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Quantum motion of atoms in the resonant field of a standing light wave

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A quantum analysis is made of the motion of atoms in the resonant field of a standing light wave. The wave functions and the eigenstate spectrum of an atom in this field are found allowing only for stimulated atomic transitions in the field. The spectrum represents two infinite systems of allowed energy bands corresponding to two generally different stationary states of an atom in the field. The edges of all the allowed energy bands are found as a function of the field intensity and detuning of the field frequency from the resonant transition frequency. The wave functions are special functions of a new type and they are identical, for zero detuning, with the eigenvalues of the Mathieu equation. For some values of the field and detuning the spectra of an atom in a standing light wave are calculated numerically. A general analysis of the spectra of quantum motion and of the wave functions of stationary states is used in considering the influence of resonant spontaneous decay on the motion of an atom in the field. In the case of positive detuning such resonant spontaneous decay increases the energy of motion of an atom, whereas in the case of negative detuning such decay reduces the energy. In the latter case a steady-state distribution of field-cooled atoms between several of the first energy bands is established. It is shown that under the influence of a test field such cold atoms may undergo resonant transitions within and between allowed energy bands.

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1. INTRODUCTION. FORMULATION OF THE PROBLEM

The interaction of a free atom with a resonant light field is known to be always accompanied by a change in the internal state of the atom and also by a change in the motion of an atom as a whole. Thus, the emission (absorption) of one photon by an atom alters the atomic momentum by an amount equal to the momentum of a resonant photon and the energy of translational motion is reduced by the recoil energy. It is also known that such recoil usually has no significant influence on the spatial motion of atoms interacting with a resonant light field. Firstly, the absorption (emission) of one resonant photon alters the momentum of an atom by an amount $\Delta P = \hbar k$ (k is the wave vector of the field) which is between four and five orders of magnitude less than the average thermal momentum of an atom $\overline{P} = (2M\varkappa T)^{1/2}$ ($\varkappa$ is the Boltzmann constant). Secondly, collisions and spontaneous decay of atoms from the states involved in a resonant transition usually result in a very rapid loss of resonance between such atoms and the field so that the atoms can reemit only a small proportion of the resonant photons.

Nevertheless, in some cases the number of reemitted resonant photons may be so high that the total change in the energy and momentum of an atom due to its interaction with the field is comparable with the average thermal values of these quantities. For example, this

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situation occurs in the case of atoms in a low-pressure gas when the lower level of a resonant transition is the ground (or metastable) state and the upper level decays with a very high probability to the lower level.

A prolonged interaction of atoms with a resonant light field, accompanied by the reemision of a large number of resonant photons, is responsible—from the classical point of view—for the forces of resonant radiation pressure exerted on moving atoms. For example, in a resonant field of a traveling light wave an atom experiences the force of spontaneous radiation pressure.[1] In a resonant field of a standing light wave an atom experiences two forces: the force of a spontaneous radiation pressure and a force of an induced radiation pressure.[2,3]

It has been suggested earlier[2,3] to use the resonant pressure of a standing light wave to cool an atomic low-pressure gas and to confine (trap) cold atoms in the zone of action of the light field.1) Such a use of the radiation pressure may, in our opinion, be of great importance in the optical spectroscopy of atoms since in many cases it can radically increase the sensitivity and resolving power of spectroscopic studies. A classical analysis[2,3] shows that tuning of the frequency of a standing light wave may cool a low-pressure atomic gas right down to temperatures of $T \sim 10^{-4} - 10^{-3}$ °K and, for a constant (negative) detuning, such cold atoms may oscillate near field antinodes. The existence of finite oscillations of cold atoms near minima of the periodic potential of a standing light wave suggests that cold atoms may be trapped for a long time in the regions of size of the same order as the wavelength of the wave in question.

However, there are two points which demonstrate the unsatisfactory nature of the classical analysis of the characteristics of the motion of cold atoms in a standing light wave. Firstly, cold atoms may not only oscillate within the periodic system of the potential field wells but also tunnel from one potential well to another. Such tunneling of atoms is a purely quantum effect and cannot be understood by classical analysis. Secondly, the cooling method proposed earlier[2,3] makes it possible, in principle, to produce a large number of atoms whose energy of motion is less than, or comparable with, the recoil energy $R = \hbar^2 k^2 / 2M$.2) In the case of ultracold atoms the de Broglie wavelength is greater than, or of the order of, the wavelength of the resonant light field and this also makes it necessary to carry out a quantum analysis of the motion of atoms in a standing light wave.

We shall give a detailed quantum-mechanical treatment of the motion of atoms in the resonant field of a standing light wave. The solution of this problem is of considerable interest, particularly as a rigorous justification of the proposed[2,3] method for cooling of atoms in a resonant light field. Moreover, the problem is of more general physical significance because it represents the first case in which the motion of free atoms in a light field is described by the propagation of "atom + field" quasiparticles. The quantum motion of atoms in a traveling light wave has been investigated in detail earlier[8-10] and some features of the quantum motion in a standing wave have been pointed out elsewhere.[11,7]

We shall consider primarily the motion of atoms due to recoil in stimulated transitions and then the motion because of recoil in the case of simultaneous stimulated and spontaneous transitions. This formulation of the problem is dictated by the fact that stimulated transitions are themselves capable of setting atoms in motion (because of the recoil effect), whereas spontaneous transitions can have the same effect only in conjunction with stimulated transitions. Moreover, in some cases it is of interest to consider the motion of atoms only under the action of recoil resulting from stimulated transitions. This is true, for example, in laser optical fields saturating a resonant transition, when the probability of stimulated transitions is considerably greater than the probability of spontaneous decay of the levels involved in a resonant transtion.

We shall conclude by considering the absorption spectra of atoms in a standing light wave and compare the results with an earlier classical analysis of the motion of atoms.

2. GENERAL EQUATIONS

We shall allow only for stimulated transitions of an atom in the field of a standing light wave and we shall assume that the resonant light field

$$\mathscr{E}(z,t) = 2\mathscr{E}_0 \cos \omega t \cos kz \tag{1}$$

causes atomic transitions between two infinitesimally narrow levels: the lower g and the upper e. The Schrödinger equation for the wave function of an atom in the field is

$$i\hbar \frac{\partial \Psi(\mathbf{r},z,t)}{\partial t} = \left[H_0(\mathbf{r}) - \frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(\mathbf{r},z,t) \right] \Psi(\mathbf{r},z,t), \tag{2}$$

where the wave function $\Psi(\mathbf{r}, z, t)$ depends on the internal coordinates of the atom $\mathbf{r}$ and on the coordinate of the center of mass of the atoms z; $H_0(\mathbf{r})$ is the Hamiltonian governing the eigenstates of a free atom with energies ε_g and ε_e; $V(\mathbf{r}, z, t) = -d(\mathbf{r}, t)\mathscr{E}(z, t)$ is the Hamiltonian of the interaction between the dipole moment of an atom $d(\mathbf{r}, t)$ and the field. Applying the standard expansion of the complete wave function of an atom in terms of the eigenfunctions $\psi_g(\mathbf{r})$, $\psi_e(\mathbf{r})$ of the states of a free atom,

