Interaction of a weak gravitational wave with the field of a rotating magnetic dipole

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The interaction between a weak gravitational wave of frequency ω and the field of a magnetic dipole rotating with a frequency $\omega_0 < \omega$ is considered. The gravitational wave source is assumed to be located at the center of the magnetic dipole. It is shown that an electromagnetic wave of frequency ω, whose amplitude is modulated at a frequency ω_0, develops as a result of the interaction. The modulation depth and the pulse shape depend on the coordinates of the point of observation and on the angle between the axis of rotation and the magnetic axis. The radiation is linearly polarized but the plane of polarization rotates with frequency ω_0. The possibility of subpulse drift is demonstrated. The distance at which the electromagnetic wave produced by the interaction may be detected is estimated with a neutron star as an example.

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A series of articles[1-7] has recently appeared which discuss various processes of emission and detection of gravitational waves (GW). Both laboratory and astrophysical objects are being investigated as possible GW emitters, but basically only devices of laboratory dimensions are considered as GW detectors. It seems to us that it is no less important to investigate the interaction of GW with objects of astrophysical dimensions, since the effect of the interaction may in this case attain the magnitude necessary for observations to be made. And although such "detectors" do not permit the experimenter to interact with them to obtain detailed and complete information about GW, nevertheless, even such purely qualitative observational studies are important for accurate and purposeful improvement of laboratory detectors of GW. We note, incidentally, that the three effects predicted by general relativity theory were first detected in an astrophysical context, and only after this was the observation of one of them carried out in a laboratory setting.

We consider in this connection the interaction of a weak GW with the field of a rotating star. According to Gal'tsov[3] and Papini and Valluri[4,5] the photoproduction of gravitons originates in the interior of stars in the Coulomb and magnetic dipole fields of the particles which comprise the substance of the star. Thus, stars are sources of weak GW, and the resulting gravitational radiation, depending on the spectra of the original photons, may belong to any range of frequencies. Many stars have fields which coincide with the field of a rotating magnetic dipole, whose magnetic axis is oriented at an angle to the rotation axis. While we do not aim to give a detailed description of the actual processes which are at work inside stars and lead to the appearance of a magnetic dipole field, we will discuss the simplest model of a rotating magnetic dipole.

Assume that the star is a conducting sphere of radius b, rotating at constant frequency $\omega_0 \ll c/b$ about an axis Z through the center of the sphere. Suppose that closed currents are circulating inside the sphere and their magnetic moment $\mathbf{m}$ is rigidly linked to the sphere and forms an angle β with the axis of rotation. After a simple calculation with standard boundary conditions we obtain the following expression for the electromagnetic field outside the sphere ($r \geq b$):

$$\begin{gathered}
\mathbf{E} = \frac{[\mathbf{r}\times\dot{\mathbf{m}}(t')]}{cr^3} + \frac{[\mathbf{r}\times\ddot{\mathbf{m}}(t')]}{c^2r^2}, \\
\mathbf{H} = \frac{3\mathbf{r}(\mathbf{m}(t')\mathbf{r}) - \mathbf{m}(t')r^2}{r^5} \\
+ \frac{3\mathbf{r}(\dot{\mathbf{m}}(t')\mathbf{r}) - \dot{\mathbf{m}}(t')r^2}{cr^4} + \frac{\mathbf{r}(\ddot{\mathbf{m}}(t')\mathbf{r}) - \ddot{\mathbf{m}}(t')r^2}{c^2r^3}, \\
\mathbf{m}(t') = |\mathbf{m}|\{\sin\beta\cos\omega_0 t', \sin\beta\sin\omega_0 t', \cos\beta\},
\end{gathered} \tag{1}$$

where $t' = t - r/c$, and the dot denotes total differentiation with respect to t.

