3}(t_1,t_2,t_3), \tag{13}$$

where we have introduced the notation

$$\Psi^{(1)}_{\mathbf{k};-\mathbf{k}'}(t')\equiv\int d\Gamma\,\rho_\mathbf{k}e^{it'L}\{\rho_{-\mathbf{k}'},D_0\}, \tag{14}$$

$$\Psi^{(2)}_{\mathbf{k};-\mathbf{k}_1,-\mathbf{k}_2}(t_1,t_2)\equiv\int d\Gamma\,\rho_\mathbf{k}e^{it_1L}\{\rho_{-\mathbf{k}_1},e^{it_2L}\{\rho_{-\mathbf{k}_2},D_0\}\}, \tag{15}$$

$$\Psi^{(3)}_{\mathbf{k};-\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_3}(t_1,t_2,t_3)\equiv\int d\Gamma\,\rho_\mathbf{k}e^{it_1L}\{\rho_{-\mathbf{k}_1},e^{it_2L}\{\rho_{-\mathbf{k}_2},e^{it_3L}\{\rho_{-\mathbf{k}_3},D_0\}\}\}. \tag{16}$$

## 3. THE LINEAR FLUCTUATION-DISSIPATION RELATION

We shall consider first the linear response of the system. It is easy to see that, because of the spatial uniformity of the system,

$$\Psi^{(1)}_{\mathbf{k};-\mathbf{k}'}(t)=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\psi^{(1)}_{\mathbf{k}}(t). \qquad (17)$$

Substituting this relation into (11) and taking the temporal Fourier transform, we obtain the following formula for the linear response:

$$\langle\rho\rangle^{(1)}_{\mathbf{k}\omega}=-\frac{i}{2\pi}\Phi_{\mathbf{k}\omega}\int_{-\infty}d\omega'\frac{\psi^{(1)}_{\mathbf{k}}(\omega')}{\omega'-\omega-i0}. \qquad (18)$$

It is not difficult to show that the quantity $\psi^{(1)}_{\mathbf{k}}(\omega)$ determines the discontinuity of the linear electric susceptibility of the plasma across the cut in the complex $\omega$-plane. In fact, in the linear approximation the induced-charge density $\langle\rho\rangle^{(1)}_{\mathbf{k}\omega}$ in the plasma is proportional to the potential $\Phi_{\mathbf{k}\omega}$ of the external field:

$$4\pi\langle\rho\rangle^{(1)}_{\mathbf{k}\omega}=-k^2\hat{\varkappa}^{(1)}(\omega,\mathbf{k})\Phi_{\mathbf{k}\omega}, \qquad (19)$$

where $\hat{\varkappa}^{(1)}(\omega,\mathbf{k})$ is the effective electric susceptibility that takes into account the polarization of the field in the plasma. The effective electric susceptibility $\varkappa^{(1)}(\omega,\mathbf{k})$ is an analytic function in the complex $\omega$-plane, with a cut along the real axis. According to the principle of causality,

$$\hat{\varkappa}^{(1)}(\omega,\mathbf{k})=\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega'\frac{\mathrm{Im}_{\omega'}\hat{\varkappa}^{(1)}(\omega',\mathbf{k})}{\omega'-\omega-i0}, \qquad (20)$$

where $\mathrm{Im}_\omega\hat{\varkappa}^{(1)}(\omega,\mathbf{k})$ is the discontinuity of the effective electric susceptibility across the cut:

$$\mathrm{Im}_\omega\hat{\varkappa}^{(1)}(\omega,\mathbf{k})=\frac{1}{2i}\{\hat{\varkappa}^{(1)}(\omega+i0,\mathbf{k})-\hat{\varkappa}^{(1)}(\omega-i0,\mathbf{k})\} \qquad (21)$$

(We note that the discontinuity of the analytic function $\hat{\varkappa}^{(1)}(\omega,\mathbf{k})$ across the cut simply coincides with the imaginary part of the electric susceptibility at real values of the frequency.) Substituting (20) into (19) and comparing the expression obtained with (18), it is not difficult to establish that

$$\mathrm{Im}_\omega\hat{\varkappa}^{(1)}(\omega,\mathbf{k})=\frac{2\pi i}{k^2}\psi^{(1)}_{\mathbf{k}}(\omega). \qquad (22)$$

We shall calculate the quantity $\psi^{(1)}_{\mathbf{k}}(\omega)$, assuming that the system is in a state of thermodynamic equilibrium in the absence of the perturbation. For the equilibrium distribution function $D_0$ the relation

$$\{A,D_0\}=-\frac{D_0}{T}\dot{A}$$

is valid ($T$ is the temperature of the system), and, therefore,

$$\psi^{(1)}_{\mathbf{k}}(\omega)=-i\frac{\omega}{T}\langle\rho^2\rangle_{\mathbf{k}\omega}, \qquad (23)$$

where $\langle\rho^2\rangle_{\mathbf{k}\omega}$ is the space-time Fourier transform of the quadratic correlation function for the charge-density fluctuations.

