Theory of degenerate ^{3}He–He II solutions

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We consider a degenerage solution of ^{3}He in superfluid ^{4}He from the point of view of the properties of fermion systems with short interaction ranges. We obtain all terms of the expansion of thermodynamic quantities and kinetic coefficients in powers of $x^{1/3}$, where x is the molar concentration, for the case where the properties of the solution are determined by specifying only a single parameter—the s-wave scattering length a. Comparison with experimental data shows that the theory proposed here describes the solution well up to concentrations of the order of 3 to 4%. We evaluate the propagation speeds of spin waves, of hydrodynamic and of high-frequency sound oscillations.

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1. INTRODUCTION

A solution of ^{3}He in superfluid ^{4}He at temperatures below the degeneracy temperature T_0 is a quantum Fermi-liquid dissolved in a superfluid background. It then becomes important to take into account effects connected with the degeneracy and with the Fermi-liquid interaction of the impurity excitations. Bardeen, Baym, and Pines[1] and a number of other authors (for references see the review by Ebner and Edwards[2]) suggested the use of arbitrarily chosen model potentials for the interaction between a pair of impurity quasi-particles to describe a degenerate solution. The parameters of the potentials were chosen in such a way that the calculated characteristics of the solutions were guaranteed to agree with experimental data. A phenomenological theory in the spirit of Landau's Fermi-liquid theory was first constructed by Khalatnikov (Ref. 3, p. 293).

The thermodynamic properties of a degenerate solution of ^{3}He in superfluid ^{4}He are comprehensively determined by specifying Landau's Fermi-liquid function $f_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}')$.[3,4] For solutions of an arbitrary concentration (less than the demixing concentration of ~6%) the explicit form of $f_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}')$ is, in general, unknown. Experimentally, however, one can only determine the first two harmonics in the expansion of the Fermi-liquid function in terms of Legendre polynomials. It is, however, clear that in the case of sufficiently low concentrations the bare ^{3}He quasi-particles dissolved in the superfluid background form a dilute degenerate Fermi gas of slow particles with a short interaction range. The thermodynamic functions of a gas of fermions in which the energy of interaction between two particles decreases rather rapidly at large distances was studied by Huang, Lee, and Yang[5,6] and also by Abrikosov and Khalatnikov (see Ref. 3).

In such an approach all properties of the solution can be described by just a single parameter—the s-wave scattering length a for the collision of two bare quasi-particles. It then turns out to be possible to evaluate theoretically the concentration dependence of all experimentally observed quantities. The author has earlier studied some properties of a degenerate solution.[7] The present paper deals in the framework of the above mentioned approach, with the thermodynamic characteristics, kinetic effects, and oscillatory processes in weak superfluid solutions, and presents a comparison with the available experimental data. Since we are dealing with a solution at sufficiently low temperatures, we shall hereafter neglect the contribution from phonons and rotons to the thermodynamic and kinetic properties. Only the impurity excitations will give the basic contribution.

The state of the solution is completely determined by three quantities: the distribution function of the excitations $n_\sigma(\mathbf{p})$ ($\mathbf{p}$ and σ are the momentum and spin of the excitation), the velocity $\mathbf{v}_s$ of the superfluid motion, and the density m_4N_4 of the Bose-component of the liquid. Here m_4 (m_3) is the mass of a ^{4}He (^{3}He) atom, N_4 (N_3) the number of ^{4}He (^{3}He) atoms per unit volume of the solution. The total energy E and momentum $\mathbf{P}$ of the system are functionals of these three variables. We define the energy $\varepsilon_\sigma(\mathbf{p})$ and momentum $\mathbf{p}$ of an excitation through the following variational relations:

$$\varepsilon_\sigma(\mathbf{p})=\left[\frac{\delta E}{\delta n_\sigma(\mathbf{p})}\right]_{\mathbf{v}_s,N_4}, \qquad \mathbf{P}=m_4N_4\mathbf{v}_s+\sum_{\mathbf{p}\sigma}\mathbf{p}n_\sigma(\mathbf{p}). \tag{1.1}$$

