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Higher orders of perturbation theory and summation of series in quantum mechanics and field theory

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The Borel method of summation of a perturbation-theory series with factorially increasing coefficients is considered. The connection between the asymptotic form of the coefficients a_k of this series for $k\to\infty$ and the nature of the singularity of the sum is established. An improved perturbation theory is constructed and the limits of its region of applicability are found. The results obtained are verified for a number of physical problems (the Lagrange function in the nonlinear electrodynamics of the vacuum, the energy levels of an electron in the Coulomb field of a nucleus with $Z>137$, the screening of the nuclear charge by the vacuum shell of a supercritical atom, and the Stark effect in the hydrogen atom) for which the coefficients of the perturbation-theory series increase factorially and for which, at the same time, (analytically or numerically) exact solutions are known. Application of the improved perturbation theory to the $g\phi^4/4!$ scalar field theory makes it possible to establish the behavior of the Gell-Mann–Low function $\psi(g)$ for $0<g/16\pi^2\lesssim 0.3$. In this interval $\psi(g)$ is a monotonically increasing function of the coupling constant g.

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1. INTRODUCTION AND FORMULATION OF THE PROBLEM

In recent years effective ways of calculating higher orders of perturbation theory (PT) in quantum mechanics and statistical physics have been found;[1-4] the structure of the PT series for the energy levels of the anharmonic oscillator has been investigated in particular detail.[2-4] In quantum field theory Lipatov[5,6] has developed a semi-classical method of calculating the functional integral, in which an important role is played by the classical solutions of the field equations, and has applied this technique to the renormalizable scalar field theory with interaction

$$H_{int}=g\int\varphi^n d^Dx/n!,\quad D=2n/(n-2).$$

In Ref. 5 the case $n\to\infty$ was considered and the first term of the $1/n$ expansion for the Gell-Mann-Low function (GLF) $\psi(g)$ was found; in Ref. 6 the asymptotic form, for $k\to\infty$, of the coefficients of the PT series

$$\psi(g)=\sum_{k=2}^{\infty}a_k(-g)^k$$

was calculated for arbitrary n. Intensive developments are being made in this direction at the present time.[7-16] Lipatov's method has been applied to the following problems: the n-dimensional anharmonic oscillator[7]

$$H=\frac{1}{2}\sum_{i=1}^{n}(p_i^2+x_i^2)+g\left(\sum_{i=1}^{n}x_i^2\right)^N,$$

the scalar theory with internal-symmetry group $O(n)$, where

$$H_{int}=\frac{g}{(2N)!}\int d^Dx\left(\sum_{i=1}^{n}\varphi_i^2\right)^N,\quad D=\frac{2N}{N-1},$$

the theory of fermions with a Yukawa interaction,[8] scalar electrodynamics[12] and reggeon field theory in the strong-coupling region.[13] The calculations of the asymptotic forms of PT series in more-realistic field-theory models (quantum electrodynamics, Yang-Mills theory, and so forth) await their turn. In this connection, the following question becomes urgent: what information can be obtained about the behavior of the exact solutions if we know the first few coefficients a_k of the PT series and their asymptotic form as $k\to\infty$? The present article is devoted to elucidating this question.

Let $f(z)$ be a function representable by a divergent power series

$$f(z)=\sum_{k=k_0}^{\infty}a_k(-z)^k,\tag{1}$$

that is asymptotic for $z\to 0$ (in field-theory problems the variable z is usually the coupling constant, but it can also have another physical meaning—see Sec. 5). In most of the theories considered[2-13] the asymptotic form of a_k is

$$a_k=(k\alpha)!\,a^kk^\beta\left(c_0+\frac{c_1}{k}+\frac{c_2}{k^2}+\ldots\right),\quad k\to\infty,\tag{2}$$

where $z!\equiv\Gamma(z+1)$, and α, β, a and c_i are calculable constants ($\alpha, a>0$). For a one-dimensional anharmonic oscillator with nonlinearity gx^4 ten coefficients c_i in the expansion (2) have been found by numerical methods.[3] In field theory, however, the calculation of these coefficients is a more complicated problem, and up to now it has been possible to obtain only the first term of the series (2), which we denote by $\tilde{a}_k$:

$$\tilde{a}_k=c_0(k\alpha)!a^k k^\beta. \tag{3}$$

Knowledge of the first N coefficients a_k and of the asymptotic form (3) makes it possible to construct not only the PT polynomials $p_N(z)$ but also the improved perturbation theory (IPT) functions $f_N(z)$:

$$p_N(z)=\sum_{k=k_0}^{k_0+N-1} a_k(-z)^k,\quad f_N(z)=\tilde{f}(z)+\sum_{k=k_0}^{k_0+N-1}(a_k-\tilde{a}_k)(-z)^k. \tag{4}$$

Here,

$$\tilde{f}(z)=\sum_{k=k_0}^{\infty}\tilde{a}_k(-z)^k$$

is the sum of the series (1) with the asymptotic coefficients $\tilde{a}_k$ (in view of the factorial increase of $\tilde{a}_k$ as $k\to\infty$ this sum must be understood in the generalized sense, applying one of the methods of summation of divergent series).[17] The first N coefficients in the function $f_N(z)$ coincide with the exact coefficients, and the distant "tail" of the PT series is also taken into account. Therefore, it is natural to expect that as N increases the functions $f_N(z)$ will give a better approximation to the exact solution than will the PT polynomials $p_N(z)$, and will enable us to establish it in a wider range of z. These qualitative arguments can be given a more exact meaning, and this will be done below.

We shall describe the content of the paper (a brief account of the results obtained was published earlier).[16] In Sec. 2 the series (1) with coefficients of the form (3) is summed using one of the variants of the Borell method; the sum $\tilde{f}(z)$ is obtained in explicit form. In Sec. 3 the connection is found between the asymptotic form of a_k as $k\to\infty$ and the character of the singularity of the sum at the point $z=0$. In Sec. 4 the limits of the region of applicability of the IPT are found. These results are verified in Sec. 5 for several examples for which the coefficients of the PT series increase factorially and for which, at the same time, exact solutions are known. In the concluding section (Sec. 6) the IPT method is applied to the $g\varphi^4/4!$ scalar field theory, and enables us to determine the GLF $\psi(g)$ in the interval $0<g/16\pi^2<0.3$.

Appendix A contains a summary of the formulas that make it possible to sum series of the form (1) with coefficients of the form (2). Appendix B contains a comparison of the analytic properties of the exact solution $f(z)$ and the function $\tilde{f}(z)$, which is the Borel sum of the series (1) with the asymptotic coefficients $\tilde{a}_k$.

2. THE BOREL METHOD OF SUMMATION

This method places the divergent series (1) in correspondence with the generalized sum

$$f(z)=\int_0^\infty e^{-t}t^\nu\varphi_{\mu\nu}(z,t)\,dt,\quad \varphi_{\mu\nu}=\sum_{k=k_0}^{\infty}\frac{a_k(-zt^\mu)^k}{(k\mu+\nu)!}. \tag{5}$$

By choosing the parameter $\mu\geqslant\alpha$ it is possible to quench the factorial growth of a_k and make the series for $\varphi_{\mu\nu}(z,t)$ convergent. For the Borel method to be applicable it is sufficient[17] that the function $\Sigma a_k(-w)^k/(k\mu)!$ be regular for small $|w|$ and not have singularities on the semi-axis $0<w<\infty$. Usually one considers the case $\nu=0$; in accordance with Hardy,[17] this method of summation is designated as the (B',μ) method. For $\mu=1$ this method has been treated by Bender and Wu[2] in an application to the anharmonic oscillator, by Shirkov[14] for the $g\varphi^4/4!$ field theory, and also in a paper by one of the authors.[15] We shall find it convenient to generalize this definition, introducing the *two* parameters μ and ν in (5). In the case when the series is summable by the Borel method (i.e., under the condition that the series for $\varphi_{\mu\nu}$ and the integral in (5) are convergent), the sum $f(z)$, naturally, does not depend on the values of μ and ν.

