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Three-boson stimulated scattering

N. E. Firsova and G. A. Skorobogatov

Chemical Research Institute of the Leningrad State University
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A quantum-mechanical justification is obtained for the three-boson stimulated scattering (TBSS) hypothesis advanced by Skorobatov and Dzevitskiĭ [Sov. Phys. Dokl. 18, 668 (1974)] in an analysis of the Boltzmann quantum equation. The resonant properties of TBSS are investigated, i.e., the increased probability of scattering of one of the two colliding bosons with the momentum belonging to the third boson located at a distance comparable with the de Broglie wavelength of the fastest of the three particles (in their mass center). Some singularities of "cumulative" production of hadrons on a nucleus are explained with TBSS taken into account.

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1. INTRODUCTION

Assume that only two- and three-particle exchanges of a certain substance take place in a spatially-homogeneous system of identical particles with the number of particles conserved and let the role of the "substance" be assumed by the momentum. Then the time evolution of the system in the N/V limit[1] is described by the balance equation[2,3]:

$$\begin{aligned}\frac{\partial f(\mathbf{p}_1,t)}{\partial t} = &\int_{-\infty}^{\infty} [\varkappa(\mathbf{p}_1',\mathbf{p}_2';\mathbf{p}_1,\mathbf{p}_2)f(\mathbf{p}_1',t)f(\mathbf{p}_2',t) \\ &-\varkappa(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')f(\mathbf{p}_1,t)f(\mathbf{p}_2,t)]d^3\mathbf{p}_2\,d^3\mathbf{p}_1'\,d^3\mathbf{p}_2' \\ &+\int_{-\infty}^{\infty} [\varkappa(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3';\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)f(\mathbf{p}_1',t)f(\mathbf{p}_2',t)f(\mathbf{p}_3',t) \\ &-\varkappa(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')f(\mathbf{p}_1,t)f(\mathbf{p}_2,t)f(\mathbf{p}_3,t)]d^3\mathbf{p}_2\,d^3\mathbf{p}_3\,d^3\mathbf{p}_1'\,d^3\mathbf{p}_2'\,d^3\mathbf{p}_3',\end{aligned} \tag{1}$$

where $f(\mathbf{p},t)\geqslant 0$ is the coarse-structure distribution function, and $\varkappa(\mathbf{p}_1,\mathbf{p}_2;\ \mathbf{p}_1',\mathbf{p}_2')$, $\varkappa(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\ \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')\geqslant 0$ are the probabilities of the elementary acts $\mathbf{p}_1,\mathbf{p}_2\to\mathbf{p}_1'$, $\mathbf{p}_2'$ and $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\to\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'$.

It was recently proved that the Uhlenbeck-Uehling equation (the Boltzmann quantum equation) for bosons

$$\begin{aligned}\frac{\partial f(\mathbf{p}_1,t)}{\partial t}\times\int &\delta(\mathbf{p}_1'^2+\mathbf{p}_2'^2-\mathbf{p}_1^2-\mathbf{p}_2^2)\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)\Lambda(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1'/|\mathbf{p}_1'|) \\ &\times\{f(\mathbf{p}_1',t)f(\mathbf{p}_2',t)[1+sf(\mathbf{p}_1,t)][1+sf(\mathbf{p}_2,t)] \\ &-f(\mathbf{p}_1,t)f(\mathbf{p}_2,t)[1+sf(\mathbf{p}_1',t)][1+sf(\mathbf{p}_2',t)]\}d^3\mathbf{p}_2\,d^3\mathbf{p}_1'\,d^3\mathbf{p}_2',\end{aligned} \tag{2}$$

where $\Lambda\geqslant 0$, $s=(2\pi\hbar)^3(2S+1)^{-1}$, and S is the particle spin, can be written in the canonical form (1) if we put