We find, as usual, the equilibrium distribution function by maximizing the entropy under the additional conditions that the total energy and the number of impurity atoms are constant:

$$n_\sigma^{(0)}(\mathbf{p})=\frac{1}{2}\left[1-\operatorname{th}\frac{\varepsilon_\sigma(\mathbf{p})-\mu_3}{2T}\right]. \tag{1.2}$$

The energy of an excitation $\varepsilon_\sigma(\mathbf{p})$ and the chemical potential of ^{3}He in the solution, μ_3, depend, in general, on $\mathbf{v}_s$ and the external magnetic field $\mathbf{H}$.

The energy of an excitation $\varepsilon_\sigma(\mathbf{p})$ is also a functional of $n_\sigma(\mathbf{p})$, $\mathbf{v}_s$, and N_4, and its first variational derivative determines the Fermi-liquid function:

$$\left[\frac{\delta\varepsilon_\sigma(\mathbf{p})}{\delta n_{\sigma'}(\mathbf{p}')}\right]_{\mathbf{v}_s,N_4}=f_{\sigma\sigma'}(\mathbf{p},\mathbf{p}'). \tag{1.3}$$

Changing to a reference system which moves with the velocity $\mathbf{v}_s$ through a Galileo transformation and using the definitions (1.1) and (1.3), we can find the energy of an excitation in the laboratory frame of reference (Ref. 3, p. 297) which close to the Fermi surface has to a

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first approximation the following form:

$$\varepsilon_\sigma(\mathbf{p})=\mu_3+\frac{p}{m^*}(p-p_0)+\left[\left(1+\frac{F_1}{3}\right)^{-1}-\frac{m_3}{m^*}\right]\mathbf{p}\mathbf{v}_s - \frac{\beta}{1+Z_0/4}\sigma\mathbf{H}+\sum_{\mathbf{p}'\sigma'} f^{(0)}_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')\delta n_{\sigma'}(\mathbf{p}'), \qquad (1.4)$$

where $\delta n_\sigma(\mathbf{p})$ is the deviation of the distribution function from the equilibrium function (1.2) for constant $\mathbf{v}_s$ and N_4, β is the Bohr magneton, $p_0=(3\pi^2N_3)^{1/3}\hbar$ is the Fermi momentum, m^* the total effective mass,

$$\frac{p_0m^*}{\pi^2\hbar^3}f^{(0)}_{\sigma\sigma'}(\theta)=F(\theta)+Z(\theta)\sigma\sigma'=\sum_k (F_k+Z_k\sigma\sigma')P_k(\cos\theta)$$

is the value of the function $f_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')$ at the Fermi surface in the zeroth approximation in $\mathbf{v}_s$ and $\mathbf{H}$, which depends on the angle θ between the vectors $\mathbf{p}$ and $\mathbf{p}'$, and $P_k(\cos\theta)$ are the Legendre polynomials. (When comparing with the results obtained by Khalatnikov[3] one must bear in mind that the variation $\delta n_\sigma(\mathbf{p})$ in Ref. 3 and in the present paper are defined differently.)

The change in the distribution function caused by the appearance of the superfluid motion and the inclusion of an external magnetic field can then in first approximation be expressed as follows:

$$\Delta n^{(1)}_\sigma(\mathbf{p})=\frac{\partial n^{(0)}}{\partial\varepsilon}\left[\left(1+\frac{F_1}{3}\right)^{-1}-\frac{m_3}{m^*}\right]\mathbf{p}\mathbf{v}_s-\frac{\partial n^{(0)}}{\partial\varepsilon}\frac{\beta}{1+Z_0/4}\sigma\mathbf{H}. \qquad (1.5)$$

We note that the chemical potential μ_3 changes only in second order in $\mathbf{v}_s$ and $\mathbf{H}$.