To calculate the sum (1) it is convenient to use a representation differing from (2) for the asymptotic form of a_k as $k\to\infty$:

$$a_k=\frac{(k\alpha)!}{k!}a^k\sum_{m=0}^{\infty}C_m(k+\beta-m)!. \tag{6}$$

The coefficients C and c are related by the linear transformation

$$c_i=\sum_{j=0}^{i}S_{ij}C_j,\quad C_0=c_0,\quad C_1=c_1-{}^1/_2\beta(\beta+1)c_0\ ,$$

etc. General formulas for the elements of the matrices S and S^{-1} are given in Appendix A. Substituting (6) into (5) and assuming that $k_0=0$, we find (see also Ref. 15)

$$\begin{gathered}\varphi_{\alpha,0}=\sum_{m=0}^{\infty}(\beta-m)!\;C_m(1+azt^\alpha)^{m-\beta-1}\\ f(z)=(az)^{-(\beta+1)}e^{1/az}\sum_{m=0}^{\infty}(\beta-m)!\;C_m(az)^m I(az;\alpha,m-\beta),\end{gathered} \tag{7}$$

where

$$I(x;\alpha,\beta)=x^{-\beta}\int_1^\infty e^{-t/x}[(t-1)^\alpha+1]^{\beta-1}dt. \tag{8}$$

This integral has the following behavior: $I\propto e^{-1/x}x^{1-\beta}$ for $x\to0$, and for $x\to\infty$

$$I\infty x^{(\alpha-1)(\beta-1)}\ \text{for}\ \beta>1-\alpha^{-1},$$
$$I\infty x^{-\beta}\ln x\ \text{for}\ \beta=1-\alpha^{-1},\quad I\infty x^{-\beta}\ \text{for}\ \beta<1-\alpha^{-1}.$$

In the frequently encountered cases $\alpha=1$ and $\alpha=2$ it is expressed in terms of familiar special functions. For example, for $\alpha=1$,

$$f(z)=(az)^{-(\beta+1)}e^{1/az}\sum_{m=0}^{\infty}(\beta-m)!\,C_m(az)^m\Gamma(m-\beta,(az)^{-1}), \tag{7'}$$

where $\Gamma(a,x)$ is the incomplete gamma function. It can be shown that the same result (7′) is given by summing the series (1) with the coefficients (6) with the aid of the Sommerfeld-Watson integral transform, as suggested by Lipatov.[5] The possibility of obtaining an answer for the sum (1) in a closed form containing standard functions is the advantage of the parametrization (6) as compated with (2), except for the case of integer values of β.

3. CONNECTION BETWEEN THE ASYMPTOTIC FORM OF a_k FOR $k\to\infty$ AND THE SINGULARITY OF THE SUM

The function $f(z)$ has a branch point $z=0$ and a cut along $z<0$. We shall calculate its discontinuity across the cut. Using the known analytic properties of the function $\Gamma(a,x)$, from (7′) we obtain

$$\Delta f(-\xi)=\frac{1}{2i}[f(-\xi+i0)-f(-\xi-i0)]$$
$$=-\pi e^{-1/a\xi}(a\xi)^{-(\beta+1)}\sum_{m=0}^{\infty}C_m(a\xi)^m, \tag{9}$$

where $\xi\equiv-z$. If the series $\Sigma_m C_m z^m$ has a finite radius of convergence, or even if it is asymptotic for $z=0$ (which, evidently, occurs in the majority of physically interesting cases), the asymptotic form of a_k for $k\to\infty$ determines the behavior of the discontinuity $\Delta f(z)$ as $z\to-0$.

In the case of arbitrary α this result arises from the following considerations.[1)] From (2) and (5) we find that the singularity of the function $\varphi_{\alpha,0}(w)$ nearest to zero is located at $w\equiv-azt^\alpha=1$:

$$\varphi_{\alpha,0}(w)\underset{w\to1}{\approx}\beta!\ (1-w)^{-(\beta+1)}\underset{t\to t_0}{\approx}\mathrm{const}\cdot(t-t_0)^{-(\beta+1)},\quad t_0=(-az)^{-1/\alpha}.$$

The integral

$$f(z)=\int_0^\infty dt\,e^{-t}\varphi_{\alpha,0}(z,t)$$

is an analytic function of z that acquires a singularity when the point $t_0(z)$ falls on the integration contour $0<t<\infty$. This happens when $z=-\xi\pm i0$, $\xi>0$. Since $t_0=(a\xi)^{-1/\alpha}\to\infty$ as $\xi\to0$, the discontinuity $\Delta f(-\xi)$ is determined by the behavior of $\varphi_{\alpha,0}(w)$ in the vicinity of the singular point $w=1$, which is known. The elementary calculation of the integral that arises gives

$$\Delta f(-\xi)=-\pi C_0\alpha^{-(\beta+1)}(a\xi)^{-(\beta+1)/\alpha}\exp\{-(a\xi)^{-1/\alpha}\},\quad \xi\to0 \tag{10}$$

(for $\alpha=1$ this formula coincides with the first term of the series (9)). Comparison of the formulas (2) and (10) shows that the faster the increase of the coefficients a_k the weaker is the singularity of the function $f(z)$ at the point $z=0$. We note that all derivatives of the discontinuity $\Delta f(-\xi)$ vanish at $\xi=0$.

The expression (10) contains the only information about the PT sum that can be extracted from the asymptotic form of the coefficients a_k, irrespective of the way of parametrizing the corrections in powers of k^{-1} (formula (2) or (6)) and of the summation method applied.[2)] We note that the asymptotic form of a_k for $k\to\infty$ can be obtained from (10) by means of dispersion relations.[1,3,8,9] The arguments presented above show that it is possible to find the discontinuity of the sum as $z\to-0$ from the form of $\tilde a_k$, i.e., the connection between the asymptotic form of a_k and the character of the singularity of $f(z)$ is reciprocal. This will be used below.

4. THE IMPROVED PERTURBATION THEORY

Turning to consider the IPT, we shall investigate the question of the intervals $0<z<z_N$ in which the functions $f_N(z)$ give good approximations to the exact solution.

Rewriting (4) in the form

$$f_N(z)=f(z)+\sum_{k=k_0+N}^{\infty}(\tilde a_k-a_k)(-z)^k,$$

we note that for $N\gg1$ we also have $k\gg1$ in this sum. Therefore, $a_k-\tilde a_k\sim\Gamma(k\alpha+\beta)a^k$, whence

$$f_N(z)\approx f(z)-(-az)^{N+k_0}\int_0^\infty\frac{e^{-t}t^{(N+k_0)\alpha+\beta-1}}{1+azt^\alpha}dt.$$

The value $z=z_N$ at which the last term is comparable in magnitude with $f(z)$ can be regarded as the upper limit of applicability of the IPT. Calculating the integral for $N\gg1$ by the method of steepest descents, we arrive at the following equation of z_N:

$$f(z)=\frac{bN^{\beta-1/2}}{1+a(N\alpha)^\alpha z}\left[az\left(\frac{N\alpha}{e}\right)^\alpha\right]^N,$$

where b is a certain constant. Hence, for $N\to\infty$ we find

$$z_N\approx\frac{1}{a}\left(\frac{e}{N\alpha}\right)^\alpha,\quad e=2.718... \tag{11}$$

We note that the value of z_N does not depend on the parameter β in (2) and is insensitive to the form of $f(z)$. The latter is explained by the fact that $[f(z)]^{1/N}\to1$ if $N\to\infty$, while $f(z)$ is finite or has a power behavior as $z\to0$.

In certain cases the corrections to the leading term $\tilde a_k$ of the asymptotic form may decrease with increase of k not by a power law (as in (2)) but exponentially:

$$a_k\underset{k\to\infty}{=}\tilde a_k(1+\gamma q^k+\dots),\quad 0<q<1 \tag{12}$$

(see the examples I and II in Sec. 5). Then the region of applicability of the IPT is expanded by a factor of q^{-1}:

$$z_N\approx\frac{1}{aq}\left(\frac{e}{N\alpha}\right)^\alpha,\quad N\gg1. \tag{11'}$$

The general conclusion is that the region in which the IPT approximates the exact solution contracts with increase of N in accordance with the law $z_N\propto N^{-\alpha}$, i.e., it contracts more rapidly the greater the parameter α

determining the factorial growth of a_k. Increase of the number of exact coefficients of the PT series and knowledge of the asymptotic form $\tilde{a}_k$ do not give the possibility of establishing $f(z)$ for $z \gtrsim z_N$ by means of (4). On the other hand, in the interval $0 < z \lesssim z_N$ the function $f_N(z)$ is closer to $f(z)$ the larger is N.