$$\begin{aligned} \Psi(\mathbf{r},z,t) = {} & \Psi_g(z,t)\psi_g(\mathbf{r})\exp\left(-\frac{i}{\hbar}\varepsilon_g t\right) \\ & + \Psi_e(z,t)\psi_e(\mathbf{r})\exp\left(-\frac{i}{\hbar}\varepsilon_e t\right), \end{aligned} \tag{3}$$

we obtain—in the resonant approximation—a system of levels for the functions $\Psi_g(z, t)$, $\Psi_e(z, t)$, describing the motion of the center of mass of an atom in the field of a standing light wave:

$$\begin{aligned} i\frac{\partial \Psi_e(z,t)}{\partial t} &= -\frac{\hbar}{2M}\frac{d^2\Psi_e(z,t)}{dz^2} - 4V_0 e^{i\omega_0 t}\cos\omega t \cos kz\, \Psi_g(z,t), \\ i\frac{\partial \Psi_g(z,t)}{\partial t} &= -\frac{\hbar}{2M}\frac{d^2\Psi_g(z,t)}{dz^2} - 4V_0 e^{-i\omega_0 t}\cos\omega t \cos kz\, \Psi_e(z,t), \end{aligned} \tag{4}$$

where $\omega_0=(\varepsilon_e-\varepsilon_g)/\hbar$ is the frequency of the resonant atomic transition, $V_0=d\mathcal{E}_0/2\hbar$, and $d=d_{eg}=d_{ge}$ is the matrix element of the projection of the dipole moment of an atom onto the direction of the field.

We can easily see that the system (4) has the stationary solutions

$$\begin{aligned}\Psi_e(z,t)&=u_e(z)\exp\left(-\frac{i}{\hbar}E_et\right),\\ \Psi_g(z,t)&=u_g(z)\exp\left(-\frac{i}{\hbar}E_gt\right),\end{aligned}\tag{5}$$

where E_e and E_g are related by

$$E_e=E_g+\hbar\Omega\tag{6}$$

($\Omega=\omega-\omega_0$ is the detuning of the frequency of the field relative to the frequency of the resonant transition) and should be determined together with the corresponding spatial wave functions $u_i(z)(i=e$ or $g)$ from the system of equations for the eigenvalues

$$\begin{aligned}&\frac{d^2u_e}{dz^2}+\frac{2M(E_g+\hbar\Omega)}{\hbar^2}u_e+\frac{4MV_0}{\hbar}\cos kz\,u_g=0,\\ &\frac{d^2u_g}{dz^2}+\frac{2ME_g}{\hbar^2}u_g+\frac{4MV_0}{\hbar}\cos kz\,u_e=0.\end{aligned}\tag{7}$$

This system of two second-order differential equations with periodic coefficients obeys the Floquet–Lyapunov theorem for systems of linear differential equations with periodic coefficients[12] [a simple substitution $du_e/dz=v_e$ and $du_g/dz=v_g$ reduces the system (7) to one such system of four equations] and for any real value of E_g it has four fundamental solutions of the type

$$u_e^{\Lambda}(z)=e^{\Lambda z}w_e^{\Lambda}(z),\quad u_g^{\Lambda}(z)=e^{\Lambda z}w_g^{\Lambda}(z),\tag{8}$$

where $\Lambda=\Lambda_j(F_g), j=1,2,3,4$ are the four generally different characteristic exponents of the system (7), each of which is known only to within the term ik, and the functions $w_i^{\Lambda}(z)$ are periodic with a period $2\pi/k=\lambda$ (where λ is the wavelength of the light field):

$$w_i^{\Lambda}(z+2\pi/k)=w_i^{\Lambda}(z).\tag{9}$$

According to Eqs. (7) and (8), these functions satisfy

$$\begin{aligned}&\frac{d^2w_e^{\Lambda}}{dz^2}+2\Lambda\frac{dw_e^{\Lambda}}{dz}+\left[\Lambda^2+\frac{2M(E_g+\hbar\Omega)}{\hbar^2}\right]w_e^{\Lambda}+\frac{4MV_0}{\hbar}\cos kz\,w_g^{\Lambda}=0,\\ &\frac{d^2w_g^{\Lambda}}{dz^2}+2\Lambda\frac{dw_g^{\Lambda}}{dz}+\left[\Lambda^2+\frac{2ME_g}{\hbar^2}\right]w_g^{\Lambda}+\frac{4MV_0}{\hbar}\cos kz\,w_e^{\Lambda}=0.\end{aligned}\tag{10}$$

We shall now consider the general properties of the fundamental solutions (8).

First of all, it follows from Eq. (8) that any fundamental solution corresponding to the characteristic exponent $\Lambda_j(E_g)$ is limited on the real axis z only for purely imaginary values of $\Lambda_j(E_g)$: $\Lambda_j(E_g)=iK_j(E_g)$, where $K_j(E_g)$ is real, and diverges for any other values of $\Lambda_j(E_g)$.

We shall call the values of E_g (and also of E_e) for which the system (7) has bounded solutions, the eigenvalues of the system (7). According to the general theorems of the stability of solutions of systems of linear differential equations with periodic coefficients,[12] these eigenvalues E_g and E_e form, for fixed V_0 and Ω, a denumerable set of coupled bands and at the edges of the bands $E_g=E_g^m$ and $E_e=E_e^m$ the system (7) has periodic solutions.

Moreover, it follows from Eq. (10) that each purely imagninary characteristic exponent $\Lambda_j=iK_j$ corresponds, for $\Omega\neq0$, to two eigenvalues E_g' and E_g'', but for $\Omega=0$ there is only one eigenvalue E_g. In fact, if the eigenvalue $E_g'(V_0,\Omega)$ corresponds to a characteristic exponent $\Lambda_j=iK_j$, where K_j is real, it then follows from Eq. (10) that such a characteristic exponent corresponds also to the eigenvalue

$$E_g''(V_0,\Omega)=E_g'(V_0,-\Omega)-\hbar\Omega,\tag{11}$$

and we can see that $E_g''=E_g'$ only for $\Omega=0$.

Finally, because the system (7) together with the solution (8), where $\Lambda=\Lambda_j$ is any characteristic exponent, always has a solution which differs in respect of the sign of the characteristic exponent,

$$u_e^{-\Lambda}(z)=e^{-\Lambda z}w_e^{-\Lambda}(-z),\quad u_g^{-\Lambda}(z)=e^{-\Lambda z}w_g^{-\Lambda}(-z),\tag{12}$$

all four solutions of the system (7) exist in pairs. For any real value of E_g one pair of solutions corresponds to the characteristic exponents $\pm\Lambda_1(E_g)$ and the other to the indices $\pm\Lambda_2(E_g)$, where Λ_1 and Λ_2 are generally unequal.

3. EIGENSTATES OF AN "ATOM + FIELD" SYSTEM

The above analysis of the fundamental solutions of the system (7) allows us to draw the following conclusions on the eigenstates and eigenfunctions of an atom in a standing light wave.