2. THE FERMI-LIQUID FUNCTION

For a single ^{3}He atom placed in superfluid ^{4}He at rest the energy spectrum of an impurity quasi-particle can be written in the form

$$\varepsilon=-\Delta+\frac{p^2}{2M}\left[1-\xi\left(\frac{p}{p_c}\right)^2\right]. \qquad (2.1)$$

Here $p_c=m_4s_0$, s_0 is the sound speed in pure ^{4}He, $M\approx 2.33\,m_3$. The quantity ξ was determined from second-sound measurements in solutions and is $\xi=0.14\pm0.05$.[8] According to Sobolev's data[9] the quantity $\xi=0$. However, numerical calculations show that taking this fact into account hardly affects the results of the subsequent calculations.

Following a method proposed in Ref. 3, we can write the total energy of the system in second-order perturbation theory in the form

$$E=E^{(0)}_4+\sum_{\mathbf{p}\sigma}\left\{-\Delta+\frac{p^2}{2M}\left[1-\left(\frac{p}{p_c}\right)^2\xi\right]\right\}n_\sigma(\mathbf{p})+\frac{8\pi a\hbar^2}{M}\sum_{i,k}n_in_kQ_{ik} - \frac{128\pi^2a^2\hbar^4}{M}\sum_{i,k,l,m}\frac{n_in_kn_lQ_{ik}}{p_i^2+p_k^2-p_l^2-p_m^2}, \qquad (2.2)$$

where the indices i, k, l, m number also the spin states, $E^{(0)}_4$ is the energy of pure ^{4}He, $Q_{ik}=\frac{1}{4}-\sigma_i\cdot\sigma_k$. Twice varying the energy (2.2) with respect to $n_\sigma(\mathbf{p})$ we get an expression for the Fermi-liquid function $f^{(0)}_{\sigma\sigma'}(\theta)$ which was first obtained in this way by Abrikosov and Khalatnikov (see Ref. 3, p. 277):

$$f_{\sigma\sigma'}(\theta)=\frac{2\pi a\hbar^2}{M}\left[1+2\lambda\left(2+\frac{\cos\theta}{2\sin(\theta/2)}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right] - \frac{8\pi a\hbar^2}{M}\sigma\sigma'\left[1+2\lambda\left(1-\frac{1}{2}\sin\frac{\theta}{2}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right], \qquad (2.3)$$

where $\lambda=p_0a/\pi\hbar$ is the small parameter of the theory. Retardation effects in the interaction of the impurity quasi-particles give a contribution to the Fermi-liquid function of order $(p_0/p_c)^2$ and we shall therefore neglect them here and in what follows.

The Fermi-liquid harmonics F_n and Z_n have opposite signs: when $n=0$

$$F_0\propto\lambda,\quad Z_0/4\propto-\lambda,$$

when $n\neq0$

$$F_n\propto(-1)^{n+1}\lambda^2,\quad Z_n/4\propto(-1)^n\lambda^2$$

and they decrease rapidly for large values of n:

$$F_n\propto(-1)^{n+1}\lambda^2n^{3/2}2^{-2n},\quad Z_n/4\propto(-1)^n\lambda^2n^{1/2}2^{-2n},\quad n\to\infty.$$

When superfluid motion and an external magnetic field are present, the expansion of the Fermi-liquid function in a power series in $\mathbf{v}_s$ and $\mathbf{H}$ takes the following form, if we take the symmetry of $f_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')$ into account:

$$f_{\sigma\sigma'}(\theta)=f^{(0)}_{\sigma\sigma'}(\theta)+h_{\sigma\sigma'}(\theta)(\mathbf{p}+\mathbf{p}')\mathbf{v}_s+g(\theta)(\sigma+\sigma')\mathbf{H}. \qquad (2.4)$$

The function $f_{\sigma\sigma'}(\theta)$ is also a functional of $n_\sigma(\mathbf{p})$, $\mathbf{v}_s$, and N_4 and its change can therefore be expressed in terms of a variation of the distribution function through the formula

$$\delta f_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')=\sum_{\mathbf{p}''\sigma''}\varphi_{\sigma\sigma'\sigma''}(\mathbf{p},\mathbf{p}',\mathbf{p}'')\delta n_{\sigma''}(\mathbf{p}''), \qquad (2.5)$$

where $\varphi_{\sigma\sigma'\sigma''}(\mathbf{p},\mathbf{p}',\mathbf{p}'')$ is simply the third variational derivative of the total energy E.