5. SOME EXAMPLES

We shall compare the PT and IPT with the exact solutions for a number of examples for which the PT coefficients have the form (2).

I. The Heisenberg-Euler lagrangian

The interaction of an electromagnetic field with the vacuum of charged particles (spin s, mass m) leads to a nonlinear correction L' to the Maxwell Lagrangian $L_0 = (E^2 - H^2)/2$. In the case of constant and uniform fields $\mathbf{E}$ and $\mathbf{H}$ exact expressions[19,20] for L' are known. We shall confine ourselves to the case of crossed fields ($\mathbf{E}\cdot\mathbf{H}=0$), when L' depends only on the invariant $z = e^2(H^2 - E^2)/m^4$:

$$L' = \frac{(2s+1)m^4}{4\pi} f_s(z), \quad f_s(z) = \int_0^\infty \frac{d\tau}{\tau^3} e^{-\tau} \chi_s(\tau z^{1/2}), \tag{13}$$

$$\chi_s(x) = \begin{cases} x/\operatorname{sh} x - 1 + {}^1/_6 x^2, & s=0 \\ 1 + {}^1/_3 x^2 - x \operatorname{cth} x, & s={}^1/_2 \end{cases}. \tag{14}$$

For $x \to 0$,

$$\chi_s(x) = \sum_{k=2}^\infty c_k^{(s)} x^{2k},$$

$$c_k^{(0)} = \frac{2(1-2^{2k-1})}{(2k)!} B_{2k}, \quad c_k^{(1/2)} = -\frac{2^{2k}}{(2k)!} B_{2k},$$

where B_n are the Bernoulli numbers (the series for $\chi_s(x)$ converges absolutely for $|x| < \pi$). Substituting these expansions into (13) and integrating term by term, we obtain the PT series:

$$f_s(z) = \sum_{k=2}^\infty a_k^{(s)} (-z)^k, \quad a_k^{(s)} = (-1)^k (2k-3)!\, c_k^{(s)}. \tag{15}$$

The asymptotic form of the coefficients $a_k^{(s)}$ does not depend on the spin s:

$$\tilde{a}_k = 2\pi^{-2k}(2k-3)! \tag{15'}$$

and, as can be seen from Fig. 1, is established very rapidly. Therefore, the IPT has good accuracy in the present case. Applying the formula (5) with $\mu = 2$, $\nu = -3$, we have

$$\tilde{f}(z) = \sum_{k=2}^\infty \tilde{a}_k (-z)^k$$

$$= \frac{2}{p^2} \int_0^\infty \frac{e^{-pt}}{1+t^2} t\, dt \tag{16}$$

$$= -\frac{2}{p^2}[\cos p \operatorname{ci} p + \sin p \operatorname{si} p],$$

$$p = \pi z^{-1/2}.$$

A comparison of $\tilde{f}(z)$ with $f_s(z)$ for example of spin

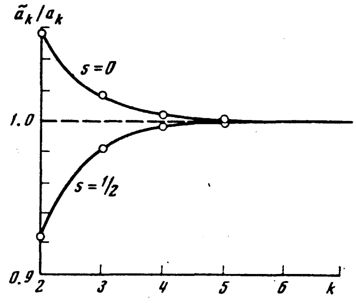

FIG. 1. Ratio of the coefficients $\tilde{a}_k/a_k$ for the Heisenberg-Euler Lagrangian.

$s = \frac{1}{2}$ is given in Fig. 2, in which the PT polynomials

$$p_N(z) = \sum_{k=2}^{N+1} a_k (-z)^k$$

and the IPT functions $f_N(z)$ are also depicted. The change from p_N to f_N extends considerably the region of approximation of the exact solution $f(z)$. We note that in the given problem the Borel sum $\tilde{f}(z)$ is close to the exact solution. Even for $z \to \infty$,

$$\tilde{f}(z) = \frac{1}{\pi^2} z \ln z + \ldots, \quad f_s(z) = \frac{2s+1}{12} z \ln z + \ldots$$

(however, the analytic properties of these functions are different- see Appendix B). Such similarity between f and $\tilde{f}$ is rarely encountered and is connected with the fact that the coefficients $\tilde{a}_k$ are very close to a_k even for small k, and corrections of order k^{-1} are absent (formula (12) holds, with $q = \frac{1}{4}$).

From (16) we find the discontinuity:

$$\Delta\tilde{f}(-\xi) = \pi^{-1}\xi \exp(-\pi\xi^{-1/2}), \quad \xi = -z = (E/E_0)^2, \tag{17}$$

which coincides with the first term of the exact expres-

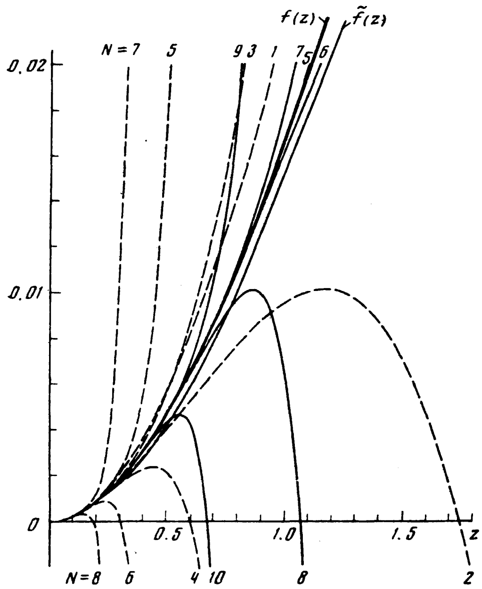

FIG. 2. Comparison of perturbation theory with the exact solution. The dashed curves are the PT polynomials and the solid curves are the IPT functions (the numbers on the curves correspond to the values of N). For $N \le 5$ the functions $f_N(z)$ coincide with the exact solution $f(z)$ within the limits of error of the Figure.

sion following from the formula[20,21] for $Im\ L'$ in the case of particles with spin s:

$$\Delta f(-\xi)=\frac{\xi}{\pi}\sum_{n=1}^{\infty}\frac{\beta_n}{n^2}\exp(-n\pi\xi^{-1/2}). \qquad (17')$$

Here, $\beta_n=(-1)^{n-1}$ for bosons, $\beta_n=1$ for fermions, and $E_0=m^2/e$ is the characteristic field intensity at which nonlinear corrections to L_0 become important and pair creation in the electric field ceases to be an exponentially small effect. When $\xi\to 0$ the formula (17′) goes over into (17). In this region ($E\ll E_0$) the discontinuity of the Lagrangian L' depends on the spin s of the particles in a trivial manner:

$$\operatorname{Im} L'=\frac{(2s+1)m^4}{4\pi^2}\xi\exp(-\pi\xi^{-1/2}).$$

II. Electron energy levels for $Z>137$

The energy ϵ of a level near the edge of the lower continuum is determined by the equation[22]

$$f(z)=(\zeta^2-1)^{-1/2}-(\zeta_{cr}^2-1)^{-1/2}, \qquad (18)$$

which is valid in the limit of small nuclear radius: $|\ln R|\gg 1$. For the lowest level $1s_{1/2}$,

$$f(z)=-\frac{1}{\pi}\left\{\psi(z^{-1/2})+\frac{1}{2}\left[\ln\frac{z}{1+z}+1+z^{1/2}-(1+z)^{1/2}\right]\right\},$$

where $z=\epsilon^{-2}-1$, $\zeta=Ze^2\simeq Z/137$, Z is the nuclear charge, $\zeta_{cr}=Z_{cr}e^2$ and $\psi(x)=\Gamma'(x)/\Gamma(x)$. Hence,

$$a_k=\frac{1}{2\pi k}\left[(-1)^k B_{2k}+\frac{\Gamma(k-1/2)}{\Gamma(k)\Gamma(-1/2)}-1\right],\quad \tilde a_k=-\frac{(2k)!}{2^{2k}\pi^{2k+1}k} \qquad (19)$$

