Analytic properties of the amplitude for elastic forward scattering of electrons by atoms

M. Ya. Amus'ya and M. Yu. Kuchiev

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences
(Submitted 22 September 1977)
Zh. Eksp. Teor. Fiz. **74**, 924–936 (March 1978)

It is shown that the exchange amplitude for elastic forward scattering has an additional cut at the left semi-axis of the complex energy plane. The branch point is located at the binding energy of the electron in the atom. For scattering by hydrogen atoms, the amplitude diverges logarithmically at $E=-1$ Ry and hence this branch point is logarithmic. On the left-hand axis of the physical sheet, no singularities have been observed besides the poles corresponding to bound states of the negative ion. Although branch points exist at energies equal to the ionization potentials of the atomic excited states, they lie in the unphysical sheets and hence do not affect the dispersion relation. The position and nature of the singularity is completely determined by the shape of the interaction potential. The singularity also exists for shortrange potentials, but it is considerably separated from the physical region, whereas, in the case of Coulomb forces, it is close to it and plays an essential role in the dispersion relation.

PACS numbers: 34.80.Bm

1. INTRODUCTION

The dispersion relation not only serves as a useful instrument that enables us to reconstruct the elastic scattering amplitude from the cross section, but also is a touchstone for any hypothesis on its analytic structure. The evolution of the dispersion relation in atomic physics begins with the work of Gerjuoy and Krall,[1,2] who proposed to write it for the amplitude of the elastic forward scattering $\bar{f}$, as if the atomic-target were simply a potential well

$$\mathrm{Re}\,\bar{f}(E)=\bar{f}_B(E)+\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,\bar{f}(E')}{E'-E}\,dE'+\sum_n \frac{d_n}{E-E_n}. \qquad (1)$$

The function $\bar{f}$ is a linear combination of the direct, f, and exchange, g, amplitudes, so chosen that $\mathrm{Im}\bar{f}$ was connected in the usual way with the total cross section; $\bar{f}_B$ is the scattering amplitude in the first Born approximation, E is the energy of the incoming electron, n enumerates the discrete levels of the compound system, i.e., of the negative ion, and E_n is the binding energy of these levels. The amplitude of the elastic forward scattering, according to (1), has poles on the physical sheet at the discrete energy levels E_n and a right-hand cut from 0 to ∞.

A check on Eq. (1), based on a comparison of theoretical and experimental data,[3,4] has shown that such a simple dispersion relation is not satisfied for a number of atoms, at any case in the low energy region. Thus, at $E=0$, for scattering by hydrogen, the left side of (1) is equal to -2.8 and the right to -2.1; for the case of helium, the corresponding figures are -1.15 and -0.5. A still greater discrepancy exists for scattering by neon. The dispersion relation in the collisions of positrons with atoms is written similar to (1). It turned out that in this case the assumption of Gerjuoy and Krall is valid.[5,6] The scattering of electrons differs from scattering of positrons in the presence of an exchange amplitude. Thus, the existing experimental data require us to explain why the relation (1) is not satisfied when it is necessary to take exchange into account or, in other words, why the analytic properties of the direct and exchange amplitudes differ.

It was shown earlier[7] that the exchange amplitude of simple forward elastic scattering of the electron by the hydrogen atom has on the left semi-axis of the complex energy plane an additional cut that begins at the energy of the incoming electron and is equal to be binding energy of the ground state of the hydrogen atom. The question naturally arises as to what amplitude singularities are caused by the excited levels of the atom in the virtual state. It is shown here that no other singularities arise on the physical sheet. Cuts exist on the left axis from the branch points at the binding energies of the discrete excited states, but they are located on the nonphysical sheets and do not make a contribution to the dispersion relation. A combined analysis of both the discrete and continuous spectra in the intermediate state leads to this conclusion. The singularity of the amplitude on the left axis is anomalous, i.e., its location and character are determined by the interaction potential. In the case of Coulomb forces, the "left" cut is adjacent to the physical region, while for the shortrange potentials it is far off and therefore makes little contribution to the dispersion relation.

The question of the possible non-analyticity of the exchange amplitude has been discussed in the literature from various points of view.[8-11] On the left energy semi-axis, poles and cuts are possible, the location and character of which do not depend on the potential of the interparticle interaction. These are called physical singularities. The complete catalog of such singularities is given in Ref. 8 for the case of three particles interacting through a potential of the Yukawa type. The physical singularities that are applicable to atomic physics have been discussed in Ref. 9 in the case of electron scattering from helium atoms. It was shown in Refs. 10 and 11 that the exchange amplitude, in contrast with the direct, can have on the energy left axis additional singularities, located to the left of the binding energy of the ground state of the atom.

 0038-5646/78/030484-07$02.40

2. EXCHANGE AMPLITUDE OF ELECTRON SCATTERING FROM A HYDROGEN ATOM

2A. Expression for the exchange amplitude in terms of the Green's function

The scattering of an electron by hydrogen is the simplest case encountered in atomic physics, for which the presence of exchange is important. The exchange amplitude of elastic scattering, by definition, describes a process in which the incoming electron, which is first located in an atom, goes to the continuous spectrum. Figure 1 shows the simple diagrams which describe such events and therefore give a contribution to the exchange amplitude. The double line indicates the propagation function of the hydrogen atom, and the thick and thin lines represent the propagation functions of the proton and electron, respectively. A circle denotes the form factor of the vertex and the wavy lines the Coulomb interaction.