Substituting the expression found into (22), we obtain the well known Kubo relation[9]

$$\mathrm{Im}_\omega\hat{\varkappa}^{(1)}(\omega,\mathbf{k})=\frac{2\pi}{k^2}\frac{\omega}{T}\langle\rho^2\rangle_{\mathbf{k}\omega}, \qquad (24)$$

which establishes the connection between the imaginary part of the effective electric susceptibility and the spectral distribution of the charge-density fluctuations in the system. In principle, the Kubo relation makes it possible to find the effective electric susceptibility if the spectral distribution of the charge-density fluctuations in the plasma is known. In fact, however, because of the absence of direct methods for calculating the spectral distribution $\langle\rho^2\rangle_{\mathbf{k}\omega}$ of the fluctuations when the Coulomb interaction between the particles is taken into account, this method of determining the electric susceptibility is found to be rather ineffective.

Usually, the relation (24) is used to determine the spectral distribution of the fluctuations from the given value of the electric susceptibility of the system (in this case, it is called a fluctuation-dissipation relation). Noting that the effective electric susceptibility $\hat{\varkappa}^{(1)}(\omega,\mathbf{k})$ is expressed in terms of the usual linear electric susceptibility $\varkappa^{(1)}(\omega,\mathbf{k})$ by

$$\hat{\varkappa}^{(1)}(\omega,\mathbf{k})=\frac{\varkappa^{(1)}(\omega,\mathbf{k})}{|\varepsilon(\omega,\mathbf{k})|^2}, \qquad (25)$$

where $\epsilon(\omega,\mathbf{k})=1+\varkappa^{(1)}(\omega,\mathbf{k})$ is the dielectric permittivity of the plasma, from (24) we find

$$\langle\rho^2\rangle_{\mathbf{k}\omega}=\frac{k^2}{2\pi}\frac{T}{\omega}\frac{\mathrm{Im}_\omega\varkappa^{(1)}(\omega,\mathbf{k})}{|\varepsilon(\omega,\mathbf{k})|^2}. \qquad (26)$$

We expand the left- and right-hand sides of the equality (26) in powers of $e^2$ and retain the principal terms. Since the linear electric susceptibility of the plasma is proportional to $e^2$, from (26) we find for the imaginary part of the electric susceptibility

$$\mathrm{Im}_\omega\varkappa^{(1)}(\omega,\mathbf{k})=\frac{2\pi}{k^2}\frac{\omega}{T}\langle\rho^2\rangle^0_{\mathbf{k}\omega}, \qquad (27)$$

where $\langle\rho^2\rangle^0_{\mathbf{k}\omega}$ is the spectral distribution of the charge-density fluctuations with neglect of the Coulomb interaction between the particles. The equality obtained can be regarded as the inverse of the fluctuation-dissipation relation (26). In the right-hand side of (27), unlike (24), there appears the quantity $\langle\rho^2\rangle^0_{\mathbf{k}\omega}$, which can be calculated easily. Therefore, the relation (27), together with the Kramers-Krönig dispersion relation, makes it possible to determine the electric susceptibility of the plasma completely, if the latter is in an equilibrium state.

By making use of the general formulas (14) and (17), it is not difficult to find a generalized relation (27) that is valid in the case of a nonequilibirum (albeit in a stable steady state) plasma too. In fact, neglecting the Coulomb interaction between the particles, we have

$$\psi^{(1)}_\omega(\mathbf{k})=2\pi i\frac{e^2}{m}\int d\mathbf{v}\,\delta(\omega-\mathbf{k}\mathbf{v})\mathbf{k}\frac{\partial f_0}{\partial\mathbf{v}}, \qquad (28)$$

where $f_0(\mathbf{v})$ is the single-particle distribution function.

Thus, in the case of a nonequilibrium plasma the inverse of the fluctuation-dissipation relation can be written in the form

$$\mathrm{Im}_{\omega}\,\varkappa^{(1)}(\omega,\mathbf{k})=-\frac{2\pi}{m}\frac{\partial}{\partial\omega}\langle\rho^2\rangle_{\mathbf{k}\omega}^{0},\tag{29}$$

where

$$\langle\rho^2\rangle_{\mathbf{k}\omega}^{0}=2\pi e^2\int d\mathbf{v}\,\delta(\omega-\mathbf{k}\mathbf{v})f_0(\mathbf{v}).\tag{30}$$

The relation (29) obtained connects the imaginary part of the linear susceptibility of the plasma (the Coulomb interaction between the particles in the plasma is taken into account) with the correlation function for the charge-density fluctuations with neglect of the Coulomb interaction between the particles. Together with (20), the relation (29) completely determines the electric susceptibility of a nonequilibrium plasma.