$$\begin{aligned}\varkappa(\mathbf{p}_1,\ \mathbf{p}_2;\ \mathbf{p}_1',\ \mathbf{p}_2') = &\delta(\mathbf{p}_1'^2+\mathbf{p}_2'^2-\mathbf{p}_1^2-\mathbf{p}_2^2)\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2) \\ &\times\Lambda(\mathbf{p}_1,\ \mathbf{p}_2;\ \mathbf{p}_1'/|\mathbf{p}_1'|),\end{aligned} \tag{3}$$

$$\begin{aligned}&\varkappa(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3') \\ &=\frac{s}{3}\sum_{\{i,j,k\}}\sum_{\{l,m,n\}}\delta(\mathbf{p}_l'-\mathbf{p}_i)\delta(\mathbf{p}_m'-\mathbf{p}_i)\varkappa(\mathbf{p}_j,\mathbf{p}_k;p_i,\mathbf{p}_n'),\end{aligned} \tag{4}$$

where $\{i,j,k\}$, $\{l,m,n\}$ are cyclic permutations of $\{1,2,3\}$. The effect of increased probability of scattering of one of the two colliding bosons with the momentum of the present "provocator" boson [see (4)] was named in Ref. 2 "three-boson simulated scattering (TBSS)." There was no complete assurance, however, of the existence of TBSS as a true three-particle process, where one could not exclude the possibility that when the Uhlenbeck-Uehling-Boltzmann Eq. (2) was derived from the Liouville-Neumann equation[1,4,5] the collective effects were not implicitly taken into account. In fact, in Refs. 1, 4, and 5 Eq. (2) is derived only in the N/V limit, and the method of the derivation consists of taking into account the successive terms of the correlation functions[4,5] or terms of the perturbation series[1] until characteristic quantum-mechanical expressions appear in the Boltzmann collision integral. The terms in the diagrams of higher order are then discarded without rigorous justification. Thus, it might turn out that the TBSS does not exist as an elementary three-particle act, and occurs only in a medium with a sufficiently large assembly of identical bosons.

Nor was it proved in Ref. 2 that, besides (3) and (4), there exist no other functions $\varkappa(\mathbf{p}_1,\mathbf{p}_2;\ \mathbf{p}_1',\mathbf{p}_2')$, $\varkappa(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\ \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$, that yield (2) when substituted in (1). To resolve these problems we derive below formula (4) from quantum-mechanical considerations. In addition, we deduce from the conservation laws resonant properties of the TBSS, which consist in the fact that the TBSS takes place only when the momentum of a provocator lies on the so-called "resonance shell" an equation for which is given below.

By the same method used in the present article to investigate TBSS, it is possible to obtain known results on simulated emission. The analogy between TBSS and

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stimulated emission are grounds for hoping that, just as lasers were developed on the basis of stimulated emission, some new technical devices will be designed on the basis of TBSS.

It is also shown in the present article that allowance for the TBSS explains some singularities of nuclear cumulative effects observed experimentally by A. M. Baldin and his co-workers (see Ref. 6 and the bibliography in Ref. 7), i.e., of hadron production on a nucleus beyond the kinematic limit. It is possible that a more detailed investigation of the role of TBSS in cumulative effects will be useful when it comes to solving the inverse problem—the study of the properties of nuclear mesons on the basis the experimentally observed momentum distribution of the cumulative hadrons.

2. QUANTUM MECHANICAL DERIVATION OF THE FORMULA FOR THE TBSS CROSS SECTION

We considered the collision of two particles having an equal rest mass μ, spin S, and momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, under the condition that over a distance exceeding the radius of the force interaction there is a third particle, "provocator," with rest mass μ, spin S, and momentum $\mathbf{p}_3$. If we do not introduce the formalism of spinor wave functions, then the wave function of the provocator can be written in the momentum representation in the form

$$\psi_{\mathbf{p}_3}(\mathbf{p}_3')=(2\pi\hbar)^{3/2}(2S+1)^{-1/2}\delta(\mathbf{p}_3'-\mathbf{p}_3)$$