A direct calculation, using Eq. (2.2) leads to the following value of the function $\varphi_{\sigma\sigma'\sigma''}(\mathbf{p},\mathbf{p}',\mathbf{p}'')$ in a solution at rest when there is no external magnetic field:

$$\varphi^{(0)}_{\sigma\sigma'\sigma''}(\theta,\eta,\zeta)=-\frac{128\pi^2a^2\hbar^4}{p_0^2M}\left[\frac{Q_{\sigma\sigma'}}{\cos\eta+\cos\zeta-\cos\theta-1}+\frac{Q_{\sigma\sigma''}}{\cos\zeta+\cos\theta-\cos\eta-1}+\frac{Q_{\sigma'\sigma''}}{\cos\theta+\cos\eta-\cos\zeta-1}\right]. \qquad (2.6)$$

Here $\cos\eta=\cos\sphericalangle(\mathbf{p}\mathbf{p}'')$, $\cos\zeta=\cos\sphericalangle(\mathbf{p}'\mathbf{p}'')$. Substituting (2.6) and (1.5) into Eq. (2.5) and using Eq. (2.4) we get

$$h_{\sigma\sigma'}(\theta)=32\frac{a^2\hbar}{p_0}\frac{\delta m}{M}Q_{\sigma\sigma'}\frac{1}{\cos^2(\theta/2)}\left(1-\frac{\cos(\theta/2)}{2}\ln\frac{1+\cos(\theta/2)}{1-\cos(\theta/2)}\right),$$
$$g(\theta)=16\frac{a^2\hbar}{p_0}\beta\frac{1}{\sin(\theta/2)}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}, \qquad (2.7)$$

where $\delta m=M-m_3$.

When obtaining Eq. (2.7) we used Eq. (2.3) for the Fermi-liquid function $f^{(0)}_{\sigma\sigma'}(\theta)$ and retained only the main terms in the concentration. One verifies easily that the function $h_{\sigma\sigma'}(\theta)$ obtained in this way is automatically the same as the relation following from the Galilean relativity principle for a solution at rest,

$$\frac{\partial f^{(0)}}{\partial \mathbf{p}}+\frac{\partial f^{(0)}}{\partial \mathbf{p}'}=\sum_{\mathbf{p}''\sigma''}\varphi^{(0)}\frac{\partial n^{(0)}}{\partial \mathbf{p}''}.$$

For the sake of simplicity we dropped here the spin indices and arguments.

For antiparallel spins Eq. (2.7) has singularities at $\theta=\pi$ and $\theta=0$. When $a>0$ the sum of the main perturbation theory terms vanishes. If, however, $a<0$ the first singularity corresponds to the Cooper pole in the scattering amplitude of the excitations, while the singularity at $\theta=0$ is a consequence of the fact that the momentum of an excitation is not invariant under a Galileo transformation when there is superfluid motion present. When summing all terms of the expansion in powers of $\mathbf{v}_s$ this corresponds simply to a shift of the Cooper pole.