The relations (27) and (29) are general and can be used to determine the permittivity not only of a hot plasma. Using (27) or (29) it is not difficult to obtain expressions for the permittivity of a degenerate plasma, a super-conducting plasma (for this, as $\langle\rho^2\rangle_{\mathbf{k}\omega}^{0}$ it is necessary to use the correlation function for a system of particles with pairing but without Coulomb interaction), a solid-state plasma (in this case, in $\langle\rho^2\rangle_{\mathbf{k}\omega}^{0}$ it is necessary to take the interaction of the electrons with the lattice into account), etc.

The spectral distribution of the charge-density fluctuations in a nonequilibrium plasma is described by the formula

$$\langle\rho^2\rangle_{\mathbf{k}\omega}=\langle\rho^2\rangle_{\mathbf{k}\omega}^{0}/|\varepsilon(\omega,\mathbf{k})|^2.\tag{31}$$

Unlike in the equilibrium case, when the spectral distribution $\langle\rho^2\rangle_{\mathbf{k}\omega}$ of the fluctuations is completely determined by the dielectric permittivity $\epsilon(\omega,\mathbf{k})$ of the plasma and the temperature $T$, in nonequilibrium conditions the spectral distribution $\langle\rho^2\rangle_{\mathbf{k}\omega}$ of the fluctuations is expressed not only in terms of the permittivity $\epsilon(\omega,\mathbf{k})$ of the plasma but also in terms of the spectral distribution $\langle\rho^2\rangle_{\mathbf{k}\omega}^{0}$ of the charge-density fluctuations in the absence of interaction between the particles. Consequently, in nonequilibrium conditions, specifying the permittivity of the plasma is not sufficient for a complete description of its electrodynamic properties—in particular, for the description of the spectral distribution of the fluctuations of the charge density and the field. However, such a description can be obtained by specifying the spectral distribution $\langle\rho^2\rangle_{\mathbf{k}\omega}^{0}$; knowing this, we can establish the permittivity $\epsilon(\omega,\mathbf{k})$ of the plasma and then determine the spectral distribution $\langle\rho^2\rangle_{\mathbf{k}\omega}$ of the fluctuations with allowance for the Coulomb interaction between the particles.

## 4. THE NONLINEAR FLUCTUATION-DISSIPATION RELATION

We turn now to the analysis of nonlinear response in a plasma. The quadratic response of the charge density to an external perturbing field is defined by (12). Taking the temporal Fourier transform and introducing the notation

$$\Psi^{(2)}_{\mathbf{k};-\mathbf{k}_1,-\mathbf{k}_2}(\omega_1,\omega_2)=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1,\omega_2).\tag{32}$$

We can rewrite the relation (12) in the form

$$\langle\rho\rangle^{(2)}_{\mathbf{k}\omega}=-\frac{1}{(2\pi)^2}\sum_{\substack{\omega_1+\omega_2=\omega\\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}}\Phi_{\mathbf{k}_1\omega_1}\Phi_{\mathbf{k}_2\omega_2}\int_{-\infty}^{\infty}d\omega_1'\int_{-\infty}^{\infty}d\omega_2'\times\frac{\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1',\omega_2')}{(\omega_1'-\omega_1-\omega_2-i0)(\omega_2'-\omega_2-i0)}.\tag{33}$$

On the other hand, the quadratic response $\langle\rho\rangle^{(2)}_{\mathbf{k}\omega}$ can be expressed in terms of the external field and the effective second-order nonlinear electric susceptibility:

$$4\pi\langle\rho\rangle^{(2)}_{\mathbf{k}\omega}=ik\sum_{\substack{\omega_1+\omega_2=\omega\\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}}k_1k_2\hat{\varkappa}^{(2)}(\omega_1,\mathbf{k}_1,\omega_2,\mathbf{k}_2)\Phi_{\mathbf{k}_1\omega_1}\Phi_{\mathbf{k}_2\omega_2}.\tag{34}$$

The effective nonlinear susceptibility $\hat{\varkappa}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)$ is an analytic function of the complex variables $\omega_1$ and $\omega_2$ and satisfies the dispersion relation

$$\hat{\varkappa}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}d\omega_1'\int_{-\infty}^{\infty}d\omega_2'\times\frac{\mathrm{Im}_{\omega_1'}\{\mathrm{Im}_{\omega_2'}\hat{\varkappa}^{(2)}(\omega_1',\mathbf{k}_1;\omega_2',\mathbf{k}_2)\}}{(\omega_1'+\omega_2'-\omega_1-\omega_2-i0)(\omega_2'-\omega_2-i0)},\tag{35}$$

where $\mathrm{Im}_{\omega_1}\{\mathrm{Im}_{\omega_2}\hat{\varkappa}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\}$ denotes the discontinuity of the nonlinear susceptibility across the cuts along the real axes in the complex $\omega_1$- and $\omega_2$-planes. Comparing (34) and (33), we find

$$\mathrm{Im}_{\omega_1}\{\mathrm{Im}_{\omega_2}\hat{\varkappa}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\}=i\frac{\pi}{k_1k_2k}\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1+\omega_2,\omega_2).\tag{36}$$

Thus, the quantity $\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1+\omega_2,\omega_2)$ directly determines the discontinuity of the effective second-order nonlinear susceptibility across the cuts in the complex $\omega_1$- and $\omega_2$-planes.