We assume that the GW source inside the star is radiating GW of frequency $\omega \gg \omega_0$ against a background of flat space-time. Since the components of the GW depend in this case on the three spatial coordinates, therefore, making use of the four conditions of Hilbert and four coordinate conditions,[8] one may cause the four components h^{0i} to vanish, and any two of the six components $h^{\alpha\beta}$ will be independent.

We will consider the simplest weak ($h \ll 1$) GW, the components of which outside the source take the form

$$\begin{gathered}
h^{11} = -h^{22} = hkb\frac{H^{(1)}_{7/2}(kr)}{(kr)^{1/2}}P_3^2(\theta)e^{-i\omega t}\sin 2\varphi, \\
h^{12} = -\frac{1}{2}\frac{\partial h^{11}}{\partial\varphi}, \qquad h^{23} = -\frac{\partial h^{13}}{\partial\varphi}, \qquad h^{33} = 0, \\
h^{13} = -hkb\left[2\frac{H_{7/2}(kr)}{(kr)^{1/2}}P_3^1(\theta) + 12\frac{H_{3/2}(kr)}{(kr)^{1/2}}P_1^1(\theta)\right]e^{-i\omega t}\sin\varphi,
\end{gathered} \tag{2}$$

where H is a Hankel function and P a Legendre polynomial.

Evaluating the intensity of the gravitational radiation by means of the Landau-Lifshitz pseudotensor,[8] after averaging over the period of the wave,[2] we find

$$\frac{dI}{d\Omega} = \frac{225h^2k^2b^2c^5}{16\pi G}\sin^4\theta. \tag{3}$$

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Expressing the amplitude h of the GW in terms of the power F of the source, we obtain

$$h=(\pi FG/30k^2b^2c^5)^{1/2}. \tag{4}$$

We presume that the interaction of the GW (2) with the field of the rotating dipole (1) takes place in a vacuum. After linearizing Maxwell's equations with respect to the small parameter h, we get

$$\Box \mathbf{a}=-\frac{4\pi}{c}\mathbf{j}_{int}, \quad \Box\Phi=-4\pi q_{int}, \tag{5}$$

where $\mathbf{a}$ and Φ are the respective corrections, linear in h, to the vector and scalar potential of the electromagnetic field, and the expressions for q_{int} and $\mathbf{j}_{int}$ have the form

$$q_{int}=-\frac{1}{4\pi}\operatorname{div}\mathbf{B}, \quad \mathbf{j}_{int}=\frac{c}{4\pi}\left\{\frac{\partial \mathbf{B}}{\partial x^0}+\operatorname{rot}\mathbf{C}\right\}, \\ C^\alpha=h^{\alpha\beta}H_\beta, \quad B^\alpha=h^{\alpha\beta}E_\beta. \tag{6}$$

The solution of Eqs. (5) will be unique, if we demand the fulfillment of the Sommerfeld radiation condition and of the condition of continuity of the tangential components of the $\mathbf{E}$ and $\mathbf{H}$ fields at the surface of the sphere. We recognize that

$$\mathbf{m}(t')=\mathbf{m}_0+\mathbf{m}_1e^{-i\omega_0(t-r/c)}+\mathbf{m}_2e^{i\omega_0(t-r/c)};$$
$$\mathbf{m}_0=|\mathbf{m}|\{0;0;1\}\cos\beta,$$
$$\mathbf{m}_1={}^1/_2|\mathbf{m}|\{1;i;0\}\sin\beta,$$
$$\mathbf{m}_2={}^1/_2|\mathbf{m}|\{1;-i;0\}\sin\beta.$$

Then, substituting expressions (1) and (2) into (6), we obtain

$$\mathbf{j}_{int}=\mathbf{j}_0(\mathbf{r})e^{-i\omega t}+\mathbf{j}_1(\mathbf{r})e^{-i(\omega+\omega_0)t}+\mathbf{j}_2(\mathbf{r})e^{-i(\omega-\omega_0)t}, \\ q_{int}=q_1(\mathbf{r})e^{-i(\omega+\omega_0)t}+q_2(\mathbf{r})e^{-i(\omega-\omega_0)t}. \tag{7}$$