The sum of graphs a and b is traditionally known as the first Born approximation. All the processes which are described by more complicated graphs can be represented as taking place in three stages. In the first, the atom virtually decays into a proton and an electron. The interaction of the atomic electron 2 with the nucleus in the initial state is completely taken into account by the form factor of the vertex. Therefore, immediately after the virtual decay of the atom, we need take into account only the interaction of the incoming electron 1 with the other particles, as is shown in Fig. 1c and 1d. In the second stage, all three particles propagate, interacting with one another in all possible ways. In the final state, the electron 1 lands in the atom, its binding with the nucleus is taken into account by the form factor, and electron 2 departs, the "last" process being the interaction of the departing electron with the "remaining" particles, as is shown in Figs. 1e and 1f.

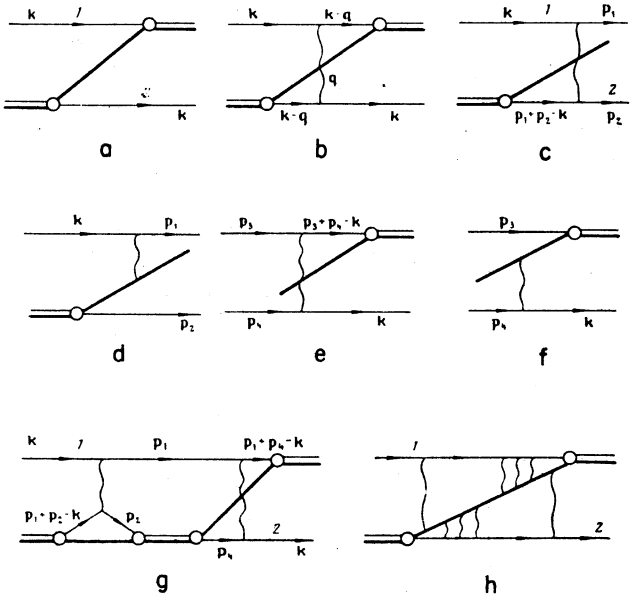

FIG. 1.

For all but the Born diagrams, there are three virtual particles in the intermediate state. Their propagation is described by the Green's function. Therefore, the sum of all the diagrams, except a and b is described by a three-particle Green's function. We obtain a representation of the amplitude in terms of the Green's function, which will be useful in what follows. For this, we note that the form factor and the Fourier transform of the wave function of the ground state of hydrogen $\varphi_0(q)$ are connected by the well known equation

$$\varphi_0(q)=\langle \mathbf{q}|V|\varphi_0\rangle/(E_0-\varepsilon_q),$$

in which $V=-1/r$ is the interaction potential of the electron with the proton (we use the atomic system of units: $e=\hbar=m=1$), the matrix element $\langle \mathbf{q}|V|\varphi_0\rangle$ is the form factor, $|\mathbf{q}\rangle=e^{i\mathbf{q}\cdot\mathbf{r}}$ describes the electron with momentum $\mathbf{q}$ and energy ε_q, $E_0=-\frac{1}{2}$ is the energy of the ground state of the hydrogen atom.

We consider the processes shown in Figs. 1c and 1d. The amplitude of the virtual emission of electron 2 from the atom is equal to the form factor. The amplitude of the propagation of the virtual electron 2 and the real electron 1 leads to the same energy denominator as in (1). Therefore, the form factor and the propagation function together give simply the wave function of the ground state. The interaction of the incoming electron with two other particles leads to a factor that is equal to the Fourier transform of the interaction potential. This means that the amplitude of the transition probability from the initial state to the virtual state with the momenta of the intermediate electrons $\mathbf{p}_1$ and $\mathbf{p}_2$, as shown in Figs. 1c and 1d, is equal to

$$\frac{4\pi}{(\mathbf{k}-\mathbf{p}_1)^2}[\varphi_0(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k})-\varphi_0(p_2)], \tag{2}$$

where k is the momentum of the incoming electron, $4\pi/(\mathbf{k}-\mathbf{p}_1)^2$ is the Fourier transform of the Coulomb potential, while the first and second terms in the brackets stem from the diagrams c and d; the signs in front of them correspond to the collision of electrons and the attraction of the electron to the nucleus.