(see curve 1 in Fig. 3). The factorial growth of a_k as $k\to\infty$ and the divergence of the PT series are explained by the fact that Eq. (18) for $\zeta>\zeta_{cr}$ describes a quasi-stationary positron level, the imaginary part of which is determined by the penetrability of the Coulomb barrier and tends to zero exponentially at the positron-creation threshold:[23]

$$\gamma=\theta(\zeta-\zeta_{cr})\gamma_c\exp\{-b[(\zeta-\zeta_{cr})/\zeta_{cr}]^{-1/2}\}. \qquad (20)$$

The Borel sum ($\mu=2$, $\nu=0$) is equal to

$$\tilde f(z)=\sum_{k=1}^{\infty}\tilde a_k(-z)^k=\frac{1}{\pi}\int_0^{\infty}dt\,e^{-t}\ln\left[1+z\left(\frac{t}{2\pi}\right)^2\right]$$
$$=-2\pi^{-1}[\cos\tau\,\mathrm{ci}\,\tau+\sin\tau\,\mathrm{si}\,\tau],\quad \tau=2\pi z^{-1/2}. \qquad (21)$$

The behavior of $p_N(z)$ and $f_N(z)$ is analogous to that in example I. The principal difference is that the function $f(z)$ and $\tilde f(z)$ are far from each other for $z>0$ (see Fig. 2 in Ref. 16). In particular,

$$f(z)=\pi^{-1}z^{1/2}+\ldots,\quad \tilde f(z)=\pi^{-1}\ln z+\ldots \text{ for } z\to\infty.$$

At the same time, the Borel sum accounts well for the discontinuity on the left cut $z=-\xi$:

$$\Delta f(-\xi)=1/2(\operatorname{cth}\pi\xi^{-1/2}-1)=(\exp\{2\pi\xi^{-1/2}\}-1)^{-1},$$
$$\Delta\tilde f(-\xi)=\exp\{-2\pi\xi^{-1/2}\}.$$

For the analytic properties of the functions $f(z)$ and $\tilde f(z)$ see Appendix B.

Equation (18) determines the dependence of the energy of a level on the nuclear charge. For Z close to Z_{cr} this dependence can be represented in the form of a PT series:

$$\varepsilon(Z)=-1+\sum_{k=1}^{\infty}\varepsilon_k\left(\frac{Z-Z_{cr}}{Z_{cr}}\right)^k. \qquad (22)$$

The coefficients ϵ_k can be expressed in terms of a_l, $1\le l\le k$. As follows from (10) and (20), the asymptotic form of ϵ_k is

$$\varepsilon_k\approx-\frac{\gamma_0}{\pi}(2k)!\,b^{-2k}k^{-1},\quad k\to\infty \qquad (22')$$

(the quantity b appearing here is determined by the slope of the level at the edge of the lower continuum).[23] Comparison with (19) shows that in the expansions of the functions $f(z)$ and $\epsilon(Z)$ the parameters α and β are the same. The factorial growth of the coefficients ϵ_k leads to the result that the PT series for $\epsilon(Z)$ has zero radius of convergence.

III. The relativistic Thomas-Fermi equation

The self-consistent potential $V(r)$ in the vacuum shell of a supercritical ($Ze^2\gg 1$) atom obeys the equation[24,25]

$$\Delta V=4\pi e^2\left[n_p(r)-\frac{1}{3\pi^2}(V^2+2V)^{3/2}\right], \qquad (23)$$

where $n_p(r)$ is the density of protons in the nucleus, and $\hbar=c=m_e=1$. We put

$$V(r)=-\frac{Z_1e^2}{r}\left[\frac{\xi(r,\mu)}{\mu}\right]^{1/2},\quad \mu=\frac{(Z_1e^2)^2}{3\pi}, \qquad (24)$$

where $Z_1=Z-N_e$ is the charge of the supercritical atom for an external observer. Let $\mu\ll 1$ (the case of weak screening[25]). The solutions of Eq. (23) in the region $r\ll 1$ possess the property of renormalizability;[26] $\xi(r,\mu)$ plays the role of the invariant charge and has a pole at $r=r_0(\mu)$, the position of which is determined by the GLF $\beta(\mu)=-\partial\xi(r,\mu)/\partial\ln r|_{r=r_a}$ ($r_a=Z_1e^2/2$ is the radi-

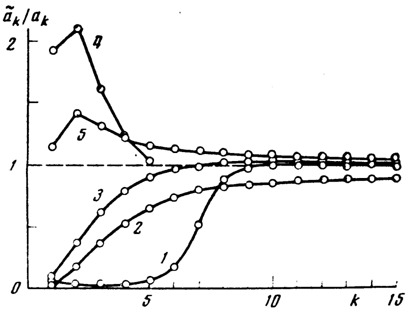

FIG. 3. Approach of the coefficients a_k to their asymptotic form for the problems II-IV.

us of the vacuum shell). For $\beta(\mu)$ it is possible to obtain a differential equation,[26] which, by means of the substitution $\mu = g^2/4$, $\varphi(g) = 2g^{-1}\beta(g^2/4)$, is brought to the form

$$d\varphi/dg = -1 + g^3/\varphi. \tag{25}$$

We shall study the properties of this equation. The boundary condition for $\varphi(g)$ follows from PT: $\xi(r,\mu) = \mu(1 - 8\mu\ln r + \ldots)$, whence $\varphi(g) = g^3 + O(g^5)$. For $g \to 0$ the functions $\varphi(g)$ and $\beta(\mu)$ can be represented in the form of PT series:

$$\varphi(g) = g^3 \sum_{k=0}^{\infty} a_k(-g^2)^k, \quad \beta(\mu) = \sum_{k=2}^{\infty} b_k(-\mu)^k, \tag{26}$$

with $b_k = 2^{2k-1}a_k$. Substituting (26) into (25) we arrive at the recursion relations

$$a_{k+1} = (k+3)\sum_{i+j=k} a_i a_j,$$

by means of which the coefficients a_k up to $k = 200$ were calculated on a computer. They increase rapidly (see Table I) and have the asymptotic form:[3)]

$$a_k = \tilde{a}_k\left(1 + \frac{21}{8k} - \frac{519}{128k^2} + \ldots\right), \quad \tilde{a}_k = c_0(k!)\cdot 2^k k^{7/2}, \tag{27}$$

with $c_0 = 0.04551\ldots$. Unlike the preceding examples, this expansion contains terms k^{-1}, $k^{-2}, \ldots$, and the slow approach of a_k to the asymptotic form $\tilde{a}_k$ (see curve 2 in Fig. 3) is connected with this. Therefore, other representations were tried:

$$\tilde{a}_k^{(\rho)} = c_0\Gamma(k + 9/2 - \rho)\cdot 2^k k^\rho$$

and the best of these, with $\rho = 2$ (curve 3 in Fig. 3), was chosen. The Borel sum $\tilde{\varphi}(g)$, corresponding to coefficients $\tilde{a}_k^{(\rho)}$ with integer ρ, is calculated from the formulas (A.6) and (A.8). The IPT functions are shown in Fig. 4; they enable us to establish $\varphi(g)$ for $g < 0.5$. We note that in this region $\tilde{\varphi}(g)$ is of the order of 10^{-4}; thus, in the examples II and III, in contrast to the example I, the Borel sums $\tilde{f}(z)$ are far from the exact solutions.