It is easy to ascertain that these expressions can be expanded in a series of spherical harmonies of the first kind with a finite number of terms. As an example, we write down the expansion for $\mathbf{j}_0(\mathbf{r})$:

$$j_{0x}=\frac{hk^2b}{14\pi r^3(kr)^{1/2}}\cos\varphi\{P_4^1(\theta)[6H_{9/2}^{(1)}(kr)+21H_{5/2}^{(1)}(kr)$$
$$+12H_{1/2}^{(1)}(kr)]+P_2^1(\theta)[12H_{9/2}^{(1)}(kr)-10H_{5/2}^{(1)}(kr)-12H_{1/2}^{(1)}(kr)]\},$$
$$j_{0z}=-\frac{hk^2b}{14\pi r^3(kr)^{1/2}}\{P_4(\theta)[24H_{9/2}^{(1)}(kr)+84H_{5/2}^{(1)}(kr)$$
$$+48H_{1/2}^{(1)}(kr)]+P_2(\theta)[12H_{9/2}^{(1)}(kr)-20H_{5/2}^{(1)}(kr)$$
$$+8H_{1/2}^{(1)}(kr)]+P_0(\theta)[16H_{5/2}^{(1)}(kr)-12H_{9/2}^{(1)}(kr)]\},$$
$$j_{0y}=-\frac{\partial j_{0x}}{\partial\varphi}.$$

We write the solution of Eqs. (5) in terms of the retarded potentials:

$$\mathbf{a}=\mathbf{a}_0(\mathbf{r})e^{-i\omega t}+\mathbf{a}_1(\mathbf{r})e^{-i(\omega+\omega_0)t}+\mathbf{a}_2(\mathbf{r})e^{-i(\omega-\omega_0)t};$$
$$\mathbf{a}_0(\mathbf{r})=\frac{1}{c}\int\frac{e^{ikR}}{R}\mathbf{j}_0(\mathbf{r}')dV', \tag{8}$$
$$\mathbf{a}_1(\mathbf{r})=\frac{1}{c}\int\frac{e^{i(k+k_0)R}}{R}\mathbf{j}_1(\mathbf{r}')dV',$$
$$\mathbf{a}_2(\mathbf{r})=\frac{1}{c}\int\frac{e^{i(k-k_0)R}}{R}\mathbf{j}_2(\mathbf{r}')dV', \quad R=|\mathbf{r}-\mathbf{r}'|. \tag{8'}$$

The right-hand sides of Eqs. (5) are defined in the entire space outside a sphere of radius b, and consequently the point of observation is located inside the integration region. Thus the integrals in expression (8) are improper integrals with a singular point at $\mathbf{r}'=\mathbf{r}$. However, these integrals converge, and, moreover, converge absolutely. From the definition of absolute convergence of improper integrals we have

$$\int_V j\,dV'=\lim_{V_\epsilon\to0}\int_{V-V_\epsilon} j\,dV',$$

and this limit does not depend on the manner of contraction of the domain V_ϵ to the singular point $\mathbf{r}'=\mathbf{r}$. For our purposes it is convenient to choose V_ϵ as the region included between two spheres with radii $r-\epsilon$ and $r+\epsilon$ and then let $\epsilon\to0$. We therefore rewrite the expression for $\mathbf{a}_0(\mathbf{r})$ in the form

$$\mathbf{a}_0(\mathbf{r})=\int_b^{r-\epsilon}r'^2dr'\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\varphi\frac{e^{ikR}}{R}\mathbf{j}_0(\mathbf{r}')$$
$$+\int_{r+\epsilon}^\infty r'^2dr'\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\varphi\frac{e^{ikR}}{R}\mathbf{j}_0(\mathbf{r}').$$

The expressions for $\mathbf{a}_1(\mathbf{r})$ and $\mathbf{a}_2(\mathbf{r})$ are written down analogously.