According to (15), the quantity $\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1,\omega_2)$ is described by the formula

$$(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1,\omega_2)=\int_{-\infty}^{\infty}dt_1e^{i\omega_1t_1}\int_{-\infty}^{\infty}dt_2e^{i\omega_2t_2}\int d\Gamma\,\rho_{\mathbf{k}}\{\rho_{-\mathbf{k}_1}(-t_1),\{\rho_{-\mathbf{k}_2}(-t_1-t_2),D_0\}\}.\tag{37}$$

Noting that the effective nonlinear susceptibility is connected with the usual nonlinear susceptibility by the relation

$$\hat{\varkappa}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)=\frac{\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)}{\varepsilon(\omega_1,\mathbf{k}_1)\varepsilon(\omega_2,\mathbf{k}_2)\varepsilon^*(\omega,\mathbf{k})},\tag{38}$$

from (36) we find

$$\mathrm{Im}_{\omega_1}\{\mathrm{Im}_{\omega_2}\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\}=i\frac{\pi}{k_1k_2k}(\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1+\omega_2,\omega_2))^0,\tag{39}$$

where the quantity $(\psi^{(2)}_{\mathbf{k}_1\mathbf{k}_2}(\omega_1+\omega_2,\omega_2))^0$ corresponds to a system in which Coulomb interaction between the particles is absent. A direct calculation gives

$$\mathrm{Im}_{\omega_1}\{\mathrm{Im}_{\omega_2}\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\}=\mathscr{L}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\langle\rho^3\rangle^{0}_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2},\tag{40}$$

where $\mathscr{L}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)$ is the differential operator

$$\mathscr{L}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)=-i\frac{\pi}{2m^2}\frac{1}{k_1k_2k}\Big\{\mathbf{k}_1\mathbf{k}\frac{\partial}{\partial\omega_1}\Big(\mathbf{k}_1\mathbf{k}_2\frac{\partial}{\partial\omega_1}+k_2^2\frac{\partial}{\partial\omega_2}\Big)+\mathbf{k}_2\mathbf{k}\frac{\partial}{\partial\omega_2}\Big(k_1^2\frac{\partial}{\partial\omega_1}+\mathbf{k}_1\mathbf{k}_2\frac{\partial}{\partial\omega_2}\Big)\Big\}\tag{41}$$

and $\langle\rho^3\rangle^0_{\mathbf{k}_1\omega_1;\,\mathbf{k}_2\omega_2}$ is the cubic correlation function for the charge-density fluctuations in the system of noninteracting particles. The relation (40) expresses the discontinuity of the nonlinear electric susceptibility across the cut in terms of a spectral correlation function for the charge-density fluctuations. Together with the dispersion formula (35), the relation (40) completely determines the second-order nonlinear electric susceptibility for the plasma.

In an analogous way, it is not difficult to show that in the general case the relation

$$\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\ldots\{\operatorname{Im}_{\omega_n}\varkappa^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)\}\ldots\}=\mathscr{L}^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)\langle\rho^{n+1}\rangle^0_{\mathbf{k}_1\omega_1;\,\mathbf{k}_2\omega_2;\ldots;\,\mathbf{k}_n\omega_n}\tag{42}$$

holds, where $\mathscr{L}^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n$ is a differential operator of $n$-th order in the variables $\omega_1,\ \omega_2\ldots,\ \omega_n$, depending on the parameters $\mathbf{k}_1,\ \mathbf{k}_2,\ldots,\ \mathbf{k}_n$:

$$\mathscr{L}^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)=-\frac{i^{(n-1)}}{n!}\frac{4\pi}{(2m)^n}\frac{1}{k_1k_2\ldots k_nk}\times\mathbf{P}\Big\{\mathbf{k}_1(\mathbf{k}_1+\mathbf{k}_2+\ldots+\mathbf{k}_n)\frac{\partial}{\partial\omega_1}\Big(\mathbf{k}_1\mathbf{k}_2\frac{\partial}{\partial\omega_1}+\mathbf{k}_2(\mathbf{k}_2+\ldots+\mathbf{k}_n)\frac{\partial}{\partial\omega_2}\Big)\Big(\mathbf{k}_1\mathbf{k}_3\frac{\partial}{\partial\omega_1}+\mathbf{k}_2\mathbf{k}_3\frac{\partial}{\partial\omega_2}+\mathbf{k}_3(\mathbf{k}_3+\ldots+\mathbf{k}_n)\frac{\partial}{\partial\omega_3}\Big)\Big(\mathbf{k}_1\mathbf{k}_n\frac{\partial}{\partial\omega_1}+\mathbf{k}_2\mathbf{k}_n\frac{\partial}{\partial\omega_2}+\ldots+k_n^2\frac{\partial}{\partial\omega_n}\Big)\Big\},\tag{43}$$

