to electron-electron scattering, can be the difference in the masses of carriers of a single sign.[6,7] The logic is probably as follows: If the masses of all the particles are the same, then conservation of momentum in the collision leads to current conservation:

$$\Delta\mathbf{J}=e\,\Delta\mathbf{v}_1+e\,\Delta\mathbf{v}_2=\frac{e}{m}(\Delta\mathbf{p}_1+\Delta\mathbf{p}_2)=0. \tag{31}$$

If the masses are different, then the current is not conserved but "relaxes":

$$\Delta\mathbf{J}=e\left(\frac{\Delta\mathbf{p}_1}{m_1}+\frac{\Delta\mathbf{p}_2}{m_2}\right)\neq 0. \tag{32}$$

However, such a "relaxation of the current," if important at all, matters only in the alternating field. In a static field, in a system of carriers of the same sign, a stationary state cannot exist (see also Ref. 1).

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# Kinetics of nucleation and stratification of dilute He$^3$–He$^4$ solutions under pressure at low temperatures

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It is possible to obtain under pressure, in the absence of a free liquid-vapor surface, a metastable solution of liquid He$^3$ in liquid He$^4$ in the stratification region in the phase diagram. The rate of formation of nucleating centers in the volume of such a solution and its dependence on temperature and concentration are computed. Classical and quantum conditions for nucleation are analyzed. The cases of the normal and the superfluid states of the Fermi component in the solution are considered in the region of the quantum regime. It is shown that the stratification of the supersaturated solution under the indicated conditions should begin not on the phase-equilibrium line, but on the rapid-nucleation line, $T_q$. The $T_q$ line is constructed.

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It is well known that the solubility of He$^3$ in liquid He$^4$ at $T<0.87$ K is finite.[1] Liquid solutions of the helium isotopes with He$^3$ contents in the range $x_l(T,P)<x<x_u(T,P)$, where $x=x_l(T,P)$ and $x=x_u(T,P)$ are the branches of the phase diagram, separate into two equilibrium phases with molar concentrations $x_l(T,P)$ and $x_u(T,P)$. At $T<0.15$ K the upper phase is virtually pure He$^3$ ($x_u\approx 1$), while the bottom phase is a solution with a He$^3$ content within the limits $0.0637<x_l<0.094$, depending on the pressure.[2]

As has been shown by Andreev,[3,4] a film of impurity He$^3$ atoms exists at the liquid solution-vapor interface. As the conditions approach the conditions for phase equilibrium a He$^3$-rich phase develops continuously out of the surface layer.[5,6] The inverse effect is observed at the liquid-solid boundary: owing to the van der Waals forces, the vessel walls and dust particles get covered with He$^4$ films,[6,7] and these surfaces cannot serve as effective nucleation centers. Thus, in the absence of a liquid-vapor boundary (such a situation arises in experiments under pressure when the vessel is filled with the solution at a temperature higher than the stratification temperature and then cooled), the stratification of the supersaturated solution should occur through the formation and growth of nuclei of the stable phase in the volume of the metastable medium.

The appearance of stable nuclei is connected with the surmounting of an energy barrier by the system. At temperatures close to absolute zero, a supercritical nucleus can arise only as a result of quantum penetration through the barrier.[8] A nucleating center is a macroscopic formation, and can be described in the quasiclassical approximation. In the same way as was done in Ref. 8, we should write down the classical Hamiltonian and then quantize it.

If the degree of metastability of the medium is not high, then the optimal forms of the density and concentration distributions correspond to a spherical nucleus of the new phase, the thickness of the transition layer being small compared to the dimensions of the nucleus, i.e., $d/R\ll 1$. This circumstance allows us to describe the transition region in terms of surface tension. On account of the slowness of the motion of the interphase boundary ($\dot{R}/u\ll 1$, $u$ is the speed of sound), energy dissipation can be neglected, i.e., it can be assumed that, in the course of the growth of a nucleus, new excitations do not arise and all the parameters of the system adia-

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batically adjust themselves to the instantaneous value of $R(t)$, as a result of which it is possible to write down the classical Lagrangian of the problem as a function of one generalized macroscopic coordinate, $R$, and its velocity, $\dot{R}$.[8]

The potential energy of a nucleus of radius $R$ is computed as the minimum work necessary for its creation, and has the standard form[9]:

$$U=AR^2-BR^3, \quad A=4\pi\sigma, \quad B=4\pi(\mu_1-\mu_3)/3v_3,$$

where $\sigma$ is the surface tension of the interphase boundary, $v_3$ is the volume occupied by a He$^3$ atom in the stable phase, $\mu_3$ and $\mu_1$ are the chemical potentials, measured under external pressure, of He$^3$ in the stable and metastable phases, respectively, and $\mu_1-\mu_3 \approx (\partial\mu_1/\partial x)\Delta x$, $\Delta x$ being the supersaturation of the solution.