(see, e.g., Ref. 8). The wave function of two particles that experience a force interaction as $t\to\infty$ will be designated $\psi_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_1',\mathbf{p}_2')$, where $\mathbf{p}_1'$, and $\mathbf{p}_2'$ are the final momenta. The wave function of the system of all three particles as $t\to\infty$ then takes the form

$$(2\pi\hbar)^{3/2}(2S+1)^{-1/2}\delta(\mathbf{p}_3'-\mathbf{p}_3)\psi_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_1',\mathbf{p}_2').$$

Assuming now that all three particles are identical and furthermore that the provocator is at a distance on the order of the de Broglie wavelength, we obtain the following wave function of the system of three particles:

$$\Psi^{(\pm)}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')=\left[\frac{(2\pi\hbar)^3}{3(2S+1)}\right]^{1/2}\sum_{\{l,m,n\}}\delta(\mathbf{p}_l'-\mathbf{p}_3)\varphi^{(\pm)}_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_m',\mathbf{p}_n'),\tag{5}$$

where

$$\varphi^{(\pm)}_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_m',\mathbf{p}_n')=[\psi_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_m',\mathbf{p}_n')\pm\psi_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_n',\mathbf{p}_m')]/\sqrt{2}.$$

The plus and minus signs pertain here respectively to the cases of three bosons and three fermions. If we designate $(2\pi\hbar)^3/(2S+1)$ by s, then we have for the square of the amplitude (5)

$$|\Psi^{(\pm)}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')|^2=\frac{s}{3}\sum_{\{l,m,n\}}[\delta^2(\mathbf{p}_l'-\mathbf{p}_3)|\varphi^{(\pm)}_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{p}_m',\mathbf{p}_n')|^2$$
$$\pm\delta(\mathbf{p}_l'-\mathbf{p}_3)\delta(\mathbf{p}_m'-\mathbf{p}_3)(|\varphi^{(\pm)}_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_m',\mathbf{p}_n')|^2+|\varphi^{(\pm)}_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_l',\mathbf{p}_n')|^2)].\tag{6}$$

Consequently the three-particle differential cross section is

$$|\Psi^{(\pm)}|^2\,d\Omega=s\delta^2(0)\,|\varphi^{(\pm)}_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_1',\mathbf{p}_2')|^2\,d\Omega,\tag{7}$$

when $\mathbf{p}_1'\neq\mathbf{p}_3$, $\mathbf{p}_2'\neq\mathbf{p}_3$, $\mathbf{p}_3'=\mathbf{p}_3$ (absence of resonance), and

$$|\Psi^{(+)}|^2\,d\Omega=4s\delta^2(0)\,|\varphi^{(+)}_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{p}_1',\mathbf{p}_2')|^2\,d\Omega,\quad |\Psi^{(-)}|^2\,d\Omega=0,\tag{8}$$

when $\mathbf{p}_1'=\mathbf{p}_3$, $\mathbf{p}_3'=\mathbf{p}_3$ or $\mathbf{p}_2'=\mathbf{p}_3$, $\mathbf{p}_3'=\mathbf{p}_3$ (resonant case). In (7) and (8) we put $d\Omega=d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3d\mathbf{p}_1'd\mathbf{p}_2'd\mathbf{p}_3'$.

Thus, TBSS exists as an elementary act and is not some manifestation of collective effects in a system of a large number of bosons. The equality $|\Psi^{(-)}|^2=0$ for the last case (8) is one of the manifestations of the Pauli principle. The reason why the probability in (8) is increased fourfold over (7) only on a set of zero measure is that we are considering an ideal model and do not take into account the uncertainty of the momenta.