Similarly, the transition amplitude of electrons with momenta $\mathbf{p}_3$ and $\mathbf{p}_4$ to the final state, as is shown in Figs. 1e and 1f, is

$$\frac{4\pi}{(\mathbf{k}-\mathbf{p}_4)^2}[\varphi_0(\mathbf{p}_3+\mathbf{p}_4-\mathbf{k})-\varphi_0(p_3)]. \tag{3}$$

Therefore, if we denote by $G(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4,E)$ the Green's function of the electrons in the field of the nucleus, then, in accord with what has been said of the three stages of the process, we obtain the following expression for the exchange amplitude $g(k)$:

$$g(k)=g_B(k)-\frac{1}{2\pi}\int\frac{4\pi}{(\mathbf{k}-\mathbf{p}_1)^2}[\varphi_0(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k})-\varphi_0(p_2)]G(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4;\varepsilon_k+E_0)\times\frac{4\pi}{(\mathbf{k}-\mathbf{p}_4)^2}[\varphi_0(\mathbf{p}_3+\mathbf{p}_4-\mathbf{k})-\varphi_0(p_3)]\frac{d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3\,d\mathbf{p}_4}{(2\pi)^{12}}, \tag{4}$$

in which $g_B(k)$ is the Born amplitude.

2B. Analytical properties of the total amplitude

It was shown in Ref. 3 that the Born exchange amplitude has a pole of third order at $k^2=-1$. For the graph of Fig. 1a, this statement was made earlier in Ref. 12. The singularities of the Born term themselves do not affect the dispersion relation (1), in which the Born term is taken into account separately. However, the unusual behavior of the Born amplitude is a sign that the total amplitude behaves in an unusual way.

We proceed to the study of the second term in Eq. (4). For this it is necessary to make a reasonable assumption on the properties of the Green's function, since it is not known. Our first assumption is that for the investigation of the analytical properties of the exchange amplitude, the most important are the collisions with small transfer of momentum in the first collision, i.e., such events in which the momentum $\mathbf{p}_1$ of the virtual electron in Figs. 1c and 1d is close to the momentum of the incoming electron $\mathbf{k}$. Subsequent calculation will confirm this assumption. The small momentum transfer corresponds to a large distance from the atom of the incoming electron. This means that after the first collision, the system does not undergo any significant changes, and we can speak separately of the incoming electron and the atom. Therefore, a very simple assumption on the intermediate state of the process, which is described; y the function G, is that the electron 1 (the "incoming" electron) propagates freely, while electron 2 (the "atomic" electron) is located in the field of the nucleus. It will be seen from further analysis at what points our second assumption will be needed.

Thus, according to the assumptions that have been made, in place of the exact Green's function, we shall use the function

$$G_1(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4; E)=(2\pi)^3\delta(\mathbf{p}_1-\mathbf{p}_3)G_q(\mathbf{p}_2, \mathbf{p}_4; E-\varepsilon_{p_1}), \tag{5}$$

which describes the propagation of the free electron $\mathbf{p}_1 - \mathbf{p}_3$ and of the second electron $\mathbf{p}_2 - \mathbf{p}_4$, interacting with the nucleus, which is taken into account by the Coulomb Green's function G_q. Substituting the Green's function (5) in (4), we obtain the following expression for the second term $\tilde{g}$ in Eq. (4)

$$\tilde{g}(k)=-\frac{1}{2\pi}\int\frac{4\pi}{(\mathbf{k}-\mathbf{p}_1)^2}\frac{4\pi}{(\mathbf{k}-\mathbf{p}_4)^2}[\varphi_0(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k})-\varphi_0(p_2)]\cdot \\ \cdot[\varphi_0(\mathbf{p}_1+\mathbf{p}_4-\mathbf{k})-\varphi_0(p_1)]G_q(\mathbf{p}_2, \mathbf{p}_4, \varepsilon_k+E_0-\varepsilon_{p_1})\frac{d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_4}{(2\pi)^9}. \tag{6}$$

Equation (6) corresponds to four graphs—corresponding to the number of possible first and subsequent collisions or, what amounts to the same thing, to the number of products of the wave functions from the square brackets. As an example, the graph corresponding to the product of the first terms in both brackets is shown in Fig. 1g. The double line in the intermediate state denotes the propagation function of the electron in the field of the nucleus with account of both the ground and excited states.

We now carry out a preliminary analysis of Eq. (6), in order to show what singularities of the amplitude $\tilde{g}(k)$ can be expected in principle on the left energy axis. For this purpose, we write out the expansion of G_q in the eigenfunctions of the Coulomb problem φ_i:

$$G_q(\mathbf{p}_2, \mathbf{p}_4; \varepsilon_k+E_0-\varepsilon_{p_1})=\sum_i\frac{\varphi_i(\mathbf{p}_2)\varphi_i^*(\mathbf{p}_4)}{\varepsilon_k+E_0-\varepsilon_{p_1}-E_i},$$

where the index i denotes the set of quantum numbers characterizing the state of both the discrete and the continuous spectra, the energy of which is equal to E_i. We consider the individual term of the sum corresponding to the discrete level with principal quantum number n, orbital l, and magnetic m. The integrand in (6) has many poles, coincidence of which can lead to the singularities of $\tilde{g}(k)$. We call attention to two of them: the poles of the wave function and of the Coulomb potential. The wave function of the discrete level $\varphi_{nlm}(\mathbf{p}_4)$ has a pole at $p_4^2=-1/n^2$, as follows from its explicit form, found by Fock;[13] the Coulomb potential gives a pole at $(\mathbf{k}-\mathbf{p}_4)^2=0$. According to the method of Landau,[14] their coincidence leads to singularities of the integral at $k^2=-1/n^2$, $n=1,2,\ldots$