$P$ denotes all possible permutations of $\mathbf{k}_1,\ \mathbf{k}_2,\ldots,\ \mathbf{k}_n$, and $\langle\rho^{n+1}\rangle^0_{\mathbf{k}_1\omega_1;\,\mathbf{k}_2\omega_2;\,\ldots;\,\mathbf{k}_n\omega_n}$ is a spectral correlation function for the charge-density fluctuations in the system of noninteracting particles:

$$\langle\rho^{n+1}\rangle^0_{\mathbf{k}_1\omega_1;\,\mathbf{k}_2\omega_2;\ldots;\mathbf{k}_n\omega_n}=(2\pi)^ne^{(n+1)}\times\int d\mathbf{v}\,\delta(\omega_1-\mathbf{k}_1\mathbf{v})\,\delta(\omega_2-\mathbf{k}_2\mathbf{v})\ldots\delta(\omega_n-\mathbf{k}_n\mathbf{v})f_0(\mathbf{v}).\tag{44}$$

If $n$ is odd, the quantity

$$\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\ldots\{\operatorname{Im}_{\omega_n}\varkappa^{(n)}(\omega_1\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)\}\ldots\}$$

is real; but if $n$ is even, it is imaginary. The dispersion relation expressing the principle of causality for the $n$-th order nonlinear susceptibility can be written in the form

$$\varkappa^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)=\frac{1}{\pi^n}\int_{-\infty}^{\infty}d\omega_1'\int_{-\infty}^{\infty}d\omega_2'\ldots\int_{-\infty}^{\infty}d\omega_n'\times\operatorname{Im}_{\omega_1'}\{\operatorname{Im}_{\omega_2'}\ldots\{\operatorname{Im}_{\omega_n'}\varkappa^{(n)}(\omega_1',\mathbf{k}_1;\omega_2',\mathbf{k}_2;\ldots;\omega_n',\mathbf{k}_n)\}\ldots\}\times[(\omega_1'+\omega_2'+\ldots+\omega_n'-\omega_1-\omega_2-\ldots-\omega_n-i0)\times(\omega_2'+\ldots+\omega_n'-\omega_2-\ldots-\omega_n-i0)\ldots(\omega_n'-\omega_n-i0)]^{-1}.\tag{45}$$

This relation enables us to establish the nonlinear susceptibility $\varkappa^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)$ for all values of the complex frequencies $\omega_1,\ \omega_2,\ldots,\ \omega_n$ from the discontinuities of the susceptibility across the cuts in the corresponding complex planes. According to (42), the discontinuities of the susceptibility across the cuts are determined by spectral correlation functions for the charge-density fluctuations of the noninteracting particles.

The formulas (42) and (45) are general and make it possible to find the nonlinear susceptibilities in a system of particles with Coulomb interaction from the correlation functions for the system without the Coulomb interaction between the particles. The relation (42) can be regarded as a generalization of the fluctuation-dissipation relation for a nonlinear electrodynamic medium. We note that the usual fluctuation-dissipation relation (29) is a particular case of (42) with $n=1$. According to (42) and (45), the electrodynamic (linear and nonlinear) properties of a plasma are completely determined if the sequence of correlation functions of different orders for the charge-density fluctuations, with neglect of the Coulomb interaction between the particles, is given.

## 5. THE SPECTRAL CORRELATION FUNCTIONS

The $n$-th order spectral correlation function for the charge-density fluctuations with the Coulomb interaction between the particles taken into account (in the polarization approximation) can be represented in the form

$$\langle\rho^{n+1}\rangle_{\mathbf{k}_1\omega_1;\,\mathbf{k}_2\omega_2;\,\ldots;\,\mathbf{k}_n\omega_n}=\frac{\langle\rho^{n+1}\rangle^0_{\mathbf{k}_1\omega_1;\,\mathbf{k}_2\omega_2;\,\ldots;\,\mathbf{k}_n\omega_n}}{\varepsilon(\omega_1,\mathbf{k}_1)\,\varepsilon(\omega_2,\mathbf{k}_2)\ldots\varepsilon(\omega_n,\mathbf{k}_n)\,\varepsilon^*(\omega_1+\omega_2+\ldots+\omega_n,\mathbf{k}_1+\mathbf{k}_2+\ldots+\mathbf{k}_n)}.\tag{46}$$

By means of the general formula (42) the spectral correlation functions of different orders for the charge-density fluctuations in a plasma can be expressed in terms of the nonlinear susceptibilities.