If we allow only for the stimulated transitions, the eigenstates of an atom in a standing light wave are the states with definite quasivector K. For a given quasivector K the eigenstate of the spatial motion of an atom is described by two-component Bloch functions

$$u_e^K(z)=e^{iKz}w_e^K(z),\quad u_g^K(z)=e^{iKz}w_g^K(z),\tag{13}$$

and the eigenstate of the internal and spatial motions is described by the functions

$$\begin{aligned}\Psi_K(\mathbf{r},z,t)=\Big\{&\psi_g(\mathbf{r})w_g^K(z)\exp\left[-\frac{i}{\hbar}(\varepsilon_g+E_g)t\right]\\ &+\psi_e(\mathbf{r})w_e^K(z)\exp\left[-\frac{i}{\hbar}(\varepsilon_g+E_g+\hbar\omega)t\right]\Big\}e^{iKz},\end{aligned}\tag{14}$$

where in both relationships the functions $w_i^K(z+2\pi/k)=w_i^K(z)$ satisfy the equations

$$\begin{aligned}&\frac{d^2w_e^K}{dz^2}+2iK\frac{dw_e^K}{dz}+\left[\frac{2M(E_g+\hbar\Omega)}{\hbar^2}-K^2\right]w_e^K+\frac{4MV_0}{\hbar}\cos kz\,w_g^K=0,\\ &\frac{d^2w_g^K}{dz^2}+2iK\frac{dw_g^K}{dz}+\left[\frac{2ME_g}{\hbar^2}-K^2\right]w_g^K+\frac{4MV_0}{\hbar}\cos kz\,w_e^K=0.\end{aligned}\tag{15}$$

If $\Omega\neq0$, the eigenfunction of the state with a given K is a superposition of two eigenfunctions (14), in one of which we have $E_g=E_g'$ and in the other $E_g=E_g''$, where $E_g'=E_g'(K)$ and $E_g''=E_g''(K)$ are two different eigenvalues corresponding to a given quasivector K. For $\Omega=0$, each state with a given K has one eigenfunction (14) with one eigenvalue E_g.

It is clear from Eq. (14) that for an atom with a quasivector K the sums $\varepsilon_g + E_g(K)$ and $\varepsilon_g + E_g(K) + \hbar\omega = \varepsilon_e + E_e(K)$ give the total energies of an atom in the states g and e. The corresponding eigenfunctions $E_g(K)$ and $E_e(K)$ clearly give the energies of the spatial motion of the same atom in the states g and e.

Thus, any state of an atom in a standing light wave is superpositional. An atom with a quasivector K can be, for $\Omega \neq 0$, in states with the total energies $\epsilon_g + E_g'(K)$ and $\varepsilon_e + E_e'(K)$ or in the states whose total energies are $\varepsilon_g + E_g''(K)$ and $\varepsilon_e + E_e''(K)$, whereas for $\Omega = 0$ it can be in states with total energies $\varepsilon_g + E_g(K)$ and $\varepsilon_e + E_e(K)$.

The dependences of the energies on the quasivector K, i.e., $E_i = E_i(K)$ (where $i = e$ or g), cannot be given in their general form. Nevertheless, the above properties of the solutions of the system (7) allow us to draw the following conclusions.

When the quasivector K is increased from 0 to $\pm\infty$, the energies $\varepsilon_i + E_i$ have values inside a denumerable number of allowed energy bands, whose set governs the spectrum of the quantum motion of an atom in a standing light wave. For $\Omega \neq 0$, this spectrum consists of two double systems ($\varepsilon_g + E_g'$ and $\varepsilon_e + E_e'$, and also $\varepsilon_g + E_g''$ and $\varepsilon_e + E_e''$), whereas for $\Omega = 0$, it consists of one double system ($\varepsilon_g + E_g$ and $\varepsilon_e + E_e$) of allowed energy bands (Fig. 1). Each double system consists of similar allowed energy bands, shifted relative to one another by an amount equal to the field quantum:

$$(\varepsilon_e + E_e) - (\varepsilon_g + E_g) = \hbar\omega_0 + \hbar\Omega = \hbar\omega.$$

So far, in analyzing the stationary solutions of Eqs. (2) and (4) we have considered the motion of atoms in a standing light wave. However, the above results show that the motion of an atom in such a wave is essentially the motion of excitation waves or "atom + field" quasiparticles. Each such quasiparticle is characterized by a specific quasimomentum $\hbar K$ and, in general, by two dispersion laws: $E_g' = E_g'(K)$ and $E_e' = E_e'(K)$.[3] The quasimomentum $\hbar K$ is always defined only to within the field momentum $\hbar k$. Therefore, the motion of "atom + field" quasiparticles can always be regarded only within the limits of the first Brillouin zone:

$$-k/2 \leqslant K \leqslant k/2. \tag{16}$$

We shall not use the quasiparticle formalism, retaining it for those problems for which the quasiparticle description is most suitable. Moreover, we shall not use the reduced representation because, from the formal point of view, it will be more convenient to use the extended representation of the Brillouin zone in which the quasivector runs to the values from $-\infty$ to $+\infty$.

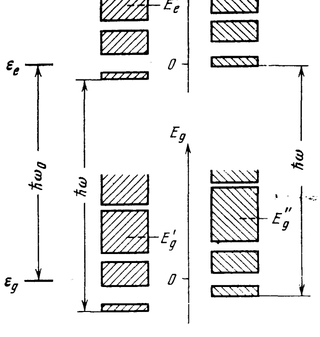

FIG. 1. Bound states of an atom in the resonant field of a standing light wave.

We shall conclude by discussing the normalization of the eigenfunctions of an atom in a standing light wave. In view of the periodicity of the spatial probability density of an atom with a quasivector K,

$$\rho_K(z) = \int |\Psi_K(\mathbf{r}, z, t)|^2 d^3r = |w_g{}^K(z)|^2 + |w_e{}^K(z)|^2 \tag{17}$$

the natural condition for the normalization of the functions (13) and (14) is the normalization to one atom whose quasivector is K and whose energies are $E_g(K)$ and $E_e(K)$ and $E_e(K)$ in a "volume" $\Delta z = \lambda$:

$$\int_0^\lambda \{|w_g{}^K(z)|^2 + |w_e{}^K(z)|^2\} dz = 1. \tag{18}$$

For such normalization in a "volume" of the order of the light wave we find, for $\Omega \neq 0$, that there are always two atoms in a volume of this kind and they are characterized by double degeneracy of each state with a given K; for $\Omega = 0$, there is only one atom in such a volume.

4. SPECTRUM OF ATOMIC ENERGIES IN A STANDING LIGHT WAVE

We shall now determine the spectra of quantum motion of an atom in the field of a standing light wave selecting specific values of the field intensity V_0 and detuning Ω of the field frequency relative to exact resonance. First of all, it is necessary to determine the eigenenergies E_g^m, which correspond to the periodic solutions of the system (7). As pointed out above, only such values of E_g give the edges of the energy bands. All the periodic solutions $u_i^m(z)$ of the system (7) and the corresponding eigenenergies E_g^m, considered as a function of the energy of the interaction of an atom with a field V_0 whose detuning is Ω, are given in the Appendix I. The eigenfunctions $u_i^m(z) = u_i^m(2\xi/k)$ and the eigenvalues $E_g^m = Rh_g^m/4$ are special functions which determine eight different types of periodic solutions of the system (7): $C_{2n}^{1,2}$, $C_{2n+1}^{1,2}$, $S_{2n+1}^{1,2}$, and $S_{2n+2}^{1,2}$ and they reduce, in the special case of $\Omega = 0$, to the Mathieu functions and the corresponding eigenvalues of the Mathieu equation. The first few eigenvalues $E_g^m = Rh_g^m/4$ (and also $E_e^m = E_g^m + \hbar\Omega = Rh_e^m/4$) are plotted in Figs. 2–4 as a function of the field $\hbar V_0 = R\theta/4$ for three values of the detuning $\hbar\Omega = Rf/4 = -0.4R$, 9, and $0.4R$. The dashed curves denote the eigenvalues $E_g^m(E_e^m)$ of the $C_{2n+1}^{1,2}$ and $S_{2n+1}^{1,2}$ types for the $\Omega \neq 0$ case. All the dependences $h_i^m = h_i^m$ are the results of numerical solutions of transcedental equations in the Appendix I.