For an accurate calculation of the retardation time we use the Gegenbauer[9] theorem ($r_2>r_1$):

$$\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}=\frac{\pi i}{2(r_1r_2)^{1/2}}\sum_{n=0}^\infty(2n+1)J_{n+1/2}(kr_1)H_{n+1/2}^{(1)}(kr_2)P_n(\cos\gamma),$$
$$P_n(\cos\gamma)=P_n(\cos\theta_1)P_n(\cos\theta_2) \tag{9}$$
$$+2\sum_{m=1}^n\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta_1)P_n^m(\cos\theta_2)\cos m(\varphi_1-\varphi_2).$$

Thus, the integrands in (8) contain products of two series of spherical harmonics of the first kind: one has a finite number of terms (7) and the other an infinite number of terms (9). But because of the orthogonality of these functions on a sphere we obtain $\mathbf{a}$ in the form of a finite series after the integration in expression (8). The scalar potential Φ is obtained from the Lorentz gauge conditions. We add to the solution obtained for the inhomogeneous d'Alembert equations the solution of the corresponding homogeneous equations to satisfy the boundary conditions.

As a result we have at $r\geqslant b$ the following expressions for the strengths of the fields $\mathbf{E}$ and $\mathbf{H}$ (only the leading terms have been retained)

$$E_\theta=\frac{15ihkbm}{4r}\left(\frac{2}{\pi}\right)^{1/2}e^{-i(\omega t-kr)}\left[\frac{1}{b^2}-\frac{1}{r^2}\right]\Big[\sin\beta\cos\theta\cdot$$
$$\cdot\cos\left(\varphi-\omega_0t+\frac{\omega_0r}{c}\right)-\cos\beta\sin\theta\Big]\sin^2\theta, \tag{10}$$
$$E_\varphi=\frac{15ihkbm}{4r}\left(\frac{2}{\pi}\right)^{1/2}e^{-i(\omega t-kr)}\left[\frac{1}{b^2}-\frac{1}{r^2}\right]\sin\beta\sin^2\theta\sin\left(\varphi-\omega_0t+\frac{\omega_0r}{c}\right),$$
$$H_\varphi=E_\theta, \quad H_\theta=-E_\varphi.$$

From (10) it is evident that the interaction of the GW (2) with the field of the rotating magnetic dipole (1) produces an electromagnetic wave (EMW) of frequency ω amplitude-modulated at a frequency $\omega_0 \ll \omega$. The degree of modulation depends both on the coordinates of the point of observation and on the angle β.

For the EMW energy flux density into a unit solid angle, we have after averaging over the period $T=\pi/\omega$.

$$\frac{dI}{d\Omega}=\frac{15FGm^2}{256c^4}\left[\frac{1}{b^2}-\frac{1}{r^2}\right]^2\sin^4\theta[1-\cos^2\beta\cos^2\theta-\sin^2\theta\cos 2(\varphi-\omega_0 t+\omega_0 r/c)\sin^2\beta-\sin 2\beta\sin\theta\cos\theta\cos(\varphi-\omega_0 t+\omega_0 r/c)]. \quad (11)$$

Therefore, the high-frequency EMW will reach the observer in pulses whose shape and frequency are functions of r, θ, φ, and β. For a stationary magnetic dipole one must set $\omega_0=0$ in formula (11).

The total radiation flux through a sphere of radius r_0 equals

$$I=\frac{FGm^2}{8c^4}\left(1-\frac{\cos^2\beta}{7}\right)\left[\frac{1}{b^2}-\frac{1}{r_0^2}\right]^2. \quad (12)$$

From formula (12) it follows that the main part of the radiation originates in the region $b<r<10b$; only $\sim 2\times 10^{-2}$ of the radiated power originates outside this region.