As an illustration, we quote the expression for the third-order spectral correlation function:

$$\langle\rho^3\rangle_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2}=\frac{\{\mathscr{L}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\}^{-1}\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\}}{\varepsilon(\omega_1,\mathbf{k}_1)\,\varepsilon(\omega_2,\mathbf{k}_2)\,\varepsilon^*(\omega_1+\omega_2,\mathbf{k}_1+\mathbf{k}_2)}.\tag{47}$$

If the plasma is in equilibrium, the $\mathscr{L}^{(n)}$ are multiplicative operators. In the case $n=2$, we have

$$\mathscr{L}^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)=i\frac{\pi}{2T^2}\frac{(\omega_1^2\mathbf{k}_2\mathbf{k}+\omega_2^2\mathbf{k}_1\mathbf{k})\mathbf{k}_1\mathbf{k}_2-\omega_1\omega_2(k_1^2\mathbf{k}_2\mathbf{k}+k_2^2\mathbf{k}_1\mathbf{k})}{k_1k_2k[k_1^2k_2^2-(\mathbf{k}_1\mathbf{k}_2)^2]}.\tag{48}$$

Then the cubic spectral correlation function for the charge-density fluctuations takes the form

$$\langle\rho^3\rangle_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2}=i\frac{2T^2}{\pi}\frac{k_1k_2k[k_1^2k_2^2-(\mathbf{k}_1\mathbf{k}_2)^2]}{(\omega_1^2\mathbf{k}_2\mathbf{k}+\omega_2^2\mathbf{k}_1\mathbf{k})\mathbf{k}_1\mathbf{k}_2-\omega_1\omega_2(k_1^2\mathbf{k}_2\mathbf{k}+k_2^2\mathbf{k}_1\mathbf{k})}\times\frac{\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\}}{\varepsilon(\omega_1,\mathbf{k}_1)\,\varepsilon(\omega_2,\mathbf{k}_2)\,\varepsilon^*(\omega_1+\omega_2,\mathbf{k}_1+\mathbf{k}_2)}.\tag{49}$$

The spectral correlation functions of higher order in the case of an equilibrium plasma can be written down in an analogous manner.

According to (40) and (48), the second-order nonlinear electric susceptibility for an equilibrium plasma can be represented in the form

$$\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)=\frac{i}{2\pi T^2}\frac{1}{\omega k_1k_2k[k_1^2k_2^2-(\mathbf{k}_1\mathbf{k}_2)^2]}\times\int_{-\infty}^{\infty}d\omega_1'\int_{-\infty}^{\infty}d\omega_2'\frac{\omega_1'+\omega_2'}{\omega_1'+\omega_2'-\omega_1-\omega_2-i0}\Big\{\frac{\omega_1'}{\omega_1'-\omega_1-i0}(\omega_1'\mathbf{k}_1\mathbf{k}_2-\omega_2k_1^2)\mathbf{k}_2\mathbf{k}+\frac{\omega_2'}{\omega_2'-\omega_2-i0}(\omega_2'\mathbf{k}_1\mathbf{k}_2-\omega_1k_2^2)\mathbf{k}_1\mathbf{k}\Big\}\langle\rho^3\rangle^0_{\mathbf{k}_1\omega_1';\mathbf{k}_2\omega_2'}.\tag{50}$$

Using this representation, by direct inspection it is not difficult to see that the following relation is valid:

$$\operatorname{Im}\{\omega\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)-\omega_1\varkappa^{(2)}(\omega,\mathbf{k};-\omega_2,-\mathbf{k}_2)-\omega_2\varkappa^{(2)}(\omega,\mathbf{k};-\omega_1,-\mathbf{k}_1)\} = \frac{\pi}{T^2}\frac{\omega_1\omega_2\omega}{k_1k_2k}\langle\rho^3\rangle^0_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2}. \tag{51}$$

Using this relation, it is not difficult to obtain an explicit expression for the discontinuity (40) of the second-order nonlinear susceptibility across the cut. As a result, the third-order spectral correlation function for the charge-density fluctuations in an equilibrium plasma is described by the formula

$$\langle\rho^3\rangle_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2} = \frac{T^2}{\pi}\frac{k_1k_2k}{\varepsilon(\omega_1,\mathbf{k}_1)\varepsilon(\omega_2,\mathbf{k}_2)\varepsilon^*(\omega,\mathbf{k})} \times\operatorname{Im}\left\{\frac{\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2\mathbf{k}_2)}{\omega_1\omega_2}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_1,-\mathbf{k}_1)}{\omega_1\omega}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_2,-\mathbf{k}_2)}{\omega_2\omega}\right\}. \tag{52}$$

This formula was obtained earlier in Ref. 10. For the spectral correlation function for the fluctuations of the electric field, from (52) we find

$$\langle E^3\rangle_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2} = -i64\pi^2\frac{T^2}{\varepsilon(\omega_1,\mathbf{k}_1)\varepsilon(\omega_2,\mathbf{k}_2)\varepsilon^*(\omega,\mathbf{k})} \times\operatorname{Im}\left\{\frac{\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)}{\omega_1\omega_2}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_1,-\mathbf{k}_1)}{\omega_1\omega}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_2,-\mathbf{k}_2)}{\omega_2\omega}\right\}. \tag{53}$$

We note that in this form the spectral correlation function for the electric-field fluctuations is valid not only for a plasma but also for any other nonlinear medium. A similar formula for the spectral distribution of the field fluctuations in the absence of spatial dispersion was obtained earlier in Ref. 11.