We consider now a situation wherein any pair out of the three colliding identical bosons can experience, with equal probability, a force interaction. Then, by virtue of (6), an event in which three bosons colliding with initial momenta $\mathbf{p}_1,\mathbf{p}_2$ and $\mathbf{p}_3$ move apart with momenta $\mathbf{p}_1'$, $\mathbf{p}_2'$ and $\mathbf{p}_3'$ have a probability $P(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\ \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$ given by

$$P(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')=\sum_{\{i,j,k\}}\frac{1}{3}|\Psi^{(+)}_{\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k}(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')|^2$$
$$=\frac{1}{3}\left[\frac{s}{3}\sum_{\{i,j,k\}}\sum_{\{l,m,n\}}\delta^2(\mathbf{p}_l'-\mathbf{p}_i)\varkappa(\mathbf{p}_j,\mathbf{p}_k;\mathbf{p}_m',\mathbf{p}_n')\right]$$
$$+\frac{2}{3}\left[\frac{s}{3}\sum_{\{i,j,k\}}\sum_{\{l,m,n\}}\delta(\mathbf{p}_l'-\mathbf{p}_i)\delta(\mathbf{p}_m'-\mathbf{p}_i)\varkappa(\mathbf{p}_j,\mathbf{p}_k;\mathbf{p}_i,\mathbf{p}_n')\right],\tag{9}$$

where

$$\varkappa(\mathbf{p}_j,\mathbf{p}_k;\mathbf{p}_m',\mathbf{p}_n')=P_{\mathbf{p}_j,\mathbf{p}_k}(\mathbf{p}_j',\mathbf{p}_k')$$
$$={}^1/_2\{|\varphi^{(+)}_{\mathbf{p}_j,\mathbf{p}_k}(\mathbf{p}_m',\mathbf{p}_n')|^2+|\varphi^{(+)}_{\mathbf{p}_k,\mathbf{p}_j}(\mathbf{p}_m',\mathbf{p}_n')|^2\}.$$

For the provocator there are two possibilities: H_1—the provocator does not change place with two interacting identical particles; H_2—the provocator changes place with one of them. Obviously, $P_{H1}=\frac{1}{3}, P_{H2}=\frac{2}{3}$. The total-probability formula then yields

$$P(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')={}^1/_3P_{H_1}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$$
$$+{}^2/_3P_{H_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3').\tag{10}$$

The probability $P_{H1}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\ \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$ corresponds to two-particle force scattering, and $P_{H2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\ \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$ to three-particle non-force scattering, i.e.,

$$P_{H_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')=\varkappa(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3').$$

Comparing (9) and (10), we obtain (4), q.e.d.

Substitution of the obtained function $\varkappa(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\ \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$ in (1) yields (2); this is equivalent to a deductive derivation of the Uhlenbeck-Uehling-Boltzmann equation from the general balance Eq. (1). We note that a deductive derivation of equations of the Boltzmann type from the Liouville equation by the method of Refs. 1, 4, and 5 also mandate the use of some non-mechanical assumptions equivalent to the introduction of the balance equation (1).[3]

It follows from the results that even if there exist functions $\varkappa$ different from expressions (3) and (4) and yielding (2) when substituted in (1), only the functions (3) and (4) are physically real.

3. RESONANT PROPERTIES OF TBSS

We describe the set of values σ that the momentum $\mathbf{p}_1'$ of a particle can assume after a pair collision, if the colliding particles are identical and their initial momenta are equal to $\mathbf{p}_1$ and $\mathbf{p}_2$, with $\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2$. From symmetry considerations it is obvious that the momentum $\mathbf{p}_2'=\mathbf{p}-\mathbf{p}_1'$ also belongs to σ. We choose for convenience the coordinate system in such a way that the vector $\mathbf{p}$ emerges from the origin and is directed along the z axis. If the projections of the $\mathbf{p}_1'$ are x, y, and z, and if

$$E=E_1+E_2,\quad E_i=\frac{\mu c^2}{[1-(v_i/c)^2]^{1/2}}=\mu c^2\left[1+\left(\frac{p_i}{\mu c}\right)^2\right]^{1/2}, \tag{11}$$