3. THERMODYNAMIC PROPERTIES

The free energy of the solution is, when we neglect terms which are quadratic in $\mathbf{v}_s$, $\mathbf{H}$, and T/T_0, given by Eq. (2.2). A direct calculation leads, as in Refs. 3, 5, and 6, to the following result:

$$F=F_4^{(0)}+N_3(-\Delta)+\frac{3}{10}\frac{p_0^2}{M}N_3\left[1+\frac{10}{9}\lambda+\frac{4}{21}\lambda^2(11-2\ln 2)-\frac{5}{7}\xi\left(\frac{p_0}{p_c}\right)^2\right] \quad (3.1)$$

Differentiating (3.1) we get at once the chemical potentials of ^{3}He and ^{4}He in the superfluid solution:

$$\mu_3=-\Delta+\frac{p_0^2}{2M}\left[1+\frac{4}{3}\lambda+\frac{4}{15}\lambda^2(11-2\ln 2)-\xi\left(\frac{p_0}{p_c}\right)^2\right], \quad (3.2)$$

$$\mu_4=\mu_4^{(0)}+N_3\frac{\partial(-\Delta)}{\partial N_4}+\frac{3}{10}\frac{\partial(\ln M)}{\partial N_4}p_0^2N_3, \quad (3.3)$$

where $\mu_4^{(0)}$ is the chemical potential of pure ^{4}He.

The change in the chemical potential of ^{3}He when the number of particles changes by δN_3 can be written in the following form[4]:

$$\delta\mu_3=\frac{p_0}{m^*}\delta p_0+\frac{\pi^2\hbar^3}{p_0 m^*}F_0\delta N_3. \quad (3.4)$$

Substituting (3.4) into Eq. (3.2) we find the total effective mass of an excitation:

$$\frac{m^*}{M}=1+\frac{8}{15}\lambda^2(7\ln 2-1)+2\xi\left(\frac{p_0}{p_c}\right)^2. \quad (3.5)$$

The osmotic pressure Π in a system with a "supergap" (membrane with fine pores through which only the superfluid component can pass) can be evaluated from the condition that the chemical potentials of the solvent must be equal on the two sides of the supergap

$$\mu_4(P,N_3)=\mu_4(P-\Pi,0), \quad (3.6)$$

where P is the pressure in the solution. Expanding both sides of Eq. (3.6) in powers of N_3 and Π and using the thermodynamic identity at $T=0$

$$dP=N_3d\mu_3+N_4d\mu_4, \quad (3.7)$$

we get the osmotic pressure in the form

$$\Pi=\int_0^{N_3}N_3\frac{\partial\mu_3}{\partial N_3}dN_3$$

or

$$\Pi=\frac{2}{5}\frac{p_0^2}{2M}N_3\left[1+\frac{5}{3}\lambda+\frac{8}{21}\lambda^2(11-2\ln 2)-\frac{10}{7}\xi\left(\frac{p_0}{p_c}\right)^2\right]. \quad (3.8)$$

We determine the excess enthalpy of the system W^E from the following equation (Ref. 10, p. 300):

$$W=-l_{30}N_3-l_{40}N_4+W^E. \quad (3.9)$$

Here $-l_{30}$ and $-l_{40}$ are the latent evaporation heats of pure ^{3}He and ^{4}He at $T=0$, per atom, and W is the enthalpy per unit volume of the solution. Up to terms of order $(T/T_0)^2$

$$W=\mu_3N_3+\mu_4N_4.$$

We can then write for the excess enthalpy

$$W^E=(\mu_3+l_{30})N_3+(\mu_4+l_{40})N_4. \quad (3.10)$$

Using (3.6) or (3.7) to evaluate μ_4+l_{40} we arrive at the final result

$$W^E=(\mu_3+l_{30})N_3-\Pi. \quad (3.11)$$

The magnetic susceptibility χ of the spin system, taking the Fermi-liquid exchange interaction into account, was first evaluated by Landau[4]:

$$\frac{\chi_{\text{id}}}{\chi}=\frac{1+Z_0/4}{1+F_1/3}, \quad (3.12)$$

where χ_{id} is the magnetic susceptibility of an ideal Fermi gas with energy spectrum (2.1). Substituting the Fermi-liquid function (2.3) into (3.12) leads to the following value of the susceptibility:

$$\chi_{\text{id}}/\chi=1-2\lambda-{}^{16}/_{15}\lambda^2(\ln 2+2). \quad (3.13)$$