IV. The Stark effect

In an electric field F the level of an atom (with energy $\epsilon_0 = -\kappa^2/2, \hbar = m = e = 1$) is transformed into a quasi-stationary state with complex energy $E = \epsilon - i\gamma/2$. For $F \ll F_0$ the Stark shift of the level can be expanded in a

TABLE I. The coefficients a_k for the function $\psi(g)$.

k	a_k	k	a_k	k	a_k	k	a_k
0	1	5	75 978	9	22 170 740 526	50	1.4470 (50)
1	3	6	1 530 720	10	630 399 084 912	100	5.5228 (193)
2	24	7	34 237 485	15	2.8980 (19)	150	1.5605 (314)
3	285	8	837 481 140	20	4.6254 (27)	200	6.6100 (441)
4	4284			30	2.0725 (45)		

Note: For $k \leqslant 10$ the exact values of a_k are given; for larger k the first five significant figures are given. The figures in brackets indicate the power of ten by which the number given must be multiplied.

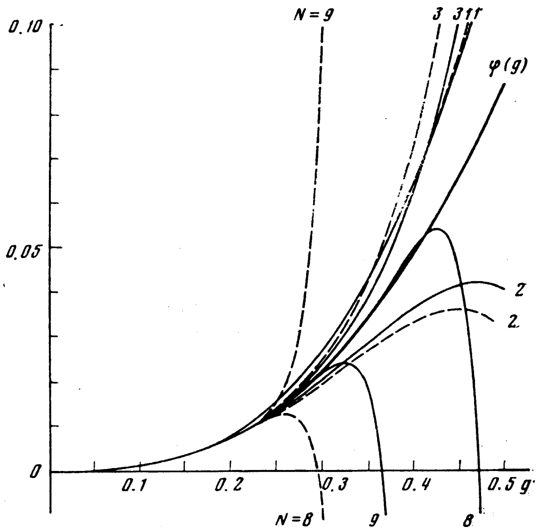

FIG. 4. The exact solution $\varphi(g)$, the PT polynomials (the dashed lines) and IPT functions (the solid curves) for the relativistic Thomas-Fermi equation.

PT series:

$$\frac{\varepsilon - \varepsilon_0}{\varepsilon_0} = \sum_{n=1}^{\infty} \alpha_n(-z)^n, \quad z = -\left(\frac{F}{F_0}\right)^2 \tag{28}$$

($F_0 = \kappa^3$ is the characteristic atomic field). The discontinuity of $E(z)$ across the cut $z < 0$ is equal to the probability γ of ionization of the level. In the semi-classical approximation,[27,28]

$$\gamma = A|\varepsilon_0|(F/F_0)^{1-2\eta}\exp(-{}^2/_3 F_0/F), \tag{29}$$

where $\eta = Z/\kappa$ is the Coulomb parameter and Z is the charge of the atomic core. The asymptotic form of α_n follows from (29) and (10):

$$\tilde{\alpha}_n = A\pi^{-1}({}^3/_4)^{2\eta-1}(2n)!({}^3/_2)^{2n}. \tag{30}$$

A comparison of (30) with numerical calculations[29] for the hydrogen atom (the 1s level, for which $\kappa = \eta = 1$, $A = 8$) is shown in Fig. 3, curve 4.

The problem of the Stark effect admits an exact solution in the case of a one-dimensional δ-potential. The energy $E = \epsilon - i\gamma/2$ is determined from the equation[30]

$$xK_{1/3}(x)[I_{1/3}(x) + I_{-1/3}(x)] + i\pi^{-1}x[K_{1/3}(x)]^2 = t, \tag{31}$$

where $t = (E/\epsilon_0)^{1/2}$ and $x = t^3(-z)^{-1/2}/3$. From this we find t^2 as a function of z, i.e., ϵ and γ as functions of the field F. We obtain the equation determining the polarizabilities α_n:

$$t = 2xK_{1/3}(x)I_{1/3}(x) = \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(2^n n!)^2}\frac{\Gamma(n+{}^5/_6)}{\Gamma(-n+{}^5/_6)}x^{-2n}, \tag{32}$$

which was solved numerically. A comparison of α_n with the asymptotic form (30) is shown in Fig. 3 (curve 5).

Thus, the asymptotic form of the high-order polarizabilities α_n is determined by the probability of ionization of the atom in weak ($F \ll F_0$) fields. Unlike the preceding examples, the PT series for the energy $\epsilon=\epsilon_0\Sigma\alpha_n(F/F_0)^{2n}$ is not alternating in sign ($\alpha_n>0$).

V. The zero-dimensional field-theory model

We consider the integral

$$f(g)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\varphi\exp\left\{-\left(\frac{\varphi^2}{2}+g\frac{\varphi^n}{n!}\right)\right\}, \qquad (33)$$

corresponding to the "zero-dimensional model" of field theory.[6] Here, $n=4,6,8,\ldots$;

$$a_k=2^{nk/2}\Gamma((nk+1)/2)/[\pi^{1/2}(n!)^k\Gamma(k+1)],$$

whence

$$\alpha=\frac{n}{2}-1,\quad \beta=-1,\quad a=\frac{1}{(n-1)!}\left(\frac{2n}{n-2}\right)^{(n-2)/2},\quad c_0=\frac{1}{\pi(n-2)^{1/2}}. \qquad (34)$$

We have investigated the case $n=4$. In this case the integral (33) can be calculated analytically:

$$f(g)=\left(\frac{3}{2\pi g}\right)^{1/2}e^{3/4g}K_{1/4}\left(\frac{3}{4g}\right),$$

and the discontinuity at $g=-\xi<0$ is equal to

$$\Delta f(-\xi)=-\frac{1}{2}\left(\frac{3}{\pi\xi}\right)^{1/2}e^{-3/4\xi}K_{1/4}\left(\frac{3}{4\xi}\right).$$

For $\xi\to0$ we obtain $\Delta f(-\xi)=-2^{-1/2}e^{-3/2\xi}$, in complete correspondence with formula (10). The convergence of the PT and IPT functions to the exact function $f(g)$ has also been investigated. The results are analogous to those obtained in the preceding examples.

With increase of g the function $f(g)$ decreases monotonically, remaining positive for $0<g<\infty$. When $g\to\infty$, $f(g)\to0$ like $g^{-1/4}$. On the other hand, the Borel sum

$$\tilde f(g)=1+\sum_{k=1}^{\infty}\tilde a_k(-g)^k=1+\frac{1}{\pi\sqrt2}\exp\left(\frac{3}{2g}\right)\mathrm{Ei}\left(-\frac{3}{2g}\right)$$

vanishes at $g\simeq230$ and then becomes negative: $\tilde f(g)=-2^{-1/2}\pi^{-1}\ln g$ for $g\to\infty$. Thus, for this example too, the functions $f(g)$ and $\tilde f(g)$ have different behavior for $g\to\infty$.

The results of the investigation of the asymptotic forms of the PT series are collected in Table II, in which the parameters α, β, and a for the examples I–V are given. For comparison, the values of these constants for the energy of the ground level of a d-dimensional anharmonic oscillator[2,7] with nonlinearity $g(\sum_{i=1}^{d}x_i^2)^2$, and also for the GLF in $g\varphi^4/4!$ scalar field theory,[6] are also given in the table. We call attention to the analogy between the structure of the series for the GLF in the $g\varphi^4/4!$ theory and in the relativistic Thomas-Fermi equation.

TABLE II. Parameters of the asymptotic forms of a_k for different problems.

Problem	α	β	a
I	2	−3	π^{-2}
II	2	−1	$(2\pi)^{-2}$
III	1	$7/2$	2
IV	2	$2(\eta-1)$	$9/4$
V	$n/2-1$	−1	see (34)
Anharmonic oscillator	1	$d/2-1$	3
$g\varphi^4/4!$	1	$7/2$	$(4\pi)^{-2}$

On the basis of the examples considered we arrive at the following conclusions.

As a rule, the functions $f_N(z)$ approximate the exact solution $f(z)$ in a wider[4)] interval of z than do the PT polynomials of the same degree N. For those z for which neighboring IPT curves (f_N and f_{N+1}) are close to each other, they approximate $f(z)$ well and thereby make it possible to reproduce the exact solution in the interval $0<z\leq z_N$. The upper bounds obtained in Sec. 4 for z_N are confirmed in all the examples we have considered (this question is analyzed in detail in Ref. 31).