Following Zel'dovich,[1] we will define the conversion coefficient α of GW into EMW energy as the ratio of the GW to EMW energy flux density:

$$\alpha=\frac{\pi Gm^2}{8c^4}\left[\frac{1}{b^2}-\frac{1}{r^2}\right]^2[1-\cos^2\beta\cos^2\theta-\sin^2\theta\sin^2\beta\cos 2(\varphi-\omega_0 t+\omega_0 r/c)-\sin 2\beta\sin\theta\cos\theta\cos(\varphi-\omega_0 t+\omega_0 r/c)]. \quad (13)$$

This coefficient reflects the characteristics of the converter, i.e. of the field (1). Up to this point it has been assumed that the source emits strictly coherent GW. However, it is more reasonable to assume that the GW will be produced by a large number of independent emitters, each of which radiates trains of GW with random phase and in an arbitrarily oriented directivity pattern (3), the result of which are incoherent GW with a spherical directivity pattern. In this case the energy flux density of the EMW becomes

$$S=F\alpha/4\pi r^2. \quad (14)$$

It should be noted that, besides the conversion of GW into EMW, a process of conversion of EMW into GW takes place. If $\alpha\ll 1$, this reverse process can be neglected; but if α is close to or greater than one, then according to Zel'dovich[1] it is necessary to consider both processes together, and also to take into account the refractive index of the matter and perturbations of other kinds, which reduce the coherence length and make the conversion of GW to EMW and vice versa incomplete. But these perturbations only reduce the energy flux density of the resulting EMW, but do not change qualitatively the effect of the amplitude modulation.

From formula (13) it follows that, if $2|\cot\beta\cot\theta|$ is considerably greater or considerably less than unity, then the pulses will arrive with frequency ω_0 or $2\omega_0$, respectively. If $2|\cot\beta\cot\theta|$ is close to unity, then the pulses will have a complicated shape, since they will be composed of pulses of frequency ω_0 and $2\omega_0$, each with amplitude and phase dependent on the coordinates of the point of observation and on the angle β. One should note that this amplitude modulation will give only a "window" through which one can observe the high frequency EMW. The GW from the source may give GW of amplitude irregular, so that an observer on Earth will register the resulting EMW only when there is a sufficiently powerful surge of high frequency GW in the window.

We will note two other features of the produced radiation. It follows from formula (10) that the radiation is linearly polarized, but consists of two parts. The plane of polarization of one part is fixed, but that of the other rotates with the rotation frequency of the star. For $\cos\theta>0$ the plane of polarization rotates clockwise, and for $\cos\theta<0$ counterclockwise. Thus, from the polarization of the EMW and the direction of rotation of the plane of polarization we can determine the angles β and θ. From expression (13) it follows that the motion of the Earth together with the observer relative to the star gives rise to a change of amplitude and of the phase difference of the pulses of frequency $2\omega_0$ and ω_0, which in turn causes a change in the pulse shape, namely a drift of the subpulses which constitute the resulting pulse. If the motion of the Earth is in the direction of $\mathbf{r}$ with speed v_r, then the drift will have a periodic character, and $N=c/v_r$ pulses will occur during a time equal to the period of drift. Assuming that $v_r=30$ km/s (the orbital velocity of the Earth around the sun) we get $N=10^4$. This value may be reduced if the radial velocity of the Earth relative to the star is larger. From formula (13) it also follows that for an observer moving toward the axis of rotation ($\sin\theta\to 0$) the effect of the amplitude modulation gradually diminishes.

We make a few estimates. Of all the known stars, neutron stars have the strongest magnetic fields. We cite therefore numerical estimates obtained by using neutron-star parameters[10]:

$$b=2\cdot 10^6 \text{ cm.} \quad H=10^{12} \text{ Oe.}$$

From formula (13) we have that in this case the conversion coefficient of GW into EMW equals $\alpha_{max}=3.2\times 10^{-14}$. According to Papini and Valluri,[5] as a result of photoproduction of gravitons, neutron stars emit GW with power $F=6.02\times 10^{34}$ erg/s, so that we obtain from formula (14)