Thus, as a result of the simplified treatment ignoring the continuous spectrum in the function G_q, we find that the amplitude $\tilde{g}(k)$ has singularities at the energies of the incoming electron, equal to the binding energies in the ground and excited states of the hydrogen atom. It will be seen that these singularities, are connected with cuts running along the entire left axis. Analytic continuation of the amplitude from the upper halfpine into the lower, which is important for obtaining the dispersion relation, turns out to be impossible in this case. However, as will be shown next, the account of the continuous spectrum leads to the result that only one singularity is present on the physical sheet, corresponding to the binding energy in the ground state of the hydrogen atom, while the singularities $k^2=-1/n^2$, $n\geq 2$ go over to the nonphysical sheet and do not affect the dispersion relation. The proof will be carried out in several stages.

1. Using the results of Schwinger,[15] we write out the expansion for the Coulomb Green's function in the form of a sum over the discrete states only; the continuous spectrum is taken into account here by the form of the terms:

$$G_q(\mathbf{p}_2, \mathbf{p}_4; \varepsilon_k+E_0-\varepsilon_{p_1})=-\frac{16p_0^4}{(p_2^2+p_0^2)(p_4^2+p_0^2)}\sum_{nlm}\frac{Y_{nlm}(\Omega)Y_{nlm}^*(\Omega')}{p_0-1/n} \tag{7}$$

In this expression p_0 is determined by the equality

$$p_0^2=-2(\varepsilon_k+E_0-\varepsilon_{p_1})=-k^2+1+p_1^2.$$

The quantity Y_{nlm} is a four-dimensional spherical function whose argument $\Omega(\Omega')$ depends on the quantities $p_2(p_4)$ and p_0. The explicit expressions for the Y_{nlm} will be insignificant. They are only important in that they do not cause singularities in the expression G_q at other points besides those determined by the zeroes of the denominator in (7). Substituting (7) in (6) we represent $\tilde{g}(k)$ in the form of a sum, each term of which is de-

termined by the corresponding term of the sum (7):

$$\tilde{g}(k)=\sum_{nlm} g_{nlm}(k).$$

2. We now investigate the analytical properties of the amplitude $g_{nlm}(k)$ and show that its singularities on the left axis are located only at $k^2=-1$ and at $k^2=-1/n^2$. The details are shown in Appendix I. We note only the general course of the discussions. According to the Landau method,[14] we need, for the study of the analytic properties of the integral, to put together a linear combination of all the denominators of the integrand, and to equate to zero their derivatives with respect to the integration variables. In the case considered, poles of the Coulomb potential, wave functions of the ground state, and poles of the component with indices nlm in formula (7) enter into this linear combination. The variables of integration in (6) are the three vectors and we obtain correspondingly three Landau vector equations. It is necessary to solve them simultaneously with the equations that determine the zeros of the denominator in the integrand of (6). As a result of the solution of this set of equations, we find that the singularities on the left axis can be located at $k^2=-1$ and $k^2=-1/n^2$.

We now show that the amplitude $g_{nlm}(k)$ is regular in some region on the real axis. According to Item 2, the singularities of $g_{nlm}(k)$ on the left axis can be located at $k^2=-1, k^2=-1/n^2$. These points can generally be branch points. The singularities of the integrand for $g_{nlm}(k)$ are such that they cannot intersect and "pinch" the multi-dimensional contour of integration over $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4$ at $k^2>-1/n^2$. Therefore, $g_{nlm}(k)$ is regular at least in the region $-1/n^2<k^2<0$.

4. From the regularity of the studied function on the segment of the real k^2 axis it follows that the boundaries of its analycity on the physical sheet are established by the Landau equations, in which the coefficients of the linear dependence should necessarily be non-negative. The derivation of this situation is contained, for example, in the book of Eden *et al.*[16] It is shown in Appendix I that the singularity of the amplitude at $k^2=-1$ satisfies this condition and consequently is present on the physical sheet, while the other singularities can appear only on the nonphysical sheets. Therefore the amplitude $g_{nlm}(k)$ on the left axis has a solitary singularity at the point $k^2=-1$ and is analytic in the region $-1<k^2<0$. We shall assume a sufficiently rapid convergence of the series for

$$\tilde{g}(k)=\sum_{nlm} g_{nlm}(k),$$

such that the analytic properties of the sum are determined by the properties of the components. Therefore, the basic conclusion from the given analysis is the following: the amplitude of $\tilde{g}(k)$ has a singularity at $k^2=-1$ and is regular at $-1<k^2<0$.