To find the total kinetic energy, $K$, of the medium, it is necessary to write down the expressions for the fluxes associated with the growth of a fluctuation, and integrate the kinetic-energy density over the volume of the metastable phase. Under the above-formulated conditions, the expression for the kinetic energy becomes parametrized: $K=\tilde{M}R^3\dot{R}^2/2$, and the Lagrangian of the system assumes the form indicated in Ref. 8:

$$L=\tilde{M}R^3\dot{R}^2/2-AR^2+BR^3, \tag{1}$$

where $\tilde{M}$ is the effective mass density to be determined.

The classical Hamiltonian for a fluctuation,

$$H=p_y^2/2My^3+\alpha y^2(1-y), \tag{2}$$

has the form of the Hamiltonian for the one-dimensional motion of a particle of variable mass, $My^3$, moving in the field $U(y)=\alpha y^2(1-y)$, $y>0$; here $y=R/R_0$ is dimensionless coordinate, $R_0=A/B$, $M=\tilde{M}R_0^5$, $\alpha=AR_0^2$, and $p_y=\partial L/\partial\dot{y}=My^3\dot{y}$ is the generalized momentum corresponding to the coordinate $y$. The oscillatory motion of the particle in the region $0<y<y_1(\epsilon)$ corresponds to quantum heterophase fluctuations; the region $y_1(\epsilon)<y<y_2(\epsilon)$ is classically forbidden, while the region $y>y_2(\epsilon)$ describes the growth of a real supercritical nucleus, $y_1(\epsilon)$ and $y_2(\epsilon)$ being the roots of the equation $\epsilon\equiv H/\alpha=y^2(1-y)$. The quantization is carried out in the manner indicated in Ref. 8.

The probability of formation of a supercritical nucleus is proportional to the coefficient of penetration through the barrier. Thus, the problem of the computation of the probability reduces to the problem of finding the effective mass of the fluxes flowing in a quantum Fermi-Bose liquid containing a growing nucleation center. At a very low temperature $T=T_c$, another phase transformation—the transition of the Fermi component into the superfluid state—should occur in the solution, and, consequently, three cases should be considered in the region of the quantum nucleation regime: $T=0$, $0<T<T_c$, and $T>T_c$.

At $T=0$ all the Fermi particles are paired, and there exist in the liquid two superfluid motions described by two-velocity hydrodynamics. In the narrow temperature interval $0<T<T_c$, there exist three motions in the liquid—two superfluid and one normal—which are describable in terms of three-velocity hydrodynamics1) [10]; at $T>T_c$ the metastable phase is a solution of the normal Fermi liquid and the superfluid Bose liquid, motion in which is described by two-velocity hydrodynamics of a type different from the one that describes the motions at $T=0$.[11] The liquid can be considered to be ideal and incompressible. In this approximation, in the case of the two-velocity hydrodynamics the velocities are uniquely determined from the continuity equations

$$\operatorname{div}\mathbf{j}_1=0, \quad \operatorname{div}\mathbf{j}_2=0, \tag{3}$$

($\mathbf{j}_1$ and $\mathbf{j}_2$ are the radially symmetric fluxes of the Fermi and Bose components, respectively) and the boundary conditions at $r=R(t)$, which express the conservation of particles in crossing the surface of a nucleus,

$$j_{1r}=-\Delta\rho_1R^2\dot{R}, \quad j_{2r}=-\Delta\rho_2R^2\dot{R}, \tag{4}$$

where $\Delta\rho_1=\rho_3-\rho_1$, $\Delta\rho_2=-\rho_2$, $\rho_3$ is the density of He$^3$ in the stable phase, and $\rho_1$ and $\rho_2$ are the densities of the Fermi and Bose components of the solution. The velocity profiles have the form $v=fR^2\dot{R}/r^2$, where the $f$'s are constants that can be determined from the boundary conditions.