We now evaluate the densities of the superfluid and normal components of the solution. When there are normal and superfluid motions present the total momentum of the liquid can be written in the form

$$\mathbf{P}=\rho^{(n)}\mathbf{v}_n+\rho^{(s)}\mathbf{v}_s, \quad (3.14)$$

where $\mathbf{v}_n$ is the velocity of the normal motion, $\rho^{(n)}$ and $\rho^{(s)}$ are, respectively, the densities of the normal and the superfluid components. As the velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ enter linearly into Eq. (3.14) it is sufficient to evaluate the flux $\mathbf{P}$ for the cases when only one of the two velocities is non-vanishing. When only the superfluid motion is present the total momentum $\rho^{(s)}\mathbf{v}_s$ is given by Eq. (1.1) in which we must substitute the change in the distribution function $\Delta n_\sigma^{(1)}(\mathbf{p})$ of (1.5). Using Eq. (3.5) to perform the calculation we get

$$\rho^{(s)}=m_4N_4-N_3\delta m-2\xi(p_0/p_c)^2MN_3. \quad (3.15)$$

In that case $\rho^{(n)}=m_4N_4+m_3N_3-\rho^{(s)}$ or

$$\rho^{(n)}=M[1+2\xi(p_0/p_c)^2]N_3, \quad (3.16)$$

which agrees with the results given in Ref. 3 when Eq.

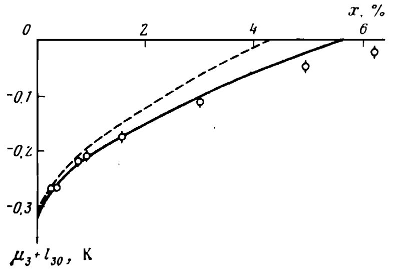

FIG. 1. Concentration dependence of the chemical potential of ^{3}He in a solution. The open circles here and in other figures are the data of Ref. 12.

(3.5) is taken into account.

We give in Figs. 1 to 5 a comparison of theoretical curves calculated using Eqs. (3.2) to (3.13) with experimental data[10–16] for the value $a=-1.51$ Å of the scattering length $(x=N_3/(N_3+N_4))$. The comparison enables us to conclude that the proposed method describes a real solution satisfactorily up to concentrations of the order of 3 to 4% (the dashed curve in Fig. 1 corresponds to an ideal gas). For higher concentrations there is qualitative agreement. As $a<0$, which corresponds to attraction between the impurity atoms, the calculations performed are suitable for the temperature range

$$T_c<T<T_0,\quad T_c\approx\frac{\gamma}{\pi}\left(\frac{2}{e}\right)^{7/3}T_0\exp\left\{\frac{\pi\hbar}{2p_0a}\right\};$$

T_c is the temperature at which the excitation spectrum undergoes a change-over which is connected with the formation of Cooper pairs and a phase transition of ^{3}He in the solution into a superfluid state.[17]

In conclusion we note that in the Boltzmann region $T>T_0$ the main concentration correction to the free energy, caused by the interaction between the impurity quasiparticles δF_{int} retains its earlier form (3.1). Indeed, δF_{int} can be written in the following form (Ref. 18, p. 261)

$$\delta F_{int}=-T\frac{8N_3^2}{g^2}\left(\frac{\pi\hbar^2}{MT}\right)^{3/2}Z_{int},\quad g=2s+1=2, \tag{3.17}$$

where Z_{int} is expressed in terms of the phase δ_l of the scattering amplitude through the formula

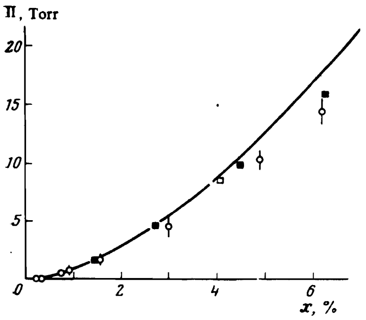

FIG. 2. Osmotic pressure in a degenerate solution. The open and filled squares are the data of Ref. 13.