Now, in order to transfer this result to the exact amplitude $g(k)$, we shall trace out how the conditions which were discussed at the beginning of Sec. 2B are satisfied. It is shown in the Appendix I that the region of small transferred momenta of the incoming electron in the first stage of the scattering makes a contribution to the singularity of the amplitude (Fig. 1c,d), and this agrees with the first assumption that was made. However, for the onset of a singularity, the region of small transferred momenta of the departing electron in the last scattering stage is also important, (Figs 1e,f). But the small momentum transfer in the last collision corresponds to large distances from the atom of the departing electron 2 and before the last collision it should be free, while electron 1 should be situated in the Coulomb field of the nucleus. This contradicts the second assumption that was made. The small momenta transferred in the latter collisions leads to the necessity of taking into account the interaction with the nucleus of electron 2 after the first collision and of electron 1 before the last, as is shown in Fig. 1h. We limit ourselves to only this class of diagrams, including graph g. The sum of the entire "ladder" of such diagrams $\bar{g}(k)$ has the form

$$\bar{g}(k)=-\frac{1}{2\pi}\int\frac{4\pi}{(\mathbf{k}-\mathbf{p}_1)^2}\frac{4\pi}{(\mathbf{k}-\mathbf{p}_4)^2}[\varphi_0(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k})-\varphi_0(p_2)]$$
$$\times[\varphi_0(\mathbf{p}_3+\mathbf{p}_4-\mathbf{k})-\varphi_0(p_3)](\varepsilon_k+E_0-\varepsilon_{p_1}-\varepsilon_{p_4})G_0(\mathbf{p}_1,\mathbf{p}_3,\varepsilon_k+E_0-\varepsilon_{p_4})$$
$$\times G_0(\mathbf{p}_2,\mathbf{p}_4,\varepsilon_k+E_0-\varepsilon_{p_1})\frac{d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3\,d\mathbf{p}_4}{(2\pi)^{12}}. \qquad (8)$$

Analysis of the analytic structure of the integral (8) differs in no way, in principle, from the corresponding study of the integral (6); we therefore give only the final result.

The amplitude $\bar{g}(k)$: there is a singularity at $k^2=-1$ and it is regular in the region $-1<k^2<0$. Here, as is shown in Appendix II, the basic role in the formation of the singularity on the left axis is played by the region of small momentum transfers by the incoming and outgoing electrons. According to the discussions that have been given, the function $\bar{g}(k)$ in this region is a good representation for the exact amplitude $g(k)$ and therefore can be stated that $g(k)$ also has a singularity on the left E axis in the case of the binding energy of the electron in the hydrogen atom. It was also shown in Appendix II that the amplitude diverges logarithmically at the point $k^2=-1$, which is therefore a logarithmic branch point.

As has already been discussed, the cut can go only to the left and, since there are no other singularities on the left, it extends to $k^2=-\infty$. The logarithmic singularity of the amplitude is due to the coincidence of the three singularities of the integrand in (8): the poles of the Coulomb integraction in the first and last collisions and the pole that is present in both Coulomb Green's functions. It then follows that the appearance of this singularity, as well as of the multiple pole of the Born amplitude, is connected with the long-range character of the potential of the inter-particle interaction, which appears at large distances from the atom of electrons in the virtual state. The pole of the amplitude which describes the discrete state of the negative hydrogen ion is connected with the atomic distances at which all three particles interact simultaneously. This region is poorly taken into account in the function $\bar{g}(k)$; therefore, we must consider the pole separately.

We now turn to the dispersion relation (1). The meromorphism of the amplitude on the segment $-1<k^2<0$ of the real axis allows us to apply Cauchy's theorem to the

contour that encloses all the singularities of the amplitude, including the left cut. As a result, the dispersion relation takes the form

$$\bar{f}(E)=\bar{f}_B(E)+\frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\bar{f}(E')}{E'-E}dE'+\frac{d_i}{E-E_i}+\frac{1}{\pi}\int_{-\infty}^{E_0}\frac{-1/2\operatorname{Im}g(E')}{E'-E}dE', \tag{9}$$

where E_i is the binding energy of the electron in the negative hydrogen ion, E_0 is the electron energy in the hydrogen ground state. The quantity $\bar{f}(E)$, in accord with Ref. 1, is connected with the direct f-amplitude and the exchange g-amplitude by the equation

$$\bar{f}(E)=f(E)-1/2g(E).$$

Here we must note that the imaginary part of the amplitude on the right is a measurable quantity. It is connected with the total scattering cross section of the electron by the atom, while $\operatorname{Im}g$ on the left cut is not measurable.

3. SINGULARITIES OF THE FORWARD SCATTERING AMPLITUDE IN THE CASE OF A SHORT-RANGE POTENTIAL

We shall show that in the case of a short-range interaction potential, the exchange amplitude also has a singularity on the left axis. For this we consider hypothetical particles—an "electron" and a "proton," which are attracted to one another by a potential of the Yukawa type:

$$V(r)=-\int_\mu^\infty e^{-\mu' r}g(\mu')d\mu'. \tag{10}$$

The electrons repelled from one another by the potential $-V(r)$. We shall assume that there exists a bound state of electron and proton, which we shall call "hydrogen." We consider the exchange amplitude of the scattering of an electron by hydrogen. In lower orders of perturbation theory, it is depicted by the diagrams a and b of Figs. 1 and 2, on which the wavy line now denotes the potential (10).