then it follows from the energy conservation law (see Fig. 1) that

$$\begin{aligned}\frac{E}{\mu c^2}&=\left(1+\frac{p_1'^2}{\mu^2c^2}\right)^{1/2}+\left(1+\frac{p_2'^2}{\mu^2c^2}\right)^{1/2}\\&=\left(1+\frac{x^2+y^2+z^2}{\mu^2c^2}\right)^{1/2}+\left(1+\frac{x^2+y^2+(p-z)^2}{\mu^2c^2}\right)^{1/2},\end{aligned} \tag{12}$$

i.e., the vector $\mathbf{p}_1'$ lies on the surface σ described in the relativistic case by Eq. (12). In the nonrelativistic limit ($p_{1,2}'^2\ll\mu^2c^2$) it follows from (12) that

$$\frac{E-2\mu c^2}{\mu c^2}=\frac{x^2+y^2+z^2}{2\mu^2c^2}+\frac{x^2+y^2+(p-z)^2}{2\mu^2c^2},$$

which yields, after transformation to canonical form and after expressing the momenta in units of μc,

$$x^2+y^2+(z-p/2)^2=\mu E_{\text{kin}}-p^2/4,\quad E_{\text{kin}}=E-2\mu c^2. \tag{13}$$

Thus, in the nonrelativistic case the vector $\mathbf{p}_1'$ lies on a sphere of radius $R=\frac{1}{2}(4\mu E_{\text{kin}}-p^2)^{1/2}$ with a center at the point $(0,0,p/2)$. From the cosine theorem we then have (see Fig. 1)

$$2R=[p^2+2(2\mu E_{\text{kin}}-p^2)]^{1/2}=[p^2-2p_1p_2\cos\alpha]^{1/2}=[p^2-2p_1'p_2'\cos\alpha']^{1/2} \tag{14}$$

Thus, if two nonrelativistic particles collide at an angle $\alpha<\pi/2$ (and consequently move apart at an angle $\alpha'<\pi/2$, then $R<p/2$; if $\alpha>\pi/2(\alpha'>\pi/2)$, then $R>p/2$, and finally if $\alpha=\alpha'=\pi/2$ then $R=p/2$ (see Fig. 2).

The momentum pairs $\mathbf{p}_1', \mathbf{p}_2'$ with which the nonrelativistic particles move apart after collision lie on opposite ends of a diameter of the sphere σ. Consequently, the probability $P(\mathbf{p}_1')=|\varphi|^2$ of scattering of a particle with momentum $\mathbf{p}_1'$ differs from zero only on the sphere σ and is symmetrical about the center of this sphere.

It follows from the foregoing that one of the particles experiencing force scattering can move away with momentum $\mathbf{p}_3$ equal to that of the provocator, provided that $\mathbf{p}_3\in\sigma$. In other words, the TBSS is resonant and can occur only if the provocator momentum lies on the "allowed" surface σ. The resonant character of the TBSS was already established earlier[9] from dimensionality considerations. However, the resonance criterion presented in the present paper reveals new singularities of TBSS, and makes it also possible to generalize the results of Ref. 9 to include the relativistic region.

It was observed in Ref. 9 that when two identical nonrelativistic bosons with energies E_1 and E_2 collide at an angle $\alpha=90°$, a "cumulative effect" is realized (i.e., an increase in the probability that one of the colliding bosons will be stopped and that the entire energy E_1+E_2 will be carried away by the second boson), provided that the collision takes place in the presence of a third resting boson—the provocator. The need for satisfying the condition $\alpha=90°$ in order to realize the cumulative effect follows also from (14). We shall generalize this result to the relativistic case, i.e., we shall find the angle α_{cum} at which two relativistic bosons must collide (in the rest system of the third boson), separated from the collision region by a distance not larger than the de Broglie wavelength of the fastest of the particles), in order that the cumulative effect take place.