## 6. SUM RULES

Using the explicit form of the spectral correlation functions for the charge-density fluctuations in the absence of Coulomb interaction between the particles, it is not difficult to establish a general integral relation connecting the linear and nonlinear electric susceptibilities of different orders for a plasma. In fact, according to (44) the equality

$$\int_{-\infty}^{\infty} d\omega_n\langle\rho^{n+1}\rangle^0_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2;\ldots;\mathbf{k}_n\omega_n}=2\pi e\langle\rho^n\rangle^0_{\mathbf{k}_1\omega_1;\mathbf{k}_2\omega_2;\ldots;\mathbf{k}_{n-1}\omega_{n-1}}. \tag{54}$$

is valid. Using (42) to express the spectral correlation functions in terms of the discontinuities of the nonlinear susceptibilities across the cuts in the complex planes of the corresponding frequencies, we obtain the following general formula:

$$\begin{aligned}\mathscr{L}^{(n-1)}(\omega_1,\mathbf{k}_1;\ldots;\omega_{n-1},\mathbf{k}_{n-1})&\int_{-\infty}^{\infty} d\omega_n\{\mathscr{L}^{(n)}(\omega_1,\mathbf{k}_1;\ldots;\omega_n\mathbf{k}_n)\}^{-1}\\ &\times\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\ldots\{\operatorname{Im}_{\omega_n}\varkappa^{(n)}(\omega_1,\mathbf{k}_1;\ldots;\omega_n\mathbf{k}_n)\}\ldots\}\\ &=2\pi e\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\ldots\{\operatorname{Im}_{\omega_{n-1}}\varkappa^{(n-1)}(\omega_1,\mathbf{k}_1;\ldots;\omega_{n-1},\mathbf{k}_{n-1})\}\ldots\}\end{aligned} \tag{55}$$

In particular, the dielectric permittivity and the quadratic nonlinear susceptibility for an equilibrium plasma are connected by the relation

$$\int_{-\infty}^{\infty} d\omega_2\operatorname{Im}\left\{\frac{\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)}{\omega_1\omega_2}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_1,-\mathbf{k}_1)}{\omega_1\omega}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_2,-\mathbf{k}_2)}{\omega_2\omega}\right\}=\pi\frac{e}{T}\frac{k_1}{k_2k}\frac{\operatorname{Im}\varepsilon(\omega_1,\mathbf{k}_1)}{\omega_1}. \tag{56}$$

Using (44), we can also obtain a number of other relations between nonlinear susceptibilities of different orders.

The nonlinear electric susceptibilities of a plasma, like the dielectric permittivity, satisfy definite sum rules. Using the formula (44), it is not difficult to integrate the spectral correlation function $\langle\rho^{n+1}\rangle^0_{\mathbf{k}_1\omega_1;\ldots;\mathbf{k}_n\omega_n}$ over all the frequencies. Then, using the relation (42) to express the correlation function in terms of the nonlinear susceptibility $\varkappa^{(n)}(\omega_1,\mathbf{k}_1;\ldots;\omega_n,\mathbf{k}_n)$, we obtain the integral sum rule

$$\int_{-\infty}^{\infty} d\omega_1\int_{-\infty}^{\infty} d\omega_2\ldots\int_{-\infty}^{\infty} d\omega_n\{\mathscr{L}^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)\}^{-1} \times\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\ldots\{\operatorname{Im}_{\omega_n}\varkappa^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)\}\ldots\}=(2\pi)^n e^{(n+1)}n_0. \tag{57}$$

This relation turns out to be of practical use if the plasma is in a state of thermodynamic equilibrium. In this case, the quantity $\{\mathscr{L}^{(n)}(\omega_1,\mathbf{k}_1:\ldots;\omega_n,\mathbf{k}_n)\}^{-1}$ is a multiplicative operator. As an example we quote the sum rule for the quadratic nonlinear susceptibility:

$$\int_{-\infty}^{\infty} d\omega_1\int_{-\infty}^{\infty} d\omega_2\operatorname{Im}\left\{\frac{\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)}{\omega_1\omega_2}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_1,-\mathbf{k}_1)}{\omega_1\omega}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_2,-\mathbf{k}_2)}{\omega_2\omega}\right\}=\frac{4\pi^3}{k_1k_2k}\frac{e^3n_0}{T^2}. \tag{58}$$