The diagram a was considered by Chew and Low[17] in connection with nuclear scattering. Its contribution $g_{B,1}$, determined according to the rule given in Sec. 2A, is equal to

$$g_{B,1}(k)=-\frac{1}{2\pi}\frac{\langle \mathbf{k}|V|\varphi_0\rangle^2}{-\varepsilon_k+E_0},$$

where V is the potential (10), φ_0 and E_0 are the wave function and the energy of the ground state of the hydrogen. The function $g_{B,1}(k)$ has a pole at $k^2=-\varkappa^2$, $\varkappa=(2|E_0|)^{1/2}$, which corresponds to the virtual state shown by the dash-dot curves in Fig. 2a.

The other singularities of $g_{B,1}(k)$ are connected with the form factor

$$\langle \mathbf{k}|V|\varphi_0\rangle=\int e^{-i\mathbf{k}\mathbf{r}}V(r)\varphi_0(r)d\mathbf{r}. \tag{11}$$

The singularities of the Fourier transform are determined by its asymptote in the coordinate representation. The quantity $\varphi(r)$ falls off exponentially as a function of the bound state:

$$\varphi_0(r)\to\alpha(r)e^{-\varkappa r},\quad r\to\infty;$$

it follows from (10) that

$$V(r)\to\beta(r)e^{-\mu r},\quad r\to\infty;$$

$\alpha(r)$ and $\beta(r)$ are the factors of the exponentials. Substituting the asymptotic forms of $\varphi_0(r)$ and $V(r)$ in (11), we find that the form factor $\langle \mathbf{k}|V|\varphi_0\rangle$ has a singularity at $k^2=-(\varkappa+\mu)^2$. According to (10), the potential $V(r)$ can contain terms that fall off as $\exp(-\mu' r)$, $\mu'>\mu$. They lead to a singularity of the form factor at $k^2=-(\varkappa+\mu')^2$, i.e., $\langle \mathbf{k}|V|\phi_0\rangle$ and also $g_{B,1}(k)$ have a cut running from $k^2=-(\varkappa+\mu)^2$ to $k^2=-\infty$.

We show that this singularity is connected with the triangular diagram. For this purpose, we note that the form factor contains all the interaction of the electron with the proton. If one of the scattering acts of these particles is considered separately, then the form factor is shown by the diagram of Fig. 2c. It is expressed in terms of the integral over the transferred momentum q. The integrand has singularities at $(\mathbf{k}-\mathbf{q})^2+\varkappa^2=0$ and $q^2+\mu^2=0$, connected with the virtual state and denoted by the dash-dot line, and with the singularity of the Fourier transform of the potential (10). The coincidence of these singularities also leads to the branch point of the form factor at $k^2=-(\varkappa+\mu)^2$, which is thus connected with all three virtual "particles" in Fig. 2c. A characteristic property of the given singularity is the fact that it is not connected with the creation of real particles, in other words, the diagram of Fig. 2c does not have a cross section for which the energy denominator would vanish at the branch point. Such singularities are known as anomalous.

Similar consideration shows that the diagram of Fig. 2b also has an anomalous cut at $k^2=-(\varkappa+\mu)^2$.

We thus see that the scattering amplitude in the first Born approximation has a pole at $k^2=-\varkappa^2$ and a cut to the left from $k^2=-(\varkappa+\mu)^2$. As μ decreases, the cut approaches the physical region and transforms into a multiple pole for the Coulomb potential. More complicated diagrams can also have anomalous singularities.

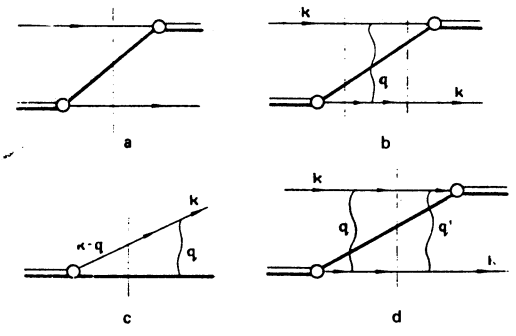

FIG. 2.

But, for short-range potentials, when $\mu \gg \varkappa$, it is usually assumed that the anomalous thresholds of the complex graphs exist far from the physical region, just as for the Born amplitude and do not make a contribution to the dispersion relation. For example, the graph of Fig. 2d has an anomalous singularity at $k^2 = -[2\mu + (2\mu^2 + \varkappa^2)^{1/2}]^2$, connected with the pole of the observed virtual state and the poles of the potential. For the Coulomb potential, as shown in Sec. 2, the situation is different: the left cut adjoins the physical region.

4. CONCLUSION

We have shown that the exchange amplitude of elastic forward scattering of the electron by hydrogen has an additional cut on the left energy axis from $k^2 = -\infty$ to $k^2 = -1$. It is regular on the interval $-1 < k < 0$, excluding the pole corresponding to the negative hydrogen ion. The latter conclusion is identical with the result of Tip.[11] In addition to the singularity at $k^2 = -1$, there are singularities at $k^2 = -1/n^2, n = 2, 3, \ldots$.

However, all the singular points corresponding to the excited states lie on the nonphysical sheets and therefore do not contribute to the dispersion relation. This is confirmed indirectly by the fact that the difference between the left and right sides of (1), according to Hutt *et al*,[4] does not have a fine structure and the scale of its change corresponds to the ionization potential of the atom. It is shown that as $k^2 \to -1$, the amplitude diverges logarithmically; therefore, the contribution of the left cut cannot be regarded as a small quantity. With this account, the dispersion relation takes on the form (9). However, analysis of such an equation, based on experimental data, becomes impossible, since the imaginary part of the amplitude on the left cut is immeasurable. There are no numerical calculations of the contribution of this singularity at the present time.