For $T=0$ the expressions for the fluxes have the form

$$\mathbf{j}_1=\rho_{s1}\mathbf{v}_{s1}+\tilde{\rho}_s\mathbf{v}_{s2}, \quad \mathbf{j}_2=\tilde{\rho}_s\mathbf{v}_{s1}+\rho_{s2}\mathbf{v}_{s2},$$

where

$$\rho_{s1}=(m_1^2/m_0^*)N_1, \quad \rho_{s2}=\rho_2-\tilde{\rho}_s, \quad \tilde{\rho}_s=m_1N_1-\rho_{s1}$$

are the density coefficients of the liquid with two superfluid components, $\mathbf{v}_{s1}$ and $\mathbf{v}_{s2}$ are the velocities of the superfluid motions, $m_1$ is the mass of the He$^3$ atom, $m_0^*=m^*/(1+F_1/3)$, $m^*$ is the effective mass of He$^3$ in the solution, $N_1$ is the Fermi-particle density, and $F_1$ is a Fermi-liquid characteristic, defined in the usual manner.[10] The kinetic-energy density is computed from the formula[12]

$$\varepsilon_K=(\rho_{s1}v_{s1}^2+2\tilde{\rho}_s\mathbf{v}_{s1}\mathbf{v}_{s2}+\rho_{s2}v_{s2}^2)/2.$$

At a temperature above the point of transition of the Fermi component into the normal state, the fluxes are given by the following expressions:

$$\mathbf{j}_1=\int\mathbf{p}n_{\mathbf{p}}\,d\tau, \quad \mathbf{j}_2=\rho_2\mathbf{v}_s.$$

The Fermi-particle distribution function has the form

$$n_{\mathbf{p}}=n_{\mathbf{q}+m\mathbf{v}_s}=n^0+m^*v_F^2\nu_1\cos\mathbf{q}\mathbf{v}_s\,\partial n_0/\partial\varepsilon,$$

where $\nu_1$ characterizes the displacement of the Fermi surface in a coordinate system moving with velocity $\mathbf{v}_s$, and determines the relative velocity of the normal component:

$$\mathbf{v}_n-\mathbf{v}_s=-(1+F_1/3)\nu_1v_F\mathbf{v}_s/v_s.$$

In this case the mass flux of the Fermi component

$$j_1=m_1N_1[1-(1+F_1/3)\nu_1 v_F/v_s]\mathbf{v}_s.$$

The kinetic-energy density in this case

$$\varepsilon_K=[\rho v_s^2+m^*N_1(v_n^2-v_s^2)/(1+F_1/3)]/2;$$

$\rho$ is the total mass density of the solution. Integrating the energy density over the volume, we obtain expressions for the effective mass density of the fluxes.

At $T>T_c$ the quantity $\tilde{M}$ does not depend on temperature, and is given by the formula

$$\tilde{M}=\tilde{M}_c=\frac{4\pi\rho}{1-c_n}\left\{\left(\frac{\Delta\rho}{\rho}\right)^2+c_n\left(\frac{\Delta N}{N_1}\right)^2+2c_n\frac{|\Delta\rho|}{\rho}\frac{|\Delta N|}{N_1}\right\},\qquad(5)$$
$$\Delta N=N_s-N_1,\quad \Delta\rho=\rho_s-\rho,\quad c_n=\frac{m^*N_1}{\rho(1+F_1/3)}.$$

In the range $T<T_c$, the quantity $\tilde{M}(T)$ varies from $\tilde{M}(0)$ to $\tilde{M}_c$. In this temperature interval the Hamiltonian, which is not given here, has a form that is different from, and more complex than, that of (2). The numerical values of the probability vary insignificantly in the entire $T$ interval. As it turns out, numerically, $|\tilde{M}_c-\tilde{M}(0)|/\tilde{M}_c\approx 1/3$.

The rate of nucleation in a unit volume of the metastable medium at $T=0$ K is estimated by the formula $W_0=J_0D_0$, where $J_0=N_0\omega_0$ is the flux flowing over a barrier with zero heterophase-fluctuation level, $N_0$ is the number of virtual nucleation centers, $\omega_0$ is the zero-point frequency of the heterophase fluctuations, and $D_0$ is the quasiclassical coefficient of penetration through a barrier with a zero level[8]:

$$N_0\sim N_1,\quad \omega_0=\gamma\frac{\alpha^{5/7}\hbar^{2/7}}{M^{2/7}},\quad \gamma=\left\{\frac{12\pi^{5/2}}{\Gamma^2(1/4)}\right\}^{4/7}$$
$$D_0=\exp\left\{-\frac{5\pi\sqrt{2}}{64}\left(\frac{M\alpha}{\hbar^2}\right)^{1/2}\right\}.$$