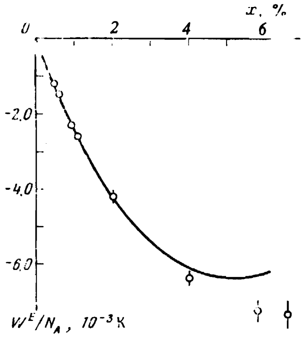

FIG. 3. Excess enthalpy of a degenerate solution.

$$Z_{int}=\frac{g}{\pi}\sum_l(2l+1)\int_0^\infty\frac{d\delta_l}{dp}\exp\left(-\frac{p^2}{MT}\right)dp. \tag{3.18}$$

For s-wave scattering of slow particles all $\delta_l=0$, apart from $\delta_0=-pa/\hbar$. Substituting δ_0 into (3.18) and bearing in mind that only particles with antiparallel spins take part in the scattering process we find

$$\delta F_{int}=\pi ah^2N_3^2/M, \tag{3.19}$$

which agrees with the corresponding term in (3.1).

4. HYDRODYNAMIC TYPES OF SOUND

The propagation of low-frequency sound waves in a solution is described by a set of hydrodynamical equations (Ref. 3, p. 214) which in the linear approximation has the form

$$m_4\frac{\partial N_4}{\partial t}+m_3\frac{\partial N_3}{\partial t}+\rho^{(n)}\operatorname{div}\mathbf{v}_n+\rho^{(s)}\operatorname{div}\mathbf{v}_s=0, \tag{4.1}$$

$$\frac{\partial N_3}{\partial t}+N_3\operatorname{div}\mathbf{v}_n=0, \tag{4.2}$$

$$\frac{\partial S}{\partial t}+S\operatorname{div}\mathbf{v}_n=0, \tag{4.3}$$

$$\rho^{(n)}\frac{\partial\mathbf{v}_n}{\partial t}+\rho^{(s)}\frac{\partial\mathbf{v}_s}{\partial t}+\nabla P=0, \tag{4.4}$$

$$\frac{\partial\mathbf{v}_s}{\partial t}+\frac{1}{m_4}\nabla\mu_4=0. \tag{4.5}$$

Here

$$S=\frac{p_0M}{3\hbar^3}\left[1+\frac{8}{15}\lambda^2(7\ln 2-1)+2\xi\left(\frac{p_0}{p_c}\right)^2\right]T$$

is the entropy per unit volume of the solution. Eliminat-

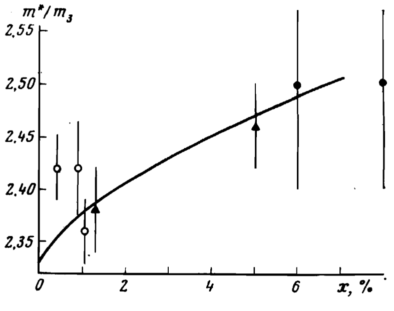

FIG. 4. Total effective mass of impurity quasi-particles. Here and in other figures the filled triangles are data of Ref. 14 and the filled circles data of Ref. 11.

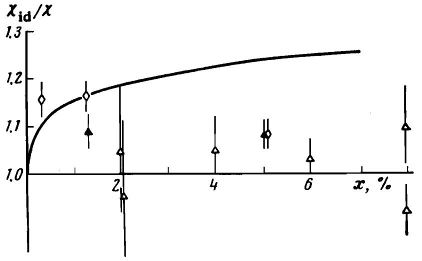

FIG. 5. Magnetic susceptibility of a solution; open triangles are data of Ref. 15, and the open diamonds the results of relative measurements of the magnetic susceptibility[16] normalized to the value for $x = 1.3\%$, evaluated using Eq. (3.13).

ing $\mathbf{v}_s$ and $\mathbf{v}_n$ from the hydrodynamical equations we get the set of equations

$$\begin{gathered}\dot S - S\dot N_3/N_3 = 0,\\ m_4\dot N_4 + m_3\dot N_3 - \Delta P = 0,\\ m_4\dot N_4 + \left(m_3 - \frac{\rho^{(n)}}{N_3}\right)\dot N_3 - \frac{\rho^{(s)}}{m_4}\Delta\mu_4 = 0.\end{gathered} \qquad (4.6)$$