The considered singularity of the amplitude on the left axis is connected with the long-range character of the Coulomb forces, which appears in the region of distances from the nucleus of electrons in the intermediate state, distances that are large in comparison with the Bohr radius. One of the electrons is assumed to be much further from the nucleus than the other. In the scattering of the electron by a complex atom, purely Coulomb forces also operate on the complex atom in a similar region. We can therefore assume that in scattering from a complex atom, the exchange amplitude has a singularity on the left axis at the energy of the incoming electron, equal to the binding energy of the atom. Other, so called physical singularities of the amplitude were considered in Ref. 9.

The demonstrated singularity of the amplitude can also appear in an important equation—the Levinson theorem. Actually, its proof is based on the analytical properties of the partial amplitudes, but is natural to assume that the singularity of the forward scattering amplitude also appears in the partial amplitudes.

In conclusion, the authors express their gratitude to V. G. Gorshkov and Yu. N. Demkov for constructive remarks and interest in the work, and to G. M. Shklyarevskii for many useful discussions.

APPENDIX I

SINGULARITIES OF THE AMPLITUDE OF g_{nlm}

The singularity of the integrand expression for the amplitude of g_{nlm}, as follows from Eqs. (6) and (7), are determined by the poles of the functions S_i, which are equal to

$$\begin{aligned} &S_1=(\mathbf{k}-\mathbf{p}_1)^2, \quad S_2=(\mathbf{k}-\mathbf{p}_4)^2, \quad S_3=(\mathbf{p}_1+\mathbf{p}_4-\mathbf{k})^2+1, \\ &S_4=(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k})^2+1, \quad S_5=p_1^2-k^2+1-1/n^2, \\ &S_6=p_1^2+p_4^2-k^2+1, \quad S_7=p_1^2+p_2^2-k^2+1. \end{aligned} \tag{I.1}$$

(We consider the product of only the first terms of each of the square brackets in (6).)

According to the general Landau method,[14] to find the singularities of the integral, it is necessary to solve the equations

$$S_i=0 \quad \text{or} \quad \alpha_i=0, \quad i=1, 2, \ldots, 7 \tag{I.2}$$

together with the Landau equations

$$\text{a) } \partial S/\partial \mathbf{p}_1=0, \quad \text{b) } \partial S/\partial \mathbf{p}_2=0, \quad \text{c) } \partial S/\partial \mathbf{p}_4=0, \tag{I.3}$$

in which

$$S=\sum_{i=1}^{7}\alpha_i S_i$$

is a linear combination of S_i with the coefficients α_i.

We first consider the equation (I.3c), with which the p_4-dependent functions S_2, S_3, S_6 are connected. We denote

$$\begin{aligned} &\mathbf{q}_1=\mathbf{k}-\mathbf{p}_4, \quad \mathbf{q}_2=\mathbf{k}-\mathbf{p}_4-\mathbf{p}_1, \quad \mathbf{q}_3=-\mathbf{p}_4, \\ &\beta_1=\alpha_2, \quad \beta_2=\alpha_3, \quad \beta_3=\alpha_6. \end{aligned}$$

In this notation, Eq. (I.3c) has the form

$$\sum_{i=1}^{3}\beta_i\mathbf{q}_i=0,$$

whence it follows that

$$\det(\mathbf{q}_i\mathbf{q}_j)=0. \tag{I.4}$$

Expressing $\mathbf{q}_i \cdot \mathbf{q}_j$ in terms of $\mathbf{k}$ and $\mathbf{p}_1$ from (I.2), and substituting in (I.4), we get

$$S_8=p_1^2+1=0.$$

Similarly, (I.3b) jointly with (I.2) leads to the equation

$$S_9=(\mathbf{k}-\mathbf{p}_1)^2+[1+(p_1^2-k^2+1)^{1/2}]^2=0.$$

The functions S_8 and S_9 take into account all the singularities of the amplitude of $g_{nlm}(k)$, which arise from the zeros of the functions S_2, S_3, S_4, S_6, S_7. Therefore the singularities of the amplitude stem from the simultaneous vanishing of the function S_1, S_5, S_8, S_9, which depend

on the single variable p_i. The functions S_1, S_8 are analogous to functions which lead to the singularities of the Born amplitude.[7] The coincidence of their zeros leads to a singularity of the function $g_{nlm}(k)$ located at $k^2=-1$. Another singularity occurs from the simultaneous vanishing of the three functions: $S_5=S_8=S_9=0$.