At $T\neq 0$ a supercritical nucleating center appears as a result of an optimal combination of thermal activation and tunneling leakage, which, for systems with a Hamiltonian of the type derived above, consists in the following: At $0<T<T^*(\Delta x)$ the nucleating centers form by purely quantum means (tunneling from the zero level), while at $T>T^*(\Delta x)$ the nucleation regime is a purely thermal-activation regime. The preexponential functions in both cases can be easily computed:

$$W(\Delta x,T)=\begin{cases}N_0\omega_0\left(1-\dfrac{T}{\theta}\right)^{-1}\exp\left(-\dfrac{\alpha\varepsilon_m}{T^*(\Delta x)}\right), & T<T^*\\[2ex] N_0\omega_0\left(\dfrac{\pi\sqrt{2}}{3\sqrt{3}}\dfrac{\nu T}{\alpha}-1\right)^{-1}\exp\left(-\dfrac{\alpha\varepsilon_m}{T}\right), & T>T^*\end{cases};\qquad(6)$$
$$\theta=\frac{4\alpha}{\pi\nu},\quad \nu=2\sqrt{2}\left(\frac{M\alpha}{\hbar^2}\right)^{1/2},\quad \varepsilon_m=\frac{4}{27}.$$

$T^*(\Delta x)$ is found from the condition for a change of regime:

$$T^*(\Delta x)=128\alpha\varepsilon_m/5\pi\nu.\qquad(7)$$

The index of the exponential function in $W$ is proportional to $\Delta x^{-7/2}$ for $T<T^*(\Delta x)$ and to the quantity $\Delta x^{-2}T^{-1}$ for $T>T^*(\Delta x)$. The critical dependence of $W(\Delta x,T)$ on $\Delta x$ leads to a situation in which on one side of the rapid-nucleation line, $T_q$, determined from the equation $W(\Delta x,T_q)=1$,[2)] i.e., in which for $\Delta x<\Delta x_q$, the rate of formation of nucleating centers is virtually equal to zero, while on the other side, i.e., for $\Delta x>\Delta x_q$, the solution breaks up practically instantaneously.

Thus, the stratification of the solutions under pressure in the $T$-$x$ plane should begin not on the phase-equilibrium line, but on the $T_q$ line (it is assumed that the liability boundary is located to the right). For the $T_q$ line we have

$$\Delta x_q\approx\left\{0.23\ln\left[N_0\omega_0\left(1-\frac{T}{\theta}\right)^{-1}\right]\right\}^{-2/7}\left(\frac{\sigma\tilde{M}}{\hbar^2}b^7\right)^{1/7},\quad T<T^*,$$
$$0.6\pi\sigma\frac{b^2}{\Delta x_q^2}\approx T\ln\left[0.1N_0\omega_0\frac{T}{T^*(\Delta x_q)}\right],\quad T>T^*;\qquad(8)$$
$$b=3\sigma v_3\left(\frac{\partial\mu_1}{\partial x}\right)^{-1}.$$

At the point where there is a change of nucleation regime, which point is determined by the relation $T_q(\Delta x)=T^*(\Delta x)$, the $T_q$ line has a kink defined by

$$\left(\frac{dx_q}{dT}\right)_+\approx-\frac{\Delta x}{2T^*},\quad \nu^{-1}\sim\left|\left(\frac{dx_q}{dT}\right)_-\right|\ll\left|\left(\frac{dx_q}{dT}\right)_+\right|.\qquad(9)$$

The shape of the $T_q$ line is shown in Fig. 1.