Since the eigenvalues E_g^m and E_e^m found in the Appendix I correspond to the periodic solutions of the system (7) and are the only possible edges of the allowed energy bands, we can find these bands (and particularly their edges) by calculating the characteristic indices

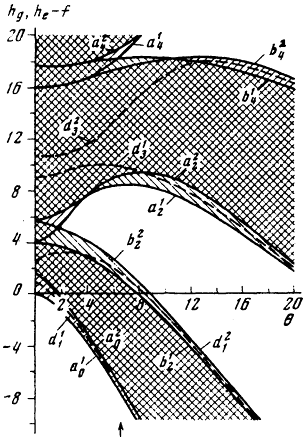

FIG. 2. Dependences of the eigenvalues h_g^m and $h_e^m = h_g^m + f$ on the parameter θ and the band of eigenvalues h_g, h_e for $f = -1.6$.

of the system (7). According to the above analysis, those values of $E_g(E_e)$ which correspond to at least two imaginary characteristic indices $\Lambda_j = \pm iK_j$ define all the allowed energy bands E_g and E_e. These values $E_g(E_e)$ govern the quasivectors $K = K(E_g)$ of an atom in the field of a standing light wave.

In general, the determination of the characteristic indices of the system (7) for given V_0 and Ω is possible only by numerical methods. However, since the characteristic indices are continuous functions of the parameter V_0 and the values of $\Lambda_j(j = 1\text{–}4)$ corresponding to small V_0 ($\hbar V_0 \ll R$) can be found by the perturbation method for the matrix equations,[12] the knowledge of the characteristic indices for small values of V_0 is sufficient for the correct determination of the allowed energy bands E_g and E_e irrespective of the value of V_0. The results of calculations of the characteristic indices in the system (7) by the perturbation theory method are given in the Appendix II.

Employing the explicit form of the characteristic indices in the case of small values of θ, we can easily show that for $f \neq 0$ the eigenvalues $h^m = a_0^1, b_2^1, a_2^2, b_4^2, a_4^1, \dots$ are the edges of the same system of allowed energy bands, whereas the eigenvalues $h^m = a_0^2, b_2^2, a_2^1, b_4^1, a_4^2, \dots$ are the edges of a different system of allowed energy bands, whereas the eigenvalues $d_1^1, d_3^1, d_5^1, \dots$ and the eigenvalues $d_1^2, d_3^2, d_5^2, \dots$ always lie within the first and second systems of the allowed bands. If $f = 0$, all the eigenvalues $a_0, b_1, a_1, b_2, a_3, b_3, a_3, b_4, a_4, \dots$ are the edges of the energy bands and the allowed bands themselves coincide with the regions of stability of the Mathieu equation.[13,14] In Figs. 2 and 4 these systems of the allowed energy bands are identified by different types of shading, and in Fig. 3 the allowed energy bands are identified by double shading (corresponding to the double coincidence of all the eigenvalues in the $\Omega = 0$ case).

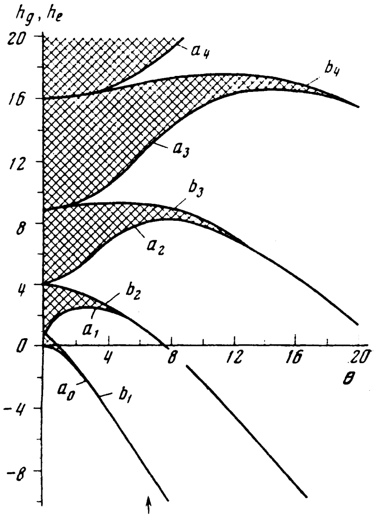

FIG. 3. Dependences $h_g^m(\theta)$ and $h_e^m(\theta)$ and the band of eigenvalues h_g, h_e for $f = 0$.

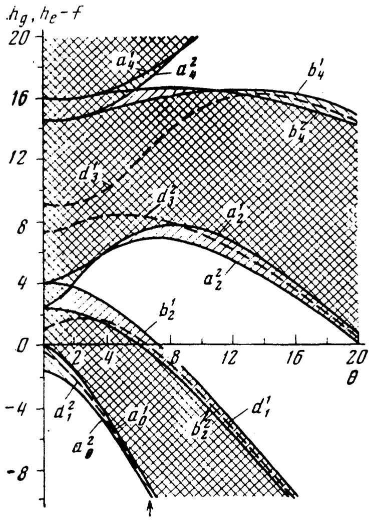

FIG. 4. Dependences of h_g^m and $h_e^m - f$ on θ and the band of eigenvalues h_g, h_e for $f = 1.6$.

Application of the results from the Appendix II readily establishes a correspondence between the values of E_g^m and E_e^m and the quasivector K. Considering, for convenience, again the normalized eigenvalues $h = 4E_g/R$, we find that when h varies from a_0^1 to b_2^1, and also from a_0^2 to b_2^2, the modulus of the quasivector K ranges from 0 to k, and for $h = d_1^1$ and d_1^2 this modulus is $|K| = k/2$. When h varies from a_2^2 to b_4^1, and also from a_2^1 to b_4^2, the value of $|K|$ varies from k to $2k$ and for $h = d_3^1$ and d_3^2 we have $|K| = 3k/2$. Continuing this process (see Table I), we can easily establish that when $E_g(E_e)$ in any of the systems of the allowed energy bands, the modulus of the quasivector ranges from 0 to $+\infty$, and the variation of $E_g(E_e)$ within the n-th band in the $\Omega \neq 0$ case corresonds to the variation of $|K|$ from $(n-1)k$ to nk, and for $\Omega = 0$ the corresponding variation is from $(n-1)k/2$ to $nk/2$.

The above analysis is used to plot in Figs. 5–7 not only the bands of the eigenvalues of E_g and E_e for $4\hbar V_0/R = 6.5$ and $4\hbar\Omega/R = -1.6$, 0, and 1.6, but also schematic dependences of the energies E_g and E_e on the quasivector K in the extended Brillouin zone representation. The dashed curves in these figures represent the dependences of the energies E_e and E_g on the quasivectors K for a free atom: $E_g = \hbar^2K^2/2M$, $E_e = \hbar^2K^2/2M + \hbar\Omega$ for

TABLE I. Correspondence between eigenvalues h^m and modulus of quasivector K.

h^m ($f \neq 0$)	h^m ($f = 0$)	$\|K\|$	h^m ($f \neq 0$)	h^m ($f = 0$)	$\|K\|$
a_0^1, a_0^2	a_0	0	d_3^1, d_3^2	b_3, a_3	$3k/2$
d_1^1, d_1^2	b_1, a_1	$k/2$	$b_4^1, b_4^2, a_4^1, a_4^2$	b_4, a_4	$2k$
$b_2^1, b_2^2, a_2^1, a_2^2$	b_2, a_2	k	d_5^1, d_5^2	b_5, a_5	$5k/2$

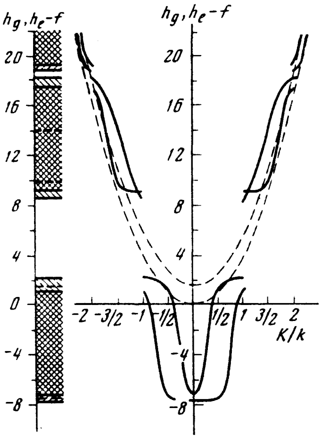

FIG. 5. Dependences of the eigenvalues h_g and $h_e = h_g + f$ on the quasivector K and the band of eigenvalues h_g, h_e for $\theta = 6.5$ and $f = -1.6$.