It follows from (12) that one of the colliding relativistic particles can turn out to be at rest after the collision, (i.e., $\mathbf{p}_1'=0$), only under the condition

$$\frac{E}{\mu c^2}=1+\left(1+\frac{p^2}{\mu^2c^2}\right)^{1/2},\quad \text{i.e. } \frac{p^2}{\mu^2c^2}=\left(\frac{E_1+E_2}{\mu c^2}\right)^2-2\frac{E_1+E_2}{\mu c^2}. \tag{15}$$

According to the cosine theorem $p^2=p_1^2+p_2^2-2p_1p_2\cos(\pi-\alpha)$, and therefore from (15) and (11) with allowance for the relation

$$\mathbf{p}_i=\mu\mathbf{v}_i/(1-v_i^2/c^2)^{1/2}$$

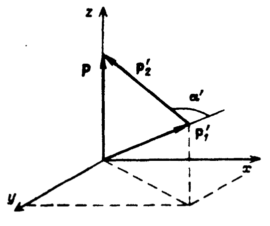

FIG. 1. Combined momentum $\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2$ and final momenta $\mathbf{p}_1'$ and $\mathbf{p}_2'$ of two scattered identical particles.

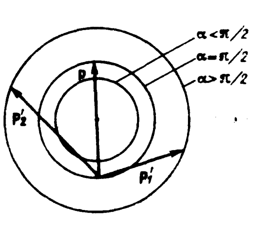

FIG. 2. Three possible variants of the location of the allowed surface (sphere) relative to the momentum $\mathbf{p}$, depending on the collision angle α between two identical nonrelativistic particles.

we obtain

$$\cos\alpha_{\rm cum}=\frac{v_1v_2}{c^2}\left[1+\left(1-\frac{v_1^2}{c^2}\right)^{1/2}\right]^{-1}\left[1+\left(1-\frac{v_2^2}{c^2}\right)^{1/2}\right]^{-1}. \qquad (16)$$

In the nonrelativistic limit ($v_1 \ll c, v_2 \ll c$) we get from (16)

$$\cos\alpha_{\rm cum}\approx v_1v_2/4c^2, \quad \text{i.e.} \quad \alpha_{\rm cum}\approx\pi/2-v_1v_2/4c^2. \qquad (17)$$

On the other hand, in the extreme relativistic limit ($v_1 \approx v_2 \approx c$) we have

$$\cos\alpha_{\rm cum}\approx 1-\left[2\left(1-\frac{v_1}{c}\right)\right]^{1/2}-\left[2\left(1-\frac{v_2}{c}\right)\right]^{1/2},$$
$$\frac{1}{2}\alpha_{\rm cum}^2\approx\left[2\left(1-\frac{v_1}{c}\right)\right]^{1/2}+\left[2\left(1-\frac{v_2}{c}\right)\right]^{1/2}. \qquad (18)$$

4. POSSIBILITIES OF EXPERIMENTALLY OBSERVING THE CUMULATIVE EFFECT

For experimental observation of the TBSS, for example on the molecules H_2, D_2, or He, it is necessary to have installations with three intersecting beams. Formula (16) gives the angle at which two beams must intersect, with v_1 and v_2 the velocities of the particles of these beams in the reference frame in which the particles of the third beam are at rest. The TBSS must be realized without fail in large assemblies of bosons with random values of the momenta; such experiments, however, would reduce to a confirmation of the Bose-Einstein statistics, from which formula (4) was in fact derived in Ref. 2. A large assembly of coherent provocator bosons is another matter. Consideration of the forced collision of two identical bosons in the presence of N coherent provocator bosons (located not far from the collision region than the de Broglie wavelength of the faster of the two colliding bosons) yields in place of (8) the following ($N+2$)-partial differential cross section

$$|\Psi^{(+)}|^2d\Omega=2(N+1)s^N\delta^{2N}(0)\,|\varphi^{(+)}(\mathbf{p}_1',\ \mathbf{p}_2')|^2d\Omega. \qquad (19)$$