6. CONCLUSION

The application of the IPT to problems in quantum field theory, where exact solutions are absent, is of special interest. Calculation of Feynman graphs gives the first few coefficients a_k of the PT series, and Lipatov's method determines their asymptotic form for $k\to\infty$. For the $g\varphi^4/4!$ scalar field theory the first three coefficients of the GLF[32] and the asymptotic form $\tilde a_k$[6] are known, and this makes it possible to calculate[5)] the functions $\tilde\psi(g)$ and $\psi_N(g)$ for $N=1,2,3$. The general picture (see Fig. 5) is analogous to the examples considered above, especially the examples II and III. For $0<g\lesssim50$ the curves $\psi_N(g)$ are extremely close to each other.[6)] From this it is possible to conclude that the exact GLF is also close to them in this range of g. It differs considerably, therefore, from the curve L obtained by Lipatov[5] by expanding in $1/n$ in the $g\varphi^n/n!$ theory (this difference can be seen clearly from Fig. 6), and this indicates the poor accuracy of the $1/n$ expansion when $n=4$. In view of this the conclusion in Ref. 5 that a zero of the GLF exists at $g\sim100$ appears to us to be doubtful.

Application of the Padé method gives results analogous to those of the IPT. Let $p_{[M,N]}(g)$ be the Padé approximate[7)] constructed from the PT coefficients a_k. On the other hand, in the IPT functions (cf. (4)) we can replace the polynomial $\Sigma(a_k-\tilde a_k)(-g)^k$ by the corresponding Padé approximant; we denote such functions by $\psi_{[M,N]}(g)$. It is obvious that

$$p_{[0,N]}(g)\equiv p_{N+1}(g),$$
$$\psi_{[0,N]}(g)\equiv\psi_{N+1}(g).$$

The three known coefficients a_2, a_3, and a_4 permit us to construct the approximants [2,0] and [1,1] for the GLF. Of these $\psi_{[2,0]}$ possesses the best accuracy (see Fig. 5). In the given case the Padé method does not lead to the determination of $\psi(g)$ in a wider range of g than does the IPT. Possibly, this is explained by the fact that the number of known coefficients a_k is too small.

The calculations presented do not enable us to reach

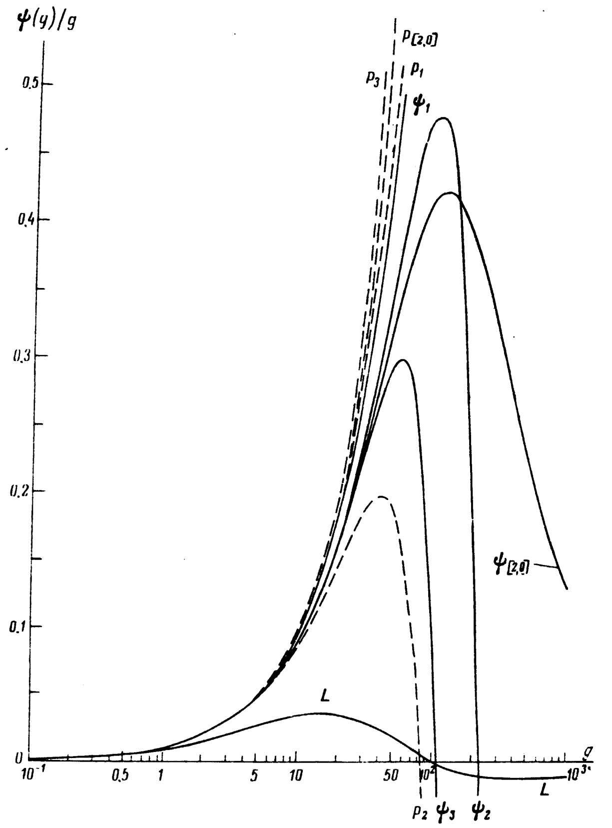

FIG. 5. The Gell-Mann–Low function in $g\varphi^4/4!$ scalar field theory. The dashed curves correspond to the PT polynomials $p_N(g)$ and the solid curves to the IPT functions $\psi_N(g)$; $p_{[2,0]}$ and $\psi_{[2,0]}$ are Padé approximants; the curve L is taken from Ref. 5.

definite conclusions about the form of the GLF for $g>50$ (in particular, one must not assign values to the zeros of the functions $\psi_2(g)$ and $\psi_3(g)$, as is clear from a comparison of Fig. 5 with Figs. 2 and 4; see also Ref. 14). Where the GLF is reliably determined with the aid of the IPT it is a monotonically increasing function of g. We note that in the case of the relativistic Thomas-Fermi equation (for which the asymptotic form of a_k has the same structure as in the $g\varphi^4/4!$ theory) the GLF increases monotonically and is without zeros for all $0<g<\infty$ (cf. Ref. 26).

The information that can be extracted unambigously from the asymptotic form of a_k concerns the character of the singularity of the GLF. For the $g\varphi^n/n!$ scalar

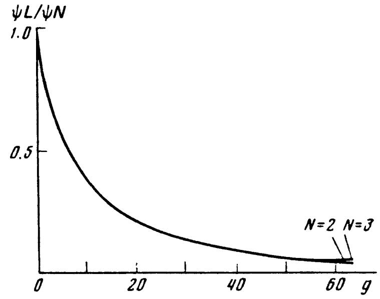

FIG. 6. Ratio of $\psi(g)$ calculated by means of the $1/n$ expansion to the IPT functions $\psi_N(g)$. For $g<50$ the curves for $N=2$ and $N=3$ merge, within the limits of error of the Figure.

field theory we have[6]

$$\alpha=\frac{n}{2}-1,\quad \beta=\frac{n^2-n+2}{2(n-2)},\quad a=\frac{(n-2)^{(n+2)/2}}{2^{n+1}\pi^{n/2}n!}\left\{\frac{\Gamma(D)}{\Gamma(D/2)}\right\}^{(n-2)/2},$$

whence, with the aid of (10), we find the discontinuity of $\psi(g)$ across the cut for $g\to -0$:

$$\Delta\psi(g)=A'|g|^{-\rho}\exp\{-A|g|^{-\sigma}\}, \tag{35}$$

where

$$\rho=\frac{n^2+n-2}{(n-2)^2},\quad \sigma=\frac{2}{n-2},\quad A=\frac{[(2\pi)^n(n!)^2]^{1/(n-2)}}{(n-2)^{(n+2)/(n-2)}}\Gamma\left(\frac{1}{2}\right)\Big/\Gamma\left(\frac{3n-2}{2n-4}\right)$$

(for $n=4$ we have $\rho=9/2$, $\sigma=1$ and $A=16\pi^2$; this case has been considered previously).[9]

According to Lipatov,[5] for $n=4$ the zero of the GLF is $g_0\simeq 103$, i.e., g_0 is not far from the weak-coupling region: $ag_0=g_0/16\pi^2\simeq 0.65$. Therefore, the situation with the zero of the GLF in the $g\varphi^4/4!$ theory can be clarified if the next few coefficients of the PT series are calculated and a method is found that enables us to establish $\psi(g)$ in a wider range of g than does the IPT (the Padé method is a possible candidate). However, the behavior of $\psi(g)$ in the strong-coupling region ($ag\gg 1$) is not connected with the asymptotic form of the coefficients a_k for $k\to\infty$.

The authors are grateful to B. L. Ioffe, M. S. Marinov and M. A. Shifman for discussing the results of the work, and also to E. B. Bogomol'nyĭ for the suggestion that we investigate the function (3.3). We should like to express special gratitude to L. B. Okun' for a detailed discussion of the work and a number of valuable comments.

APPENDIX A

The formulas that make it possible for series with factorially growing coefficients to be summed effectively are collected here. These formulas were used in the calculation of the functions $\tilde{f}(z)$ and $f_N(z)$ presented in Figs. 2, 4 and 5.