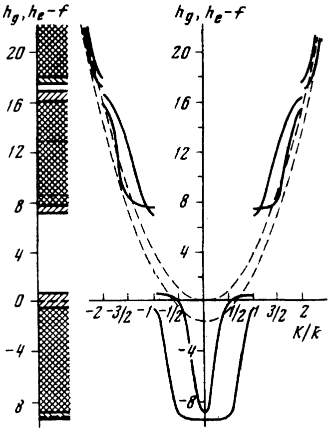

FIG. 7. Same as Fig. 5, but for $f = 1.6$.

one system of allowed energy bands and $E_g = \hbar^2K^2/2M - \hbar\Omega$, $E_e = \hbar^2K^2/2M$ for another system (see the Appendix II). For convenience in comparison of Figs. 2–4 and 5–7, which apply to the same detuning, the values of $\theta = 6.5$ are denoted by arrows in Figs. 2–4.

These results allow us to calculate the spectra of the quantum motion of atoms in a standing light wave for any value of the wave field and detuning. For example, Fig. 8 shows the spectra of an atom as the dependences $E_g = E_g(K)$ and $E_e = E_e(K)$ for the values $\hbar V_0 = 1.25R$, $\hbar\Omega = 0.5R$ in the extended Brillouin zone representation.

5. SPONTANEOUS TRANSITIONS. COOLING OF ATOMS

We shall use the analysis of the motion of atoms under the action of recoil in induced transitions (Secs. 2 and 3) and consider the influence of resonant spontaneous decay on the spatial motion of atoms in a standing light wave. Our aim will be primarily to give a quantitative description of the role of spontaneous transitions and we shall assume that the probability of spontaneous decay 2Γ of the upper level e to the lower level g is much less than the probability of induced transitions V_0. Subject to this condition, resonant spontaneous decay firstly mixes the degenerate states of an atom with a given value of K and secondly, gives rise to atomic transitions between states with different quasivectors, i.e., it results in an accelerated motion of atoms. We can easily determine the sign of this acceleration. In fact, in view of the isotropy of spontaneous radiation, on the average each resonant decay of an atom results in a transition from a state with energies $E_g = E - \hbar\Omega$, $E_e = E$ to a state with energies $E_g = E$, $E_e = E + \hbar\Omega$. Since the dependence of E_i on the modulus of the quasivector is also a monotonically rising function, it follows that for $\Omega > 0$ the atoms are on the average accelerated but for $\Omega < 0$ they are retarded by resonant spontaneous decay, going over, respectively, to higher and lower allowed energy bands.

A simple analysis showed also that the force which accelerates (decelerates) atoms is resonant if $|K| \gg k$. In fact, once again allowing for the fact that—on the average—there is no change in the atomic momentum as a result of resonant spontaneous decay, we can easily show that the highest (in the absolute sense) acceleration is experienced by those atoms for which the change in the quasivector in the subsequent stimulated transition is

$$|K_2 - K_1| \approx k. \tag{19}$$

Let us assume that the quasivector K_1 corresponds to the atomic energies $E_g = E - \hbar\Omega$ and $E_e = E$, whereas the quasivector K_2, as pointed out above, corresponds to the energies $E_g = E$ and $E_e = E + \hbar\Omega$. Taking any pair of the values of K_1 and K_2

$$K_1 = [2M(E - \hbar\Omega)]^{1/2}/\hbar, \quad K_2 = (2ME)^{1/2}/\hbar$$

or

$$K_1 = (2ME)^{1/2}/\hbar, \quad K_2[2M(E + \hbar\Omega)]^{1/2}/\hbar,$$

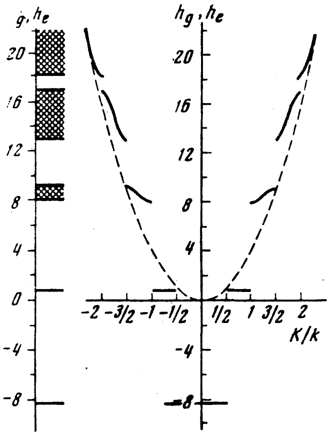

FIG. 6. Same as Fig. 5, but for $f = 0$.

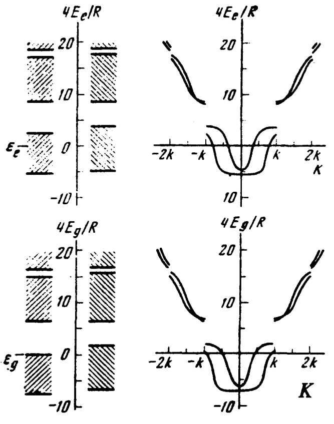

FIG. 8. Systems of allowed bands and dependences of energies on the quasivector K for $V_0 = 1.25\ R/\hbar$ and $\Omega = 0.5R/\hbar$.

we find from Eq. (19) that the highest acceleration is experienced by the atoms for which

$$k|\hbar \bar{K}/M| = k|\bar{v}| \approx |\Omega|, \tag{20}$$

where $\bar{v} = \hbar\bar{K}/M$ is the average velocity of the atoms ($\bar{K} \approx K_1 \approx K_2 \gg k$), i.e., the atoms in resonance with the field are accelerated most.

These features of the quantum motion of atoms in the case of simultaneous stimulated and resonant continuous transitions in the field of a light wave are entirely due to the isotropic nature of the spontaneous radiation. Allowance for the fluctuations of the direction and the number of spontaneously emitted photons gives rise to new features involving the spreading of atoms over the allowed energy bands. Clearly, if $\Omega > 0$ this acceleration and spreading of atoms occur simultaneously and the motion of atoms is far from quasisteady. If $\Omega > 0$, fluctuations of the spontaneous radiation prevent cooling of atoms. Therefore, for a negative detuning a steady-state distribution of cold atoms is always established.

The earlier classical analysis[3] yielded the distribution of the velocities of cold atoms ($|v| = \hbar|K|/M \ll |\Omega|/k$). The analysis given there[3] remains valid in the quantum-mechanical approach and it shows that the energy distribution of cold atoms is described by the exponential distribution

$$f(E) \sim \exp[-(E - E^{min})/\bar{E}], \tag{21}$$

where $\bar{E} = \frac{1}{2}\hbar\Gamma(|\Omega|/\Gamma + \Gamma/|\Omega|)$ is the average energy of the field-cooled atoms when the detuning is $\Omega = -|\Omega|$, and E and $E^{\min}$ are the values of the energies $E_g(K)$, $E_g(0)$, $E_e(K)$, $E_e(0)$ in the two systems of allowed energy bands. Thus, cooled atoms are always distributed over the first few allowed energy bands right up to energies $\sim E(0) + \bar{E}$, where the mimimum width of the distribution $\bar{E} = \hbar\Gamma$ is reached for a detuning $\Omega = -\Gamma$.