The presence of the factor ($N+1$) in (19) means that, just as lasers were developed on the basis of stimulated emission of atoms, a possibility exists of using TBSS for the development of some new experimental or technical devices. Namely, if two boson beams of the same sort, with equal intensities and with equal energies, are aimed on an assembly of coherent bosons at an angle $\alpha_{\rm cum}$ (16), then the TBSS will cause, with high probability, half of the bosons from the beams to replenish the assembly of the coherent bosons, while the other half will leave the region of intersection with double the energy (in the reference frame in which the assembly of the coherent bosons is at rest). Such a cumulative effect will occur, naturally, only if the de Broglie wavelength of the bosons in each of the initial intersecting beams greatly exceeds the distance between the bosons of the coherent assemblies.

The TBSS for pions has apparently already been observed in nuclear reactions accompanied by multiple production of pions and other particles. In a number of experiments (see Ref. 6 and the bibliography in Ref. 7), Baldin and co-workers have observed and investigated cumulative formation of mesons and hadrons on nuclei, i.e., particle production on a nucleus beyond the kinematic limit which is attainable in collisions with an individual nucleon. The experimental data led to an empirical formula for the invariant cross section

$$E\frac{d^3\sigma}{d\mathbf{p}^3}=\text{const}\cdot A^d\exp\left(-\frac{T}{T_0}\right), \qquad (20)$$

where A is the atomic mass of the fragmented nucleus, T is the kinetic energy of the produced particles, d takes on values from 1 to 2, depending on the type of the bombarding particle (p, d, or t), and $T_0=55\ldots60$ MeV for light nuclei ($A<50$) and $T_0=60\ldots65$ MeV for heavy ones ($A\geqslant50$). These processes are characterized by four principal singularities: a) The cross section does not depend on the energy of the incident hadron, starting with 5 GeV/nucleon and higher ("scaling"). b) The energy spectrum of the produced particles has the exponential form (20), and T_0 increases with increasing rest mass of the particle. c) The cross section has a power-law dependence on the atomic number of the nucleus ($d\geqslant1$), and d increases with increasing cumulativity number $\eta\equiv \mathbf{p}_{\rm cum}/(\mathbf{p}_{\rm in}/A)$, where $\mathbf{p}_{\rm cum}$ is the momentum of the cumulative particle and $\mathbf{p}_{\rm in}$ is the momentum of the initial relativisitc particle. d) The cross section is approximately isotropic at shell momenta and decreases with increasing angle at large momenta. None of the models proposed by different authors[6,7,10-13] is capable of accounting simultaneously for all four singularities.

Thus, the statistical bootstrap model[12,13] offers a simple explanation for the scaling, for the exponential spectrum of the produced particle, for the growth of T_0 with increasing rest mass of the particle, and for the value $d=1$ in the power-law dependence of the cross section on A. On the other hand if it is recognized that the pions produced in fireball decay should experience TBSS, then the dependence of the cross section on the angle [see (d)], the values of the parameter d exceeding unity, and the increase of d with increasing cumulativity number [see (c)] are also explained. In addition, allowance for the TBSS explains why the experimentally observed number of cumulative hadrons is 10...20 times larger than the theoretical value calculated in Ref. 13.

We shall assume that the nuclear matter contains an appreciable number of pions whose momenta are small in absolute value compared with the momenta of the fireball pions. Then, when a pair of fireball pions collide at an angle $\alpha_{\rm cum}$ (accurate to the Heisenberg uncertainty), one of them has a higher probability of going over into the assembly of the nuclear provocator pions, while the other (the cumulative one) is emitted from the nucleus at double the energy. The table shows, by way of example, the values of $\alpha_{\rm cum}$ (in the rest system of the provocator pion) for fireball-pair pions with kinetic energies T_π equal to 300, 3,000, and 10,000 MeV. We note that formula (16) for $\alpha_{\rm cum}$ was derived from relation (11), which is valid in vacuum. The vacuum rest mass of the pion is $\mu c^2=135$ MeV, and the effective rest mass in nuclear matter can be obtained in first-order approximation from the formula (see, e.g., Ref. 14)

$$\mu^* c^2=\mu c^2-V. \qquad (21)$$

If the depth of the potential well is $V\ll\mu c^2$ (i.e., we can

TABLE I. Calculation of the TBSS parameters for relativistic pions as a function of their kinetic energy (T_π) in the rest system of the provocator pions.