Using the results obtained in measurements of the surface energy of the interphase boundary, $\sigma=0.023$ erg-cm$^{-2}$,[6] the chemical potential of He$^3$ in the solution, $\partial\mu_1/\partial x\approx 0.39\times 10^{-16}$ erg,[7] and the other parameters at $x=0.07$ and $P=0.5$ atm,[13] we obtain the numerical estimates:

$$\omega_0\approx 10^{11}\ \text{sec}^{-1},\quad R_0/a\approx 0.5\Delta x^{-1},$$
$$T^*(\Delta x)\approx 0.2\cdot\Delta x^{3/2}\ [\mathrm{K}],$$
$$U_{max}\approx 3\cdot 10^{-2}\Delta x^{-2}\ [\mathrm{K}],$$
$$T^*(\Delta x_q)\approx 14\cdot 10^{-3}\ \mathrm{K},\quad \Delta x_q(0)\approx 0.15.$$

On the basis of the fact that, for concentrations of up to 16%, the derivative of the chemical potential of He$^3$ in the solution does not vanish:

$$\partial\mu_1/\partial x>0,\quad 0<x<0.16,$$

Seligman *et al.*[7] suggested that, in the absence of a liquid-vapor boundary, He$^3$–He$^4$ solutions are metastable right up to concentrations not lower than the in-

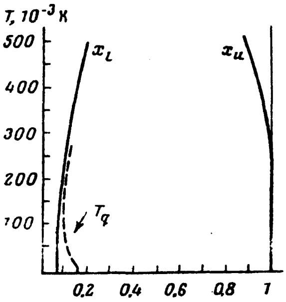

FIG. 1. Dependence of the limiting concentration of a supersaturated He$^3$-He$^4$ solution under a pressure of 0.5 atm on temperature (the $T_q$ curve); $x_l$ and $x_u$ are the branches of the stratification diagram.

dicated concentration. A $\Delta x = 0.3\%$ supercooled solution was obtained by Landau *et al.*[14] at $P = 0.5$ atm and $T = 0.03$ K. In an experiment performed by Watson *et al.*[2] a 9.28% solution, after the disappearance of the interphase boundary at $P = 7$ atm, again split up into two phases at $P = 16$ atm on the stratification line. However, as was noted by the authors, the mixture did not hold out until full equilibrium was reached.[2] Unfortunately, as far as we know, no systematic experimental investigations of the considered phenomenon with whose results we could have compared the results of the present work have so far been performed.

The experimental investigation of the phenomenon involves the necessity to penetrate fairly far into the metastable region. This can, apparently, be done by forcing the solution to flow out slowly from the vessel through a "porous membrane" (a capillary or a system of capillaries). In this case $He^4$ will mainly flow out, and the solution in the vessel will grow richer in $He^3$. This corresponds to displacement parallel to the horizontal axis in the $x$-$T$ plane.

In conclusion, we express our gratitude to A. F. Andreev and S. T. Boldarev for useful discussions.

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1) Strictly speaking, for $0 < T < T_c$ there exists a temperature region in which, for high supersaturations, the mean free path, $\lambda \approx \hbar v_F \epsilon_F T^{-2}$, of the Fermi excitations turns out to be large compared to the nucleus dimensions (i.e., $\lambda \gg R$) and in which, consequently, the hydrodynamic description of the normal component is incorrect. It can be shown, however, that allowance for this circumstance changes the result insignificantly, and the hydrodynamic approximation yields virtually the correct answer in the dangerous region also.

2) The rate, $W(\Delta x, T)$, of nucleation contains the dimensional factor $N_0\omega_0$. In the equation $W(\Delta x, T) = 1$ and the formulas (8) following from it, the value of $N_0\omega_0$ is taken in the cgs system, which corresponds to the characteristic scales of the quantities for the experiments in this region and gives an enormous value: $\ln N_0\omega_0 \approx 80$. Therefore, a change of the time scale by several orders of magnitude virtually does not change the position of the "rapid-nucleation" curve.

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# Quasitwo-dimensional electron-hole liquid in strong magnetic fields

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A study is made of the formation of an electron–hole liquid in quasitwo-dimensional systems in magnetic fields strong enough to ensure that the one-particle spectrum is completely discrete. Thermodynamic functions of this liquid are also calculated. In contrast to other systems, the exchange interaction is sufficient to form a liquid in the case considered and the correlation effects associated solely with virtual transitions to higher levels are negligible. The phase diagram is obtained and a "van der Waals" theory of liquid–gas phase transitions is developed for the investigated system. The possibility of experimental observation of the predicted effects is discussed.

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The behavior of electron and electron–hole systems in strong magnetic fields is of considerable interest. In particular, there have been many investigations of the kinetic properties (see, for example, [1–4]) and Wigner crystallization[5–8] of quasi two-dimensional electron systems in strong magnetic fields. We shall consider the formation of a liquid in a quasi two-dimensional electron–hole system in a transverse strong magnetic field. The system may be, for example, a quantized semimetal film with equilibrium electrons $e$ and holes