6. ABSORPTION SPECTRUM OF COLD ATOMS

The existence of a band structure of atomic levels in a standing light wave can be deduced from atomic transitions within and between the allowed energy bands, and also from transitions of atoms from allowed energy bands to states which are not involved in resonant transitions.[7]

Let us assume that atoms in a standing light wave are subject to a test field, whose wave vector is parallel to the wave vector of the standing light wave:

$$\mathscr{E}_\mu = \mathscr{E}_\mu{}^0 \exp[i(k_\mu z - \omega_\mu t)]. \tag{22}$$

If the initial and final states of an atom belong to some allowed energy bands, the probability of a transition of an atom from a state "K" to a state "K'" (band-band transition) will be governed by matrix elements of the type

$$M_{K'K} = V_\mu{}^0 \langle w_g{}^{K'} | \exp[i(K + k_\mu - K')z] | w_e{}^K \rangle, \tag{23}$$

where $V_\mu^0 = d\mathscr{E}_\mu^0/\hbar$. It is clear from Eq. (23) that the recoil of a test field results most probably in transitions between states whose quasivectors differ by an amount equal to the wave vector of the test field.

$$K' \approx K + k_\mu. \tag{24}$$

Therefore, for $k_\mu < k (\omega_\mu < \omega)$ these transitions occur mainly within the same band or between neighboring energy bands, whereas for $k_\mu > k (\omega_\mu > \omega)$ they occur mainly between neighboring energy bands.

When the detuning of the field frequency from the frequency of an atomic transition is negative, the energy spectrum of atoms in a field cannot be deduced from the absorption of test radiation by cold atoms distributed over the lower energy bands. A characteristic feature of this absorption spectrum is the existence of finite absorption at any frequency ω_μ of the test field because for any wave vector $k_\mu = \omega_\mu/c$ we can have transitions of cold atoms either inside or between allowed energy bands.

In addition to the intraband and interband transitions in a standing light wave, there may be also transitions of atoms from allowed energy bands to discrete states which are not the g and e states of a resonant transition. Such transitions occur also mainly subject to the condition (24), but in this case K' denotes the wave vector of a free atom. If $\Omega < 0$, the absorption spectrum for a neighboring transition consists of a series of wide absorption lines, each of which represents one of the lowest filled energy bands. Such an absorption spectrum differs qualitatively from the usual spectrum of a neighboring transition when a traveling wave interacts with the g–e resonant transition. For example, if $V_0 \gg \Gamma$, the absorption spectrum of a neighboring transition may consist of lines whose separations and widths are governed by the field and not by the radiative decay of levels.

7. CONCLUSIONS. COMPARISON WITH THE CLASSICAL ANALYSIS

Having established the main features of the quantum motion of atoms under the influence of recoil in stimulated and resonant spontaneous transitions, we shall now consider the correspondence between the quantum and classical treatments. It is known[2,3] that in the classical approach each atom in a standing light wave is subject to an induced radiation pressure force, which modulates the velocity of the atom, and to a spontaneous radiation pressure force, which accelerates or decelerates the atom. We can now see that the classical modulation of the velocity of an atom by the induced radiation pressure force corresponds, in the quantum treatment, to the motion of atoms with specific quasivector and energies associated with specific allowed energy bands of an atom in the light-wave field. The spontaneous radiation pressure force retains its meaning in the quantum-mechanical approach, namely, this force is responsible for the acceleration (or deceleration, i.e., cooling) of atoms so that they are transferred from certain allowed energy bands to other when the energy of motion is increased or reduced.

Thus, the agreement between the main results of the quantum and classical analyses confirms completely the possibility of cooling of atoms in the field of a standing light wave with a negative detuning in the case of adiabatic scanning of the field frequency. As established earlier,[3] such cooling may occur in a time of $\sim 10^{-4}-10^{-3}$ sec when the field frequency is varied from the initial value $\omega_i=\omega_0-\Delta\omega_D$ to the final value $\omega_f=\omega_0-\Gamma$, and in this way an atomic gas can be cooled to temperatures of $T=\hbar\Gamma/k\sim 10^{-4}-10^{-3}$ °K.

However, our quantum-mechanical analysis shows also that when the detuning is $\Omega=-|\Omega|\neq 0$, cold atoms cannot be trapped by antinodes of a standing light field. In fact, if $\Omega\neq 0$, the spatial density of the probability of finding an atom is periodic in space and the minimum period is $\Delta z=\lambda$. Therefore, for $\Omega\neq 0$, a cold atom is generally equally likely to be at a node or antinode of the field, and cannot be localized in a region whose dimensions are less than the wavelength of the resonant light field. This follows also from the structure of the lower energy bands. Since the width of the allowed energy bands in the $\Omega\neq 0$ case is generally of the same order as the gaps between the bands, the probability of spreading of the wave packet of a cold atom is of the same order as the probability of the transition of an atom from an antinode to a node of the field.

In the special case of $\Omega=0$ the spatial density of the probability of finding an atom is periodic and the period is $\Delta z=\lambda/2$. Therefore, if the field V_0 is sufficiently high, in the $\Omega=0$ case a cold atom may be localized either at an antinode or at a node of the field. The latter is also self-evident from the spectrum of quantum motion in the $\Omega=0$ case.

In spite of the fact that atoms cannot be trapped in a region of the order of the wavelength of the light field, cold atoms may remain for a long time in a macroscopic region of size governed by the transverse dimensions of the light beam. It has been shown[15] that the gradient forces in a three-dimensional standing light wave may trap all the cold atoms in the region of intersection of light waves.

It follows that the results of the present investigation together with those reported earlier[2,3,15] confirm the possibility of cooling and prolong the trapping of atoms in a standing light wave. Moreover, this discussion shows that, in contrast to free atoms, cold atoms in a standing light wave may absorb effectively microwave radiation going over from lower to higher allowed energy bands.

The authors are grateful to A. T. Matachun for her great help in the numerical calculations of the eigenvalues of the main equation in the present paper.

APPENDIX I. PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATION SYSTEM (7)

Using the dimensionless variables

$$\xi=kz/2,\quad h_g=h=4E_g/R,\quad f=4\hbar\Omega/R,\quad \theta=4\hbar V_0/R,$$

we shall rewrite the system (7) in the form

$$\begin{aligned}&\frac{d^2u_e}{d\xi^2}+(h+f)u_e+2\theta\cos 2\xi\, u_g=0,\\ &\frac{d^2u_g}{d\xi^2}+hu_g+2\theta\cos 2\xi\, u_e=0.\end{aligned}\tag{I.1}$$

The known methods for finding periodic solutions of systems of differential equations with periodic coefficients[12,13] show that the system (I.1) with $f\neq 0$ has eight types of independent periodic solutions. We shall denote them as solutions of the type $C_{2n}^{1,2}$, $C_{2n+1}^{1,2}$, $S_{2n+1}^{1,2}$, and $S_{2n+2}^{1,2}$ and for each of them we shall give the type of expansion and infinite transcendental equation which define all the eigenvalues h^m for this type of solution.