T_π, MeV	$\mu^* c^2$, MeV	v/c	$\lambda_\perp$, 10^{-13} cm	$\lambda_\parallel$, 10^{-13} cm	Number of nucleons in the volume $4\pi\lambda_\perp^2\lambda_\parallel/3$	α_{cum}, deg	
						T_π, T_π	T_π, 3000 MeV
300	85	0.9758	5.5	3.8	32	50.3	37.3
300	14	0.9990	13.5	4.0	470	23.8	17.7
3000	85	0.9996	1.7	0.4	1	18.9	–
3000	14	0.99999	4.3	0.4	15	7.8	7.8
10000	135	0.99991	0.95	0.12	0.16	13.1	10.6

neglect the polarization of vacuum). It follows from the optical model of the nucleus[15] that $V \approx 50 \pm 5$ MeV, so that $\mu^* c^2 \approx 85$ MeV at T_π equal to 300 and 3,000 MeV. If $T_\pi = 10$ GeV, we get $V = 0$, i.e., $\mu^* c^2 = \mu c^2 = 135$ MeV.

In the energy region of the Δ resonances, a strong interaction with the nucleon medium leads to a much greater decrease of the rest mass.[16,17] When account is taken of the natural width, the Δ resonances overlap the entire energy region from 300 ± 125 to 3230 ± 440 MeV.[18] Since exact calculations of the effective rest mass of the pions near the Δ resonances were not carried out by anyone, we have listed, in the table, at the critical energies $T_\pi = 300$ and $T_\pi = 3,000$ MeV, the angles α_{cum} for two values: $\mu^* c^2 = 85$ MeV, and $\mu^* c^2 \approx 0.1\ \mu c^2 \approx 14$ MeV. For $T_\pi =$ GeV we used the vacuum rest mass of the pion, inasmuch as at high energies[19] the length of formation of the fireball exceeds the diameter of the nucleus, so that the fireball pions move already in vacuum.

It was observed in experiment that the pions of the decaying fireball travel predominantly forward and backward.[20] This anisotropy, with the TBSS taken into account, explains the aforementioned property, [see (d)] of cumulative pions, inasmuch as with increasing energy of the colliding pions we have $\alpha_{cum} \to 0$ (see the table).

It is seen from (19) that the TBSS cross sections proportional to the number of provocator pions and, consequently, to the number of nucleons separated by a distance equal to the de Broglie wavelength from the fireball pion pair colliding at the angle α_{cum}. Since the average number of nucleons encountered "in transmission" in the nucleus is proportional to $A^{1/2}$, it follows that $N \propto A^{1/3}$ for pions with a transverse de Broglie wavelength $\lambda_\perp$ smaller than the radius of the nucleon, i.e., in the region $T_\pi > 3$ GeV. Consequently, in single TBSS we have $d = 4/3$ in the energy region $T_\pi > 3$ GeV. In n-fold TBSS for fast pions ($T_\pi > 3$ GeV) we have $d = 1 + n/3$, which agrees with the increase of D when the cumulativity number is increased.

Although the TBSS mechanism of production of cumulative particles holds only for bosons, the appearance of cumulative fermions can be readily attributed to energy transfer in collisions. A more detailed allowance for the TBSS can yield a more detailed picture of the variation of the momentum spectrum of the particles of a fireball that decays in a nucleus. Conversely, by investigating in experiment the momentum spectrum of cumulative hadrons it is possible in principle to extract information on the momentum distribution of the nuclear pions.

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