1. We shall find the sum of the series (1) with $k_0=0$ and coefficients

$$a_k=(k+\beta)!a^k k^n, \tag{A.1}$$

where n is an integer. If $n=0$, application of the Borel method (5) with the parameters $\mu=1$ and $\nu=0$ gives

$$\varphi(z,t)=\sum_{k=0}^{\infty}(k+\beta)!(-azt)^k=\beta!(1+azt)^{-(\beta+1)},$$
$$f(z)=\beta!(az)^{-(\beta+1)}\exp[(az)^{-1}]\Gamma(-\beta,(az)^{-1}), \tag{A.2}$$

where

$$\Gamma(\alpha,x)=\int_x^{\infty}e^{-t}t^{\alpha-1}\,dt$$

is the incomplete gama function.[34] We note the particular cases

$$\Gamma(1,x)=e^{-x},\quad \Gamma(1/2,x)=\pi^{1/2}\operatorname{Erfc}(x^{1/2}),\quad \Gamma(0,x)=-\operatorname{Ei}(-x), \tag{A.3}$$

where $\mathrm{Ei}(-x)$ is the integral exponential function and $\mathrm{Erfc}(z)$ is the error integral:

$$\mathrm{Ei}(-x)=-\int_x^\infty e^{-t}\frac{dt}{t}, \quad \mathrm{Erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt.$$

For $m=1,2,3,\ldots$, the formulas

$$\sum_{k=0}^\infty (k+m)!(-z)^k=(-z)^{-(m+1)}\left\{e^{1/z}\mathrm{Ei}(-1/z)-\sum_{p=1}^m (p-1)!(-z)^p\right\}, \tag{A.4}$$

$$\sum_{k=0}^\infty \Gamma(k+m+1/2)(-z)^k=(-z)^{-m}\left\{\pi z^{-1/2}e^{1/z}\,\mathrm{Erfc}(z^{-1/2})-\sum_{p=0}^{m-1}\Gamma(p+1/2)(-z)^p\right\} \tag{A.5}$$

hold (for $m=0$ the sum over p in the right-hand side must be omitted).

2. Turning in (A.1) to the case $n\geq 1$, we denote

$$f_n(z,\beta)=\sum_{k=0}^\infty (k+\beta)!\,k^n(-z)^k=\left(z\frac{d}{dz}\right)^n f_0(z,\beta). \tag{A.6}$$

The function $f_0(z,\beta)$ is given by formulas (A.2). Differentiating successively, we obtain

$$f_n(z,\beta)=(-1)^n[P_n(z^{-1})f_0(z,\beta)-Q_n(z^{-1})],$$

where P_n and Q_n are polynomials of degree n, depending parametrically on β.[31] In the important particular cases $\beta=0,-\frac{1}{2}$, these formulas can be transformed to a more convenient form in which all the coefficients in the polynomials are integers:

$$\sum_{k=0}^\infty k!\,k^n(-z)^k=(-1)^{n+1}[U_n(t)e^t\,\mathrm{Ei}(-t)+V_n(t)], \tag{A.7}$$

$$\sum_{k=0}^\infty \Gamma(k+1/2)k^n(-z)^k=(-1/2)^n[\pi t^{1/2}u_n(2t)e^t\,\mathrm{Erfc}(t^{1/2})-v_n(2t)]. \tag{A.8}$$

Here $t=z^{-1}$, and the polynomials U_n, etc., are determined successively by the recursion relations

$$\begin{aligned}U_{n+1}(t)&=t(U_n+U_n'), & V_{n+1}(t)&=U_n+tV_n',\\ u_{n+1}(t)&=(1+t)u_n+2tu_n', & v_{n+1}(t)&=t(u_n+2v_n').\end{aligned} \tag{A.9}$$

We give the first few polynomials in explicit form:

$$\begin{aligned}&u_0=1,\quad v_0=0,\quad u_1=t+1,\quad v_1=t,\\ &u_2=t^2+4t+1,\quad v_2=t^2+3t,\end{aligned} \tag{A.10}$$

$$\begin{aligned}&u_3=t^3+9t^2+13t+1,\quad v_3=t^3+8t^2+7t,\\ &u_4=t^4+16t^3+58t^2+40t+1,\quad v_4=t^4+15t^3+45t^2+15t;\end{aligned}$$

$$\begin{aligned}&U_0=t,\quad V_0=0,\quad U_1=t^2+t,\quad V_1=t,\\ &U_2=t^3+3t^2+t,\quad V_2=t^2+2t,\\ &U_3=t^4+6t^3+7t^2+t,\quad V_3=t^3+5t^2+3t,\\ &U_4=t^5+10t^4+25t^3+15t^2+t,\quad V_4=t^4+9t^3+17t^2+4t.\end{aligned} \tag{A.11}$$

Thus, the sum $\Sigma(k+\beta)!\,k^n(-z)^k$ for integer and half-integer values of β is expressed in terms of the standard functions $\mathrm{Ei}(-x)$ and $\mathrm{Erfc}(x^{1/2})$, and this is convenient for numerical calculations. In the case of arbitrary β the sum is expressed in terms of the confluent hypergeometric function $\Psi(a,c;x)$.

3. The representation of a_k in the form (6) is convenient for summing series, while the parametrization (2) is more intuitive. The transformation from (2) to (6) is effected by matrices S and S^{-1}, the elements of which have the form

$$S_{ij}=p_{i-j}(\beta-j),\quad (S^{-1})_{ij}=q_{i-j}(\beta-j), \tag{A.12}$$

where $p_k(x)$ and $q_k(x)$ are polynomials of degree $2k$:

$$\begin{aligned}&p_0=q_0=1,\quad p_1=-q_1=\tfrac{1}{2}x(x+1),\\ &p_2=\tfrac{1}{8}x(x^2-1)(x+\tfrac{2}{3}),\quad q_2=\tfrac{1}{8}x(x^2-1)(x-\tfrac{2}{3}),\ldots;\end{aligned} \tag{A.13}$$

$$\begin{aligned}p_k&=\frac{1}{2^k k!}\left[x^{2k}-\frac{k(2k-5)}{3}x^{2k-1}+\ldots\right],\\ q_k&=\frac{(-1)^k}{2^k k!}\left[x^{2k}-\frac{k(4k-7)}{3}x^{2k-1}+\ldots\right].\end{aligned} \tag{A.14}$$

The values of these polynomials for integer $x=-1,0,1,2,\ldots$ are equal to

$$p_k(x)=(-1)^k s(x+1,x+1-k),\quad q_k(x)=(-1)^k S(x+1,x+1-k), \tag{A.15}$$

where $s(n,k)$ and $S(n,k)$ are Stirling numbers of the first and second kinds, known from combinatorics. The determination of the polynomials p_k and q_k for arbitrary k, and recursion relations and explicit expressions for them for $k\leq 4$, can be found in Ref. 31.

APPENDIX B

ANALYTIC PROPERTIES OF THE BOREL SUM *f(z)* AND EXACT SOLUTIONS

The analytic properties of functions of the form (1), (2) are of great interest. We shall consider them for the examples of Sec. 5, which has exact solutions.

