Gravitational waves and the limiting stability of self-excited oscillators

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For observation of low frequency gravitational radiation bursts arising from the collapse of galactic nuclei or the formation of black holes in the pre-galactic era of the universe, self-excited electromagnetic oscillators are required which possess a relative frequency instability of less than $1\cdot 10^{-18}$. The instability can be reduced by coupling the oscillator to a high Q cavity which is thermally stabilized at a temperature below 4°K. An expression for the lowest frequency instability at an optimal power can be obtained by taking into account the thermodynamic fluctuations of the cavity, the shot thermal effect, the shot ponderomotive effect, and also purely quantum self-oscillation fluctuations. Numerical estimates indicate that a relative instability level of $1\cdot 10^{-20}$ can be attained.

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There are two currently known astrophysical scenarios that should result in comparatively frequent bursts of long wavelength gravitational radiation with a duration τ_{gr} between 10^2 and 10^5 sec arriving in the region of the Earth.[1,2] The sources of the bursts must be the asymmetric collapse of galactic nuclei at a relatively late cosmological era, and also the formation of small black holes in the pre-galactic era of the universe (when its age was $\approx 10^6-10^8$ years). According to these scenarios, the bursts of gravitational radiation must perturb the metric of space with an amplitude h ranging from 10^{-18} to 10^{-15}, and the frequency of the bursts may exceed one per day. These scenarios have provided the justification for a program to look for such bursts using heliocentric satellites.[3,4] At an Earth—satellite distance $L \gtrsim c\tau_{gr}$, the variations of the metric must lead to variations in the frequency of an electromagnetic signal transmitted from a satellite to the Earth (or vice versa) of relative magnitude $(\Delta\omega/\omega)_{gr} = h$. It is obvious that a key problem in the realization of such a program is the development of a self-excited oscillator whose frequency instability $\Delta\omega/\omega$ over the time τ_{gr} is appreciably smaller than the expected value of h.

To stabilize microwave oscillators, Khaĭkin[5] proposed that in them one should use cavities made of superconducting materials placed in a thermal bath with a subcritical temperature. Development of this method recently enabled Turneaure and Stein[6] to achieve an instability level $\Delta\omega/\omega = 6\cdot 10^{-16}$ over 10^3 sec at a temperature around 1.9 °K. To estimate the instability level that could in principle be achieved in this method, the following simple arguments are usually used: the cavity should be made of a dielectric material with the highest possible Debye temperature and small microwave losses,[7,8] and it should be covered with a layer of a superconductor with the highest possible critical temperature. Then when it is cooled to ~1.5 °K, the influence of the temperature dependence of the impedance of the superconductor on the cavity frequency will be small and the main contribution to the frequency instability will be made by the temperature instability δT due to the finite coefficient of linear thermal expansion α. The eigenfrequency ω_e of a chosen mode of the cavity must then vary by a relative amount equal to

$$\Delta\omega_e/\omega_e = \alpha\delta T. \tag{1}$$

For sapphire, $\alpha = 7\cdot 10^{-13}$ deg^{-1} at $T = 1.7$ °K; setting $\delta T = 10^{-6}$ °K, we can reckon to achieve an instability level $\Delta\omega_e/\omega \approx 10^{-18}$. On the other hand, a cryogenic technique was developed a comparatively long time ago that makes it possible to maintain the temperature of a relatively large mass around $5\cdot 10^{-2}$ °K with a continuous removal of heat at the rate ~10^4 erg/sec. And since α decreases as T^3, one could obtain at least in principle much smaller values for $\Delta\omega_e/\omega_e$.

The aim of this paper is to show that there is a limit to the attainable instability $\Delta\omega/\omega$ for self-excited oscillators of a fairly general form.