1) Solutions of type C_{2n}^{1}:

$$\begin{aligned}c_{2n}^{1}u_e&=\sum_{r=0}^{\infty}{}_{2n}^{1}A_{4r+2}^{c}\cos(4r+2)\xi,\\ c_{2n}^{1}u_g&=\sum_{s=0}^{\infty}{}_{2n}^{1}A_{4s}^{g}\cos 4s\xi;\end{aligned}\tag{I.2}$$

$$h\equiv a^{1}=\frac{-2\theta^2}{4-h-f-}\,\frac{\theta^2}{16-h-\ldots-}\,\frac{\theta^2}{4k^2-h-}\,\frac{\theta^2}{4(k+1)^2-h-f-\ldots}.\tag{I.3}$$

2) Solutions of type C_{2n}^{2}:

$$\begin{aligned}c_{2n}^{2}u_e&=\sum_{r=0}^{\infty}{}_{2n}^{2}A_{4r}^{e}\cos 4r\xi,\\ c_{2n}^{2}u_g&=\sum_{s=0}^{\infty}{}_{2n}^{2}A_{4s+2}^{g}\cos(4s+2)\xi;\end{aligned}\tag{I.4}$$

$$h\equiv a_{2n}^{2}=-f-\frac{2\theta^2}{4-h-}\,\frac{\theta^2}{16-h-f-\ldots-}\,\frac{\theta^2}{4k^2-h-}\,\frac{\theta^2}{4(k+1)^2-h-f-\ldots}.\tag{I.5}$$

3) Solutions of type C_{2n+1}^{1}:

$$\begin{aligned}c_{2n+1}^{1}u_e&=\sum_{r=0}^{\infty}{}_{2n+1}^{1}A_{2r+1}^{e}\cos(2r+1)\xi,\\ c_{2n+1}^{1}u_g&=\sum_{s=0}^{\infty}{}_{2n+1}^{1}A_{2s+1}^{g}\cos(2s+1)\xi;\end{aligned}\tag{I.6}$$

$$\begin{aligned}h\equiv d_{2n+1}^{1}=1&-\frac{\theta^2}{9-h-f-}\,\frac{\theta^2}{25-h-\ldots-}\,\frac{\theta^2}{(2k+1)^2-h-}\,\frac{\theta^2}{(2k+3)^2-h-f-\ldots}\\ &-\frac{\theta^2}{1-h-f-}\,\frac{\theta^2}{9-h-}\,\frac{\theta^2}{25-h-f-\ldots-}\,\frac{\theta^2}{(2k+1)^2-h-}\,\frac{\theta^2}{(2k+3)^2-h-f-\ldots}.\end{aligned}\tag{I.7}$$

4) Solutions of type C_{2n+1}^{2}:

$$\begin{aligned}c_{2n+1}^{2}u_e&=\sum_{r=0}^{\infty}{}_{2n+1}^{2}A_{2r+1}^{e}\cos(2r+1)\xi,\\ c_{2n+1}^{2}u_g&=\sum_{s=0}^{\infty}{}_{2n+1}^{2}A_{2s+1}^{g}\cos(2s+1)\xi;\end{aligned}\tag{I.8}$$

$$\begin{aligned}h\equiv d_{2n+1}^{2}=1-f&-\frac{\theta^2}{9-h-}\,\frac{\theta^2}{25-h-f-\ldots-}\,\frac{\theta^2}{(2k+1)^2-h-}\,\frac{\theta^2}{(2k+3)^2-h-f-\ldots}\\ &-\frac{\theta^2}{1-h-}\,\frac{\theta^2}{9-h-f-}\,\frac{\theta^2}{25-h-\ldots-}\,\frac{\theta^2}{(2k+1)^2-h-}\,\frac{\theta^2}{(2k+3)^2-h-f-\ldots}.\end{aligned}\tag{I.9}$$

5) Solutions of type S_{2n+1}^{1}:

$$\begin{aligned}s_{2n+1}^{1}u_e&=\sum_{r=0}^{\infty}{}_{2n+1}^{1}B_{2r+1}^{e}\sin(2r+1)\xi,\\ s_{2n+1}^{1}u_g&=\sum_{s=0}^{\infty}{}_{2n+1}^{1}B_{2s+1}^{g}\sin(2s+1)\xi;\end{aligned}\tag{I.10}$$

$$h\equiv d_{2n+1}^{1}.\tag{I.11}$$

6) Solutions of type S^{2}_{2n+1}:

$$s^{2}_{2n+1}u_e=\sum_{r=0}^{\infty}{}^{\;2}_{2n+2}B^{e}_{4r+4}\sin(2r+1)\xi,$$

$$s^{2}_{2n+1}u_g=\sum_{s=0}^{\infty}{}^{\;2}_{2n+1}B^{g}_{2s+1}\sin(2s+1)\xi; \qquad \text{(I.12)}$$

$$h=d^{2}_{2n+1}. \qquad \text{(I.13)}$$

7) Solutions of type S^{1}_{2n+2}:

$$s^{1}_{2n+2}u_e=\sum_{r=0}^{\infty}{}^{\;1}_{2n+2}B^{e}_{4r+4}\sin(4r+4)\xi,$$

$$s^{1}_{2n+2}u_g=\sum_{s=0}^{\infty}{}^{\;1}_{2n+2}B^{g}_{4s+2}\sin(4s+2)\xi; \qquad \text{(I.14)}$$

$$h=b^{1}_{2n+2}=4-\frac{\theta^2}{16-h-f-}\,\frac{\theta^2}{36-h-\ldots-}\,\frac{\theta^2}{4k^2-h-}\,\frac{\theta^2}{4(k+1)^2-h-f-\ldots}. \qquad \text{(I.15)}$$

8) Solutions of type S^{2}_{2n+1}:

$$s^{2}_{2n+2}u_e=\sum_{r=0}^{\infty}{}^{\;2}_{2n+2}B^{e}_{4r+2}\sin(4r+2)\xi,$$

$$s^{2}_{2n+2}u_g=\sum_{s=0}^{\infty}{}^{\;2}_{2n+2}B^{g}_{4s+4}\sin(4s+4)\xi; \qquad \text{(I.16)}$$

$$h=b^{2}_{2n+2}=4-f$$

$$-\frac{\theta^2}{16-h-}\,\frac{\theta^2}{36-h-f-\ldots-}\,\frac{\theta^2}{4k^2-h-}\,\frac{\theta^2}{4(k+1)^2-h-f-\ldots}. \qquad \text{(I.17)}$$

APPENDIX II. CHARACTERISTIC EXPONENTS OF THE EQUATION SYSTEM (7)

We shall not describe simple but time-consuming calculations and give directly, for $f\neq 0$ and $\theta\ll 1$, the characteristic exponents $\lambda_j(j=1,2,3,4)$ of the system (I.1), which are related to the characteristic exponents Λ_j of the system (7) by the relationship $\Lambda_j=\lambda_j k/2$.

For $h\neq m^2$ and $h+f\neq n^2(m,n=1,2,3,\ldots)$, in the first approximation with the small parameter θ, the characteristic exponents λ_j are governed by the relationships

$$\lambda_{1,2}=\pm i\left(h+\frac{2(4-f)\theta^2}{(4-f)^2-16h}\right)^{1/2},$$
$$\lambda_{3,4}=\pm i\left(h+f+\frac{2(4+f)\theta^2}{(4-f)^2-16h}\right)^{1/2}, \qquad \text{(II.1)}$$

if $h\neq(1-f/4)^2$, and by the relationships

$$\lambda_{1,2}=\pm i\left(h-\frac{\theta^2}{h+f}\right)^{1/2},\quad \lambda_{3,4}=\pm i\left(h+f-\frac{\theta^2}{h}\right)^{1/2}, \qquad \text{(II.2)}$$

if $h=(1-f/4)^2$.