1. We write formula (13) in the form

$$\begin{aligned}&f_s(z)=zg_s(z^{-1/2}),\\ &g_s(x)=\int_0^\infty e^{-xt}\chi_s(t)t^{-3}\,dt,\quad s=0,\ 1/2.\end{aligned} \tag{B.1}$$

Differentiating twice with respect to x and using formula 3.554 (4) from Ref. 34, we obtain in the case of spin $s=\frac{1}{2}$:

$$g_{1/2}''(x)=\psi(x/2)-\ln(x/2)+1/x+1/3x^2,$$

where $\psi(z)$ is the logarithmic derivative of the gamma function. Hence,

$$g_{1/2}'(x)=2\ln\left[\frac{\Gamma(1+x/2)}{(\pi x)^{1/2}(x/2e)^x}\right]-\frac{1}{3x}.$$

Integrating this equality and using the expansion of $\ln\Gamma(1+x)$ near $x=0$ (see Ref. 34), we obtain for $z\to\infty$ (the region of strong fields $(H^2-E^2)^{1/2}\gg m^2c^3/e\hbar$)

$$f_{1/2}(z)=h(z)\ln z+\sum_{n=0}^\infty c_n z^{1-n/2}, \tag{B.2}$$

where $h(z)=(z+3z^{1/2}+3/2)/6$ and, for $n>3$,

$$c_n=(-1)^{n+1}\zeta(n-1)/2^{n-2}n(n-1) \tag{B.3}$$

(the coefficients c_0, c_1, and c_2 have a different form, but this is unimportant for the following). For scalar particles the answer is analogous to (B.2), with

$$h(z)=\frac{1}{12}(z-3),\quad c_n=(-1)^n\frac{1-2^{1-n}}{n(n-1)}\zeta(n-1),\quad n\geqslant 3. \tag{B.4}$$

The series appearing in (B.2) can be represented without difficulty in the form

$$\sum_{k=1}^{\infty}F\left(-\frac{1}{2k}z^{-1/2}\right),$$

where

$$F(t)=\sum_{n=1}^{\infty}\frac{2t^n}{n(n+1)}=2[1+t^{-1}(1-t)\ln(1-t)].$$

From this it follows that the sum has singularities on the second sheet, of the type $(z-z_k)\ln(z-z_k)$, at the points $z_k=1/4k^2$ (for scalar particles, $z_k=(2k+1)^{-2}$). These singularities bunch toward zero; the complicated character of the singularity of $f_s(z)$ at this point is connected with this.[8)]

On the other hand, the Borel sum (16) can be rewritten in the form

$$\tilde{f}(z)=\frac{z}{\pi^2}\left\{\cos\frac{\pi}{\sqrt{z}}\ln z+\pi\sin\frac{\pi}{\sqrt{z}}+\varphi(z^{-1})\right\}, \tag{B.5}$$

where φ is an entire function. The only singularities of $\tilde{f}(z)$ are the points $z=0$, ∞. Thus, the singularities at the points $z=z_k$ are lost when we go from the exact function $\Sigma a_k(-z)^k$ to the Borel sum $\Sigma\tilde{a}_k(-z)^k$. We note that the representation (B.5) for $\tilde{f}(z)$ makes it easy to find its discontinuity across the out $-\infty<z<0$, and gives formula (17).

II. For the problem of the electron levels for $Z>137$, it can be seen directly from (19) that the singularities of $f(z)$ are the simple poles of the function $\psi(z^{-1/2})$, located at the points $z_k=k^{-2}$ on the second sheet. Thus, in this case too, $z=0$ is a point of bunching of singularities. The Borel sum $\tilde{f}(z)$ does not have singularities at $z=z_k$.

III. For the relativistic Thomas-Fermi equation it can be shown that the point $g=\infty$ is a complicated singularity of the function $\varphi(g)$, which has the following expansion:

$$\varphi(g)=g^2\sum_{k=0}^{\infty}\alpha_k^{(0)}g^{-k}-\frac{\ln g}{g^2}\sum_{k=0}^{\infty}\alpha_k^{(1)}g^{-k}+\ldots+\frac{(-\ln g)^n}{g^{4n-2}}\sum_{k=0}^{\infty}\alpha_k^{(n)}g^{-k}+\ldots \tag{B.6}$$

The coefficients $\alpha_k^{(i)}$ are determined from a complicated coupled system of recursion relations:

$$\alpha_0^{(0)}=1/\sqrt{2},\quad \alpha_1^{(0)}=-1/3,\quad \alpha_2^{(0)}=-\sqrt{2}/9;$$

$$\alpha_0^{(1)}=4\sqrt{2}/27,\quad \alpha_1^{(1)}=32/81;\quad \alpha_0^{(2)}=32\sqrt{2}/3645,\ldots$$

We shall show, for the given example, how the behavior of $\tilde{f}(z)$ for $z\to\infty$ depends on the method of parametrization of the asymptotic form of the coefficients a_k. If

$$\tilde{a}_k^{(\rho)}=c_0\Gamma(k+9/2-\rho)\cdot 2^k k^\rho,\quad c_0=0.04551\ldots, \tag{B.7}$$

it can be shown that, for $0<\rho<9/2$ and $g\to\infty$,

$$\tilde{\varphi}_\rho(g)=g^2\sum_{k=0}^{\infty}\tilde{a}_k^{(\rho)}(-g^2)^k\approx\begin{cases}Ag^3(\ln g)^{-\rho}, & \rho-\text{ is not an integer}\\ Bg, & \rho=1,\ 2,\ 3,\\ Cg^2, & \rho=4\end{cases} \tag{B.8}$$

where

$$A=-2^{-\rho}c_0\Gamma(9/2-\rho)/\Gamma(1-\rho),\quad B=1/2(-1)^\rho\Gamma(7/2-\rho)c_0,\quad C=1/8(\pi/2)^{1/2}c_0.$$

The behavior of the functions $\tilde{\varphi}_\rho(g)$ in the region of large g depends essentially on ρ, although the coefficients (B.7) for $k\gg 1$ differ only by terms of order $1/k$.

Note added in proof (December 15, 1977). We have now considered a further example: the perturbation-theory series for the energy $E_0(g)$ of the ground level of an anharmonic oscillator with nonlinearity $g|x|^4$ in a space of d dimensions. For $d=1$ the coefficients of this series were calculated by Bender and Wu,[2] and this permits us to construct the PT polynomials and the IPT functions. A comparison of these with the exact values of $E_0(g)$ obtained by numerical solution of the Schrödinger equation leads to the same conclusions as in Sec. 5. For $N\to\infty$ the region of applicability of the N-th approximation of perturbation theory contracts: $g_N\sim e/3N$. On the other hand, the calculation by the Padé-approximant method of the energy $E_{[N,N]}(g)$ leads to the result that the region in which the exact solution $E_0(g)$ is approximated increases with increase of N. This is an important advantage of the Padé method in the summation of series with factorially increasing coefficients. However, this advantage is manifested only in cases when a sufficient number of the first coefficients a_k of the series are known.

1) See Ref. 18, in which an analogous method was used to establish the relationship between the nearest singularity of the scattering amplitude $f(s,t)$ and the asymptotic form of the partial amplitudes $f_l(s)$ for $l\to\infty$.

2) This statement assumes the absence in the exact solution of terms of the type $\Phi(z)=\varphi(z)\exp(-z^{-\nu})$, where $\nu>0$ and $\varphi(z)$ is a function that does not have an essential singularity at $z=0$. It is obvious that the *PT* series does not give any information about the presence of such terms in the exact solution, since all derivatives of the function $\Phi(z)$ vanish as $z\to+0$. The presence of such terms should be established from independent considerations (see, e.g., Ref. 7). It is known that the existence of instantons, which are not present in the problems we are considering, lead to the appearance of such terms.

3) The quantities α, β, a and c_i/c_0 were obtained from the recursion relations for a_k given above. The coefficient c_0 was found by comparing the asymptotic form (27) with the exact values of a_k for $k=150-200$.

4) It should not be thought that the good approximation of $f(z)$ by the IPT functions is due to the fact that the Borel sum $\tilde{f}(z)$ is close to the exact solution. This occurs only in exceptional cases (example I); usually, these functions are very far from each other.

5) In the calculation of $\tilde{\psi}(g)$ the asymptotic form of the coefficients was written in the form $\tilde{a}_k=c\Gamma(k+\frac{1}{2})a^k k^4$ and formula (A.8) from Appendix A was used.

6) On the other hand, the polynomials $p_N(g)$ with $N=1$, 2 and 3 are close to each other only in the interval $0<g<20$. For

larger values of g they differ considerably, both from the IPT functions and amongst themselves (e.g., for $g=50$ we have $p_3/\psi_3=2.76$ and $p_3/p_2=4.22$, while $\psi_3/\psi_2=0.86$). The advantage of the IPT over the PT is clear from this.

[7)] I.e., the fraction $P_N(g)/Q_M(g)$, where N and M are the degrees of the numerator and denominator (33).

[8)] For $|z|>(2s+1)^{-2}$ the series in (B.2) is convergent, but the first term $h(z)\ln z$ has a branch point at infinity. The possibility of obtaining a convergent expansion for L' in inverse powers of the external field was noted earlier (35).

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