We consider first the instability of the eigenfrequency of electromagnetic oscillations of a cavity made from a dielectric of volume V covered with a superconducting layer. Suppose that this volume has a fairly good thermal contact with the thermal bath, so that the time τ_T^* of thermal relaxation is short (if the cavity has the shape of a cube, one of whose faces touches the thermal bath, then $\tau_T^* \approx V^{1/3}/v$, where v is the velocity of sound). In this case, the thermodynamic fluctuations of the temperature[9]

$$\Delta T^2_{\text{thermod}} = \varkappa T^2/C_V V$$

(C_V is the specific heat of the dielectric) must determine the level of the attainable instability $\Delta\omega_e/\omega_e$. Bearing in mind that $\alpha \approx \gamma C_V/3E$ (where E is Young's modulus for the dielectric and γ is the Grüneisen constant), and

$$C_V = \frac{2\pi^2}{5}\frac{\varkappa^4 T^3}{(\hbar v)^3}$$

(see, for example,[9]), then with allowance for thermodynamic fluctuations and an averaging time $\hat{\tau} \gg \tau_T^*$ we find that the instability of the mode eigenfrequency of the cavity is

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$$\left(\frac{\Delta\omega_e}{\omega_e}\right)_{\text{thermod}} \approx \frac{\gamma}{3EV}\left[\frac{2\pi^2}{5}\frac{(\varkappa T)^5}{(\hbar v)^3}V\frac{\tau_T^*}{\hat{\tau}}\right]^{1/2}. \quad (2)$$

On the other hand, during the operation of the oscillator, some of its total power W must be dissipated in the cavity, which raises the mean temperature in the cavity and in the adjoining region of the thermal bath to some level T_{stat}. If $T_{\text{stat}} > 2T$, and the power dissipated in the cavity is approximately equal to the total power, then Eq. (2) takes the form

$$\left(\frac{\Delta\omega_e}{\omega_e}\right)_{\text{thermod}} \approx \frac{2\gamma}{3EV}\left[\frac{\varkappa T_{\text{stat}}W(\tau_T^*)^2}{\hat{\tau}}\right]^{1/2}, \quad (3)$$

$$(\varkappa T_{\text{stat}})^4 \approx \frac{10}{\pi^2}\frac{W\tau_T^*(\hbar v)^3}{V}. \quad (4)$$

Hitherto, we have assumed that the fluctations of the temperature due to the mean temperature T_{stat} are continuous. But the absorption of every quantum $\hbar\omega_e$ in the cavity from the general flux of power W gives rise to jumps in the temperature of a "shot" kind (shot thermal effect). Since the spectral density of the quantum fluctuations of the power is equal to $2\hbar\omega_e W$, the fluctuations of the temperature ΔT_{ST} can be written as

$$\Delta T_{\text{ST}} \approx \frac{1}{C_V V}\left[\frac{2\hbar\omega_e W(\tau_T^*)^2}{\hat{\tau}}\right]^{1/2}. \quad (5)$$

Substituting this expression in (1) and using the connection between C_V and α, we obtain a formula analogous to (3) for the shot thermal effect causing fluctuations in the eigenfrequency of the cavity mode:

$$\left(\frac{\Delta\omega_e}{\omega_e}\right)_{\text{ST}} \approx \frac{\gamma}{3EV}\left(\frac{2\hbar\omega_e W(\tau_T^*)^2}{\hat{\tau}}\right)^{1/2}. \quad (6)$$

It is obvious that when $\varkappa T_{\text{stat}} < \frac{1}{2}\hbar\omega_e$ the second effect will be predominant.

We now consider the role of the self-oscillation fluctuations. The coefficient in the diffusion law of the phase drift of the self-excited oscillator depends strongly on the method of regeneration. If it is assumed that the fluctuations of the frequency are due solely to the zero-point fluctuations of the field, and accordingly $\varkappa T_{\text{stat}} < \frac{1}{2}\hbar\omega_e$, then one can use the well-known expression of Townes (for example, see[10])

$$(\Delta\omega/\omega)_{\text{self-osc}} = (\hbar\omega_e/4WQ_e^2\hat{\tau})^{1/2}, \quad (7)$$

where Q_e is the electric Q of the cavity, and the mean frequency of the self-oscillations is $\bar{\omega} = \omega_e$. It is important to note that the derivation of (7) presupposes that ω_e is strictly constant.