The characteristic exponents λ_j for the values $h=m^2$, and also for h close to the squares of integers can be calculated, on condition that $h+f\neq n^2$, only separately for each of the values $m=1,2,3,\ldots$.

We shall assume that the difference between h and m^2 is of the order of the small parameter θ^2:

$$h=m^2+\mu\theta^2+\nu\theta^4+\ldots, \qquad \text{(II.3)}$$

where $\mu,\nu,\ldots$ are arbitrary numbers satisfying the conditions $\mu\theta^2\ll m^2,\nu\theta^2\ll\mu,\ldots$. For $f\neq 0$, the eigenvalues h^m are even relative to the parameter θ so that the odd powers of θ in Eq. (II.3) are of no interest. Then, in the first approximation containing the small parameter θ, we find for $m=1$ that

$$\lambda_{1,2}=-i\pm i\left(\frac{\mu}{2}-\frac{1}{2f}-\frac{1/2}{f-8}\right)\theta^2,$$
$$\lambda_{3,4}=\pm i(1+f)^{1/2}\pm\frac{i\theta^2}{2(1+f)^{1/2}}\left\{\mu-\frac{1}{2f}+\frac{3}{f-8}\right.$$
$$\left.\mp\frac{(1+f)^{1/2}}{2f}+\frac{1}{2[1\pm(1+f)^{1/2}]}\right\} \qquad \text{(II.4)}$$

and for $m=2$,

$$\lambda_{1,2}=-2i\pm\frac{i\theta^2}{4}\left[\left(\mu+\frac{1}{12-f}\right)\left(\mu-\frac{2}{4+f}+\frac{1}{12-f}\right)\right]^{1/2},$$
$$\lambda_{3,4}=\pm i\left[4+f+\left(\mu-\frac{2}{12-f}\right)\theta^2\right]^{1/2}, \qquad \text{(II.5)}$$

and for higher numbers $m=3,4,5,\ldots$, we obtain

$$\lambda_{1,2}=-im\pm\frac{i\theta^2}{2m}\left(\mu+\frac{1}{4m+4-f}-\frac{1}{4m-4+f}\right),$$
$$\lambda_{3,4}=\pm i\left[m^2+f+\left(\mu-\frac{(m+2)/m}{4m+4-f}+\frac{(m-2)/m}{4m-4+f}\right)\theta^2\right]^{1/2}. \qquad \text{(II.6)}$$

The characteristic exponents λ_j for the values of $h+f$ close to m^2 are

$$h+f=m^2+\mu\theta^2+\nu\theta^4+\ldots, \qquad \text{(II.7)}$$

on condition that $h\neq n^2$ and these are also given by Eqs. (II.4)–(II.6) if we replace f with $-f$.

The characteristic exponents in those cases when h and $h+f$ are both squares of integers are of interest only in a denumerable set of the values of $f=n^2-m^2=0,\pm1,\pm2,\ldots$ and these are of very little practical importance. Therefore, we shall not give the relevant results. We shall simply point out that the indices λ_j for $n^2=m^2$ are identical with the characteristic exponents of the Mathieu equations[13,14] calculated, for example, in the book by Yakubovich and Starzhinskiĭ[12] right up to the second order in θ.

Finally, for $f=0$ and $\theta\ll 1$, the characteristic exponents λ_j for the values of $h\neq m^2$ and $h=m^2$ can be found in the same book by Yakubovich and Starzhinskiĭ.[12]

[1] The idea of the trapping of atoms was first put forward by Letokhov[4] for a nonresonant standing light wave. A detailed classical analysis of this possibility was made by Letokhov and Pavlik.[5] Cooling of atoms was predicted earlier by Hansch and Schawlow[6] for an isotropic light field.

[2] Following Letokhov and Minogin,[7] we shall call these ultracold atoms.

[3] In a different approach an "atom + field" quasiparticle can be described by a quasimomentum $\hbar K$ and two effective masses $M_g=M_g(K)$ and $M_e=M_e(K)$ with quadratic dispersion laws.

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Temperature dependence of the refractive index in condensed media

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A method of direct measurement of the classical constant of physical optics $(\partial n/\partial T)_\rho$ is proposed and developed. The essential element of the experimental set-up, which makes such a technique possible, is a single-mode laser with a pulse width of 3×10^{-9} sec and energy 0.3 J. A detailed substantiation of the proposed method is presented. Values of $(\partial n/\partial T)\rho$ were measured for a number of substances. The results are discussed and compared with the results of calculations carried out by Raman.

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Investigation of the temperature dependence of the refractive index is one of the fundamental tasks of physical optics. Up to now the solution of this problem was restricted to the measurement of the so-called total derivative dn/dT of the refractive index with respect to temperature. It is generally customary to express the dependence of the refractive index on the various thermodynamic parameters by derivatives with respect to these parameters. In this traditional manner the temperature dependence of the refractive index $n(T)$ can be represented in the general case in the form

$$n(T)-n_0=\int\left\{-\alpha(T)\rho\left(\frac{\partial n}{\partial\rho}\right)_T+\left(\frac{\partial n}{\partial T}\right)_\rho\right\}dT,\qquad(1)$$

where n_0 is the refractive index at a certain fixed temperature, $\alpha(T)$ the coefficient of thermal expansion, and ρ the density. The sum under the integral sign is the total derivative dn/dT. The first term of this sum is due to the effect of thermal expansion, and the second to the temperature dependence of molecular polarizability.[1]

Knowledge of the total derivative dn/dT alone is often insufficient. Thus, for instance, if one of the terms in the expression under the integral sign in (1) is not known in addition to dn/dT, it is impossible to establish a correspondence between the quantity Δn measured experimentally and the temperature change ΔT in the case when ΔT depends on a space coordinate. In fact, thermoelastic stresses arise in this case and the thermal expansion coefficient, with fringe effects disregarded, must be replaced by the quantity

$$\alpha_{\text{eff}}\approx\alpha\frac{1+\nu}{3(1-\nu)},\qquad(2)$$

where ν is Poisson's ratio. It is then impossible to determine ΔT from the measured Δn and the known dn/dT.

In recent years the study of the temperature dependence of the refractive index has become a problem of timely interest due to the creation of sources of short high-power light pulses, the lasers. As the laser radiation passes through condensed media, additional refraction occurs in the latter, due to heating of the substance by absorption of radiant energy. This phenomenon may be utilized as a research tool. Thus in[2] a method was proposed and developed for the measurement of small local absorption coefficients, which is based on high-speed interferometric recording of change of the refractive index. On the other hand, in most cases the same phenomenon leads to undesirable nonlinear effects such as self-focusing or defocusing.[3] In either case the change of the refractive index is determined according to formula (1) with account taken of (2) or, for a sufficiently short time of action, according to the same formula (1) with account taken of the time dependence of ρ (nonstationary process). It is obvious that for a correct solution of problems of the type described above knowledge of each of the terms in the expression under the

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