$$4\pi r^2 S=1.9\cdot 10^{21} \text{ erg/s.}$$

We assume following Dyson and ter Haar,[10] we can detect an RF EMW with

$$S=10^{-16} \text{ erg/cm}^2\text{s,}$$

and, therefore, we can detect the EMW resulting from the interaction at a distance $r\leqslant 1.2\times 10^{18}$ cm from the neutron star. This distance increases with increasing power of the GW source. It is interesting to note that at this distance from a neutron star one can also detect GW. In fact, at this distance the GW energy flux density equals

$$S=3.1\cdot 10^{-3}\ \text{erg/cm}^2\text{s}.$$

According to Braginskiĭ et al.,[2] one can, with the help of an electromagnetic detector, register GW in the radio spectrum with an energy flux density

$$S=10^{-4}\ \text{erg/cm}^2\text{s}.$$

This makes it possible to select an object for setting up an experimental detection of GW on the basis of the singularities of the EMW (10) resulting from the interaction and coming from a neutron star.

In conclusion we note that, in the case of GW incident from the outside on a rotating magnetic dipole, the expressions for $\mathbf{j}_{\text{int}}$ and q_{int} (6) can be expanded in a series with an infinite number of terms of spherical harmonics of the first kind. Consequently the solution of Eq. (5) for the potentials $\mathbf{a}$ and Φ obtained in this case also as an infinite series of spherical harmonics of the first kind, while the coefficients of this series have a form which significantly complicates further analysis of the obtained solution.

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Theory of hadron plasma

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The characteristics are calculated of a new state of matter corresponding to densities n and temperatures T such that $\hbar c n^{1/3}$, $T \gtrsim 1$ GeV. This is the hadron plasma in which quarks that are the components of hadrons under ordinary conditions are collectivized. The calculations are performed within the framework of the so-called quantum chromodynamics, i.e., the theory of strong interactions. The results obtain for cold plasma with high density of baryon charge are applied to the collapse of neutron stars and to the problem of the repulsive core of nucleons. Results on the properties of hot neutral plasmas are applied to cosmology and to hadron collisions at high energies.

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1. INTRODUCTION

The present paper is concerned with the description of the properties of matter at densities much greater than the density of atomic nuclei, or temperatures exceeding the characteristic hadron mass $m\sim 1$ GeV. It is based on the theory of strong interactions with non-Abelian gauge fields[1] which is frequently called quantum chromodynamics[2] (see also the review by Politzer[3]). Under the above conditions, the separation between the hadrons becomes smaller than their dimensions, and they cease to behave as individual objects, i.e., the quarks of which they are made up are collectivized.[4,5] By analogy with the similar behavior of atomic electrons, this state of matter can be referred to as hadron plasma, in contrast to the normal state, i.e., the hadron gas. This analogy is considerably enhanced by the similarities between chromodynamics and electrodynamics: in both cases, the interaction is transported by massless vector fields.

To explain the notation used below, we recall that the Lagrangian density in chromodynamics has the form

$$L=-\frac{1}{4}F_{\mu\nu}{}^aF_{\mu\nu}{}^a+\sum_{N=1}^{N_K}\bar{\Psi}_N\left(i\hat{\partial}+\frac{g}{2}\lambda^a b_\mu{}^a\gamma_\mu-m_N\right)\Psi_N,$$
$$F_{\mu\nu}{}^a=\partial_\mu b_\nu{}^a-\partial_\nu b_\mu{}^a+gf^{abc}b_\mu{}^b b_\nu{}^c, \qquad (1)$$

where the fields b_μ^a correspond to the gauge vector fields, the gluons, and Ψ_N represent quarks of flavor N, i.e., ordinary quarks u and d, the strange quark s, the charmed quark c, and, possibly, other quarks up to flavor N_k. Greek subscripts correspond to Lorentz subscripts and run over values between 0 and 3, and Latin superscripts are color indices running over values 1–8; λ^a are the Gell-Mann matrices of the group SU_3, and