Comparison of (6) and (7) shows that from the point of view of raising the frequency stability they impose opposite requirements on the power W. Regarding (6) and (7) as equations, we obtain the optimal power,

$$W_{\text{opt}} = \frac{3}{2\sqrt{2}}\frac{EV}{\gamma Q_e \tau_T^*}, \quad (8)$$

at which the smallest frequency instability is attained:

$$\left(\frac{\Delta\omega}{\omega}\right)_{\min} \approx \left[\frac{\gamma\sqrt{2}}{3}\frac{\hbar\omega_e}{EVQ_e}\frac{\tau_T^*}{\hat{\tau}}\right]^{1/2}. \quad (9)$$

If $W > W_{\text{opt}}$, the instability is basically determined by Eq. (6), and if $W < W_{\text{opt}}$ by Eq. (7), and in both cases it is greater than $(\Delta\omega/\omega)_{\min}$.

As can be seen from (9), the frequency stability of the self-excited oscillator increases with decreasing time τ_T^*. It was assumed in the derivation of this formula that the only source of frequency fluctuations is the shot heating of the cavity due to the absorption in it of photons. However, if the Q factor $Q_e = \omega_e\tau_e^*/2$ is sufficiently high, there will be another effect—the change in the cavity eigenfrequency ω_e as a result of the ponderomotive effect. The point is that the force of the radiation pressure in the cavity, which changes its size (and therefore ω_e as well), increases with increasing Q_e:

$$F \approx WQ_e/2\omega_e l = \bar{N}\hbar\omega_e/2l; \quad (10)$$

here, $\bar{N}$ is the mean number of photons in the cavity and l depends on the shape of the cavity and the chosen mode (in the simplest case, the fundamental mode of the cube with $l \approx V^{1/3}$).

The fluctuations F due to the discrete absorption and emission of photons also lead to fluctuations of ω_e (shot ponderomotive effect). Since

$$\Delta F \approx \frac{\hbar\omega_e\Delta\bar{N}}{2l} = \frac{\hbar\omega_e}{2l}\left(\bar{N}\frac{(\tau_e^*)^2}{\hat{\tau}}\right)^{1/2}$$

for $\hat{\tau} \gg \tau_e^*$ and $\Delta\omega_e/\omega_e \approx \Delta l/l = -\Delta F/ES$,

$$\left(\frac{\Delta\omega_e}{\omega_e}\right)_{\text{SP}} \approx \frac{1}{4EV}\left(\frac{2W\hbar\omega_e(\tau_e^*)^2}{\hat{\tau}}\right)^{1/2}. \quad (11)$$

Comparing (6) and (11) and bearing in mind that $\gamma = 1$ at low temperatures, we see that the shot ponderomotive effect is predominant for $\tau_e^* > \tau_T^*$.

Thus, we can use (8) and (9) only for $\tau_T^* \gg \tau_e^*$. Assuming that the opposite inequality is satisfied ($\tau_e^* \gg \tau_T^*$), we can in the same way as for the shot thermal effect use (7) and (11) and obtain the smallest frequency instability of the oscillator attainable at the optimal power:

$$\left(\frac{\Delta\omega}{\omega}\right)_{\min} \approx \left(\frac{\hbar\omega_e\tau_e^*}{2\sqrt{2}EVQ_e\hat{\tau}}\right)^{1/2} = \left(\frac{\hbar}{\sqrt{2}EV\hat{\tau}}\right)^{1/2}, \quad (12)$$

$$W_{\text{opt}} = EV\omega_e/\sqrt{2}Q_e^2. \quad (13)$$

The condition $\varkappa T_{\text{stat}} < \frac{1}{2}\hbar\omega_e$ with allowance for the expressions (4) and (13) can be rewritten in the form

$$\omega_e > \left[\frac{E\tau_T^* v^3}{\hbar Q_e^2}\frac{5\sqrt{2}}{\pi^2}\right]^{1/5}. \quad (14)$$

Calculations show that if in addition to quantum fluctuations there are other sources of noise in the oscillator (thermal, electron shot noise), then other expressions can be obtained similarly for $(\Delta\omega/\omega)_{\min}$, but these always lead to larger values of the smallest instability for the same E, V, and $\hat{\tau}$.