Solving these equations, we find the position of the singularity: $k^2=-1/n^2$. However, the Landau equation

$$\frac{\partial}{\partial \mathbf{p}_i}(\alpha_5 S_5+\alpha_8 S_8+\alpha_9 S_9)=0$$

leads to the equality

$$\alpha_5+\alpha_8/n^2+\alpha_9(2\pm 1/n)=0,$$

which is unsolvable at positive $\alpha_5, \alpha_8, \alpha_9$. Reviewing the other possibilities such as the vanishing of some α_i in Eqs. (I.2) or the product of the other factors from the square brackets in (6), we verify that the amplitude $g_{nlm}(k)$ does not have singularities at other points on the left axis.

APPENDIX II

LOGARITHMIC SINGULARITY OF $g(k)$

The proof that shows that the integral (8) has a singularity only at $k^2=-1$, is similar to that given for (6). We shall show that the simultaneous vanishing of the three functions

$$S_1=(\mathbf{k}-\mathbf{p}_1)^2, \quad S_2=(\mathbf{k}-\mathbf{p}_4)^2, \quad S_3=p_1^2+p_4^2-k^2+1 \tag{II.1}$$

leads to a logarithmic divergence of $\bar{g}(k)$ at $k^2=-1$. For this purpose, we note that as $S_3\to 0$ the singularities of the function $G_q(\mathbf{p}_1, \mathbf{p}_3, \epsilon_k+E_0-\epsilon_{p4})$ are similar to the singularities of the function G_0:[15]

$$G_0(\mathbf{p}_i, E)=\frac{1}{E-\varepsilon_{p_i}}\exp\left[-i\eta\frac{E-\varepsilon_{p_i}}{4E}\right]\left(\frac{2\pi\eta}{e^{2\pi\eta}-1}\right)^{1/2}, \tag{II.2}$$

where

$$E=\epsilon_k+E_0-\epsilon_{p4}, \quad \eta=(2E)^{-1/2}.$$

From (II.2), we find the estimate

$$G_q(\mathbf{p}_1, \mathbf{p}_3, \varepsilon_k+E_0-\varepsilon_{p_1})\to\frac{f(\mathbf{p}_3)}{(p_1^2+p_4^2-k^2+1)^2(k^2-p_4^2)^{1/2}} \text{ for } p_1^2, p_4^2, k^2\to -1. \tag{II.3}$$

Similarly, we find for the second Green's function

$$G_q(\mathbf{p}_2, \mathbf{p}_4, \varepsilon_k+E_0-\varepsilon_{p_1})\to\frac{f(\mathbf{p}_2)}{(p_2^2+p_4^2-k^2+1)^2(k^2-p_1^2)^{1/2}}. \tag{II.4}$$

We substitute (II.3), (II.4) in the integral (8) and make the change of variables

$$|\mathbf{k}-\mathbf{p}_1|=q, \quad |\mathbf{k}-\mathbf{p}_4|=qx.$$

We obtain the result that $g(k)$, as $k^2\to -1$ diverges as

$$\bar{g}(k)=C\int\frac{dq}{q}=C\ln q, \quad q\to 0,$$

which also indicates a logarithmic singularity.

[1]E. Gerjuoy, Phys. Rev. **109**, 1806 (1958).
[2]E. Gerjuoy and N. A. Krall, Phys. Rev. **119**, 705 (1960).
[3]F. W. Byron, Jr., F. J. de Heer and C. J. Jochain, Phys. Rev. Lett. **35**, 1147 (1975).
[4]P. K. Hutt, M. M. Islam, A. Rabheru and M. R. C. McDowell, J. Phys. B: Atom. Molec. Phys. **9**, 2447 (1976); F. J. de Heer, M. R. C. McDowell and R. W. Wagenaar, J. Phys. B: Atom. Molec. Phys. **10**, 1945 (1977).
[5]B. H. Bransden and P. K. Hutt, J. Phys. B: Atom. Molec. Phys. **8**, 603 (1975).
[6]P. G. Coleman, T. C. Griffith, R. R. Heyland and T. L. Killeen, Electron and Photon Interactions with Atoms; H. Kleinpoppen and M. R. C. McDowell, Eds., New York, 1976, p. 181.
[7]M. Ya. Amus'ya and M. Yu. Kuchiev, ZhTF Pis. Red. **3**, 440 (1977) [Sov. Phys. Tech. Phys. Lett. **3**, 178 (1977)].
[8]M. N. Rubin, R. L. Sugar and G. Tiktopoulos, Phys. Rev. **162**, 1555 (1967).
[9]R. D. Peccei and R. D. Violier, Ann. Phys. **103**, 29 (1977).
[10]A. Tip, J. Phys. B: Atom. Molec. Phys. **10**, L11 (1977).
[11]A. Tip, J. Phys. B: Atom. Molec. Phys. **10**, L295 (1977).
[12]V. G. Gorshkov and S. G. Sherman, ZhETF Pis. Red. **17**, 519 (1973) [JETP Lett. **17**, 374 (1973)].
[13]V. Fock, Z. Physik **98**, 145 (1935).
[14]L. D. Landau, Nuclear Phys. **13**, 181 (1959).
[15]J. Schwinger, J. Math. Phys. **5**, 1606 (1959).
[16]R. J. Eden, P. V. Landshoff, D. I. Olive, H. C. Polkinghorne, The Analytic S Matrix, Cambridge, England, 1966, p. 66.
[17]G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

Translated by R. T. Beyer