For an equilibrium plasma, sum rules of the form

$$\int_{-\infty}^{\infty} d\omega_1\int_{-\infty}^{\infty} d\omega_2\ldots\int_{-\infty}^{\infty} d\omega_n\,\omega_1\omega_2\{\mathscr{L}^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)\}^{-1} \times\operatorname{Im}_{\omega_1}\{\operatorname{Im}_{\omega_2}\ldots\{\operatorname{Im}_{\omega_n}\varkappa^{(n)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\ldots;\omega_n,\mathbf{k}_n)\}\ldots\} =(2\pi)^n e^{(n+1)}n_0\frac{T}{m}\mathbf{k}_1\mathbf{k}_2 \tag{59}$$

also hold. In particular, for the quadratic nonlinear susceptibility we obtain

$$\int_{-\infty}^{\infty} d\omega_1\int_{-\infty}^{\infty} d\omega_2\,\omega_1\omega_2\operatorname{Im}\left\{\frac{\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)}{\omega_1\omega_2}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_1,-\mathbf{k}_1)}{\omega_1\omega}-\frac{\varkappa^{(2)}(\omega,\mathbf{k};-\omega_2,-\mathbf{k}_2)}{\omega_2\omega}\right\}=\frac{4\pi^3}{k_1k_2k}\frac{e^3n_0}{mT}\mathbf{k}_1\mathbf{k}_2. \tag{60}$$

In an analogous way we can sum rules for the cubic nonlinear susceptibility.

[1)]The applicability of the fluctuation-dissipation relation for nonlinear systems was demonstrated in Ref. 6 (cf. the discussion of this question in Ref. 7).


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# Polarization effects in stimulated scattering of electromagnetic waves

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A study is made of the influence of polarization on the stimulated scattering of electromagnetic waves in an isotropic plasma. The nonlinear interaction in this scattering results in a coherent distribution of the polarizations, i.e., the radiation becomes completely polarized. It is shown that the distribution of elliptically polarized waves in the **k** space may be singular, i.e., it may be concentrated in streamlines. The degree of stability of such distributions is governed by the degree of circular polarization. In the case of linear polarization, the distribution is singular in a plane perpendicular to the polarization vector.

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## INTRODUCTION

Electromagnetic waves in an isotropic plasma have—in contrast to, for example, Langmuir waves—an additional degree of freedom, which is their polarization. Allowance for this polarization is important and sometimes fundamental in the nonlinear interaction of electromagnetic waves (see, for example, Berkhoer and Zakharov[1] and Manakov[2]), particularly in the stimulated scattering,[3-5] which is the main nonlinear mechanism when the wave intensity is sufficiently low. The usual approach to the kinetics of the stimulated scattering is based on the polarization averaging,[3,4] which—strictly speaking—is valid only for isotropic distributions of waves in the **k** space.

It is well known that in the Thomson scattering of polarized light there is a correlation between the scattering angle and the scattered-wave polarization. For example, the scattering of a wave at an angle of $\pi/2$ produces completely polarized light. It is therefore clear that the polarization effects are as important in the stimulated scattering of electromagnetic waves.

The present paper is concerned with the influence of these effects on the stimulated scattering kinetics. The kinetics will be described by a polarization density matrix whose diagonal elements represent numbers of waves of specific polarization and the nondiagonal elements are the anomalous averages associated with the polarization degeneracy, which is only possible in an isotropic plasma.

The stimulated scattering of electromagnetic waves results in the polarization distribution of the waves at each point in **k** space to approach a coherent state, i.e., a completely polarized distribution.

This important property largely determines the structure of steady-state spectra and their stability. In the anisotropic excitation case these spectra are the same as the spectra of the Langmuir turbulence of an isothermal plasma,[5] being singular in the **k** space: the wave distribution is concentrated in streamlines. The degree of stability of such streamline distributions is governed by the degree of circular polarization. In the case of linearly polarized waves the spectra are again singular but this time in a plane perpendicular to the polarization vector.

## 1. BASIC EQUATIONS

It is known that the stimulated scattering of electromagnetic waves in an isotropic plasma is, like the scattering of the Langmuir waves in an isothermal plasma, the main nonlinear mechanism if the wave intensity is sufficiently low. This interaction represents the scattering by low-frequency density fluctuations $\delta n$, induced by the high-frequency pressure of the hf waves. Therefore, the stimulated scattering of electromagnetic waves can be described satisfactorily by a simplified scheme based on the averaging over the short time $1/\omega_k$, where $\omega_k=(\omega_p^2+k^2c^2)^{1/2}$ is the natural frequency of the electromagnetic waves.

Following earlier work,[6] we shall introduce quantities $a_{k\lambda}$, which are the amplitudes of electromagnetic waves corresponding to different polarizations $\mathbf{s}_{k\lambda}$ and normalized in such a way that the total energy of the waves $\mathcal{H}_0$ is

$$\mathcal{H}_0=\sum_\lambda\int\omega_k|a_{k\lambda}|^2\,d\mathbf{k}.$$

The behavior of the amplitudes $a_{k\lambda}$ is described by

$$\frac{\partial a_{k\lambda}}{\partial t}+i\omega_k a_{k\lambda}=-i\int A_{kk_1}\langle \mathbf{l}\lambda|\mathbf{l}_1\lambda_1\rangle a_{k_1\lambda_1}\delta n_\varkappa\delta(\varkappa-\mathbf{k}+\mathbf{k}_1)\,d\varkappa\,d\mathbf{k}_1, \qquad (1)$$

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