To obtain estimates, let us set $E = 4\cdot10^{12}$dyn/cm^2 (sapphire), $V = 1$ cm^3, $\omega_e = 6\cdot10^{10}$ rad/sec, $Q_e = 1\cdot10^{10}$ in (12) and (13). Then

$$\left(\frac{\Delta\omega}{\omega}\right)_{min} \approx 1\cdot10^{-20}(\hat{\tau})^{-1/2}, \quad W_{opt} \approx 2\cdot10^{3}\ \text{erg/sec}.$$

These estimates show that the problem of measuring $h \approx 10^{-18}$ can in principle be solved. A high frequency stability of an electromagnetic oscillator is required not only for the gravitational experiments we have described. Therefore, the limiting relations may also be helpful for planning other high-precision physics experiments.

Note added in proof (January 12, 1978). Equation (12) can be obtained on the basis of general arguments (as has been pointed out by Yu. I. Vorontsov): the relative error $\Delta\omega/\omega$ is equal to $\Delta l/l$; for continuous measurement of the coordinate of a mechanical oscillator the smallest error is $\Delta l_{min} = (\hbar/m\omega_M)^{1/2}(1/\omega_m\hat{\tau})^{1/2}$ (where m and ω_M are the mass and frequency of the oscillator). Taking into account only the fundamental mode of the mechanical vibrations of the cavity and expressing m and ω_M in terms of E and V, we readily obtain $\Delta\omega/\omega = \Delta l/l \approx (\hbar/EV\hat{\tau})^{1/2}$.

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Scale-invariant solutions in the hydrodynamic theory of multiple production

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The scale-invariant solutions in the hydrodynamic theory of multiple production are considered. The question of the behavior of a hadron system at points on its boundary with a vacuum is investigated. It is shown that the requirement of conservation of the energy and the momentum of the system indicates the necessity for the introduction of particle-like states at the periphery of the hadron liquid. These states are identified with the leading particles. The solutions to the equations of motion are found and physically analyzed.

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1. INTRODUCTION

Recently there has been a marked increase in interest in the study of the space-time picture of the processes of multiple production. In particular, in the investigation of high-energy hadron-nucleus collisions the question of the temporal evolution of the hadron systems turns out to be closely associated with the experimentally observable characteristics. In view of this, of special interest is an in-depth analysis of the hydrodynamic theory of high-energy collisions—in essence the only model that allows a detailed space-time description of the process of multiple production of particles.

One of the main achievements of the hydrodynamic approach is the fairly successful explanation of the experimental data on the structure of the secondary particles and their transverse-momentum distribution. These results, which justify the basic idea that a hydrodynamic phase exists in the course of multiple generation of particles, are insensitive to the specific choice of the equation of state of the hadron liquid and the initial conditions for its expansion. Let us recall that in the basic paper by Landau[1] the initial conditions were chosen to correspond to a homogeneous quiescent liquid in a Lorentz-contracted disk with transverse dimensions $l_\perp \approx 1/\mu$ and longitudinal dimensions $l_\parallel \sim 1/E_0$ (μ is the π-meson mass and E_0 is the total energy of the colliding particles in the CM system), while the equation of state was chosen in the form $p = \epsilon/3$ (p is the pressure and ϵ is the energy density). Khalatnikov[2] has obtained an exact analytic solution to the problem of the one-dimensional motion with Landau's initial conditions.[1]

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