We choose as independent variables N_3, N_4, and T. We consider a weakly excited state of the solution. We shall assume that small perturbations for all equilibrium quantities vary in the field of the sound wave as $e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}$. Neglecting terms of order T/T_0 by using the thermodynamic identity (3.7) we get:

$$\begin{gathered}(m_4u^2 - A_4)\,\delta N_4 + (m_3u^2 - A_3)\,\delta N_3 = 0,\\ \left(m_4u^2 - \frac{\rho^{(s)}}{m_4}\frac{\partial\mu_4}{\partial N_4}\right)\delta N_4 + \left[\left(m_3 - \frac{\rho^{(n)}}{N_3}\right)u^2 - \frac{\rho^{(s)}}{m_4}\frac{\partial\mu_4}{\partial N_3}\right]\delta N_3 = 0,\end{gathered} \qquad (4.7)$$

where $A_i = N_3\partial\mu_3/\partial N_i + N_4\partial\mu_4/\partial N_i$, $i = 3, 4$; $u = \omega/k$ is the propagation speed of the acoustic wave. From the compatibility condition of the set (4.7) we find the required dispersion equation

$$m_4\frac{\rho^{(n)}}{N_3}u^4 - u^2\left\{\rho^{(s)}\left(\frac{m_3}{m_4}\frac{\partial\mu_4}{\partial N_4} - \frac{\partial\mu_4}{\partial N_3}\right) + m_4A_3 - \left(m_3 - \frac{\rho^{(n)}}{N_3}\right)A_4\right\} + \left\{A_3\frac{\partial\mu_4}{\partial N_4} - A_4\frac{\partial\mu_4}{\partial N_3}\right\}\frac{\rho^{(s)}}{m_4} = 0. \qquad (4.8)$$

Equation (4.8) determines the velocities of first and second sound in a degenerate solution. Substituting Eqs. (3.2), (3.3), (3.15), (3.16) into the dispersion relation and solving it we get finally (compare Ref. 3, p. 309, and Ref. 19)

$$u_1^2 = s_0^2 + s_0^2\left[\frac{m_4}{M}\left(\alpha + \frac{\delta m}{m_4}\right)^2 + \gamma - \frac{\delta m}{m_4}\right]c + \frac{p_0^2}{M^2}\left\{\frac{1}{3}\frac{m_4}{M}\left(\alpha + \frac{\delta m}{m_4}\right)^2 - \varkappa\left(\alpha + \frac{\delta m}{m_4}\right) - 2\xi\frac{M}{m_4}\left(\alpha - \frac{m_3}{m_4}\right)^2 + \frac{3}{10}\eta\frac{M}{m_4}\right\}c, \qquad (4.9)$$

$$u_2^2 = \frac{p_0^2}{3M^2}\left[1 + 2\lambda + \frac{8}{15}\lambda^2(11 - 2\ln 2) - 4\xi\left(\frac{p_0}{p_c}\right)^2\right] - \frac{m_4}{M}\alpha^2 s_0^2 c, \qquad (4.10)$$

where

$$\alpha = \frac{N_4}{m_4 s_0^2}\frac{\partial(-\Delta)}{\partial N_4} \approx 1.28\ [^{20}],\quad \gamma = \frac{N_4^2}{m_4 s_0^2}\frac{\partial^2(-\Delta)}{\partial N_4^2},$$

$$\varkappa = \frac{\partial\ln M}{\partial\ln N_4} \approx 1.25\ [^{20}],\quad \eta = MN_4^2\frac{\partial^2\ln M}{\partial N_4^2},\quad c = \frac{N_3}{N_4}.$$

The last term in Eq. (4.9) describes the non-linear concentration dependence of the first sound speed in a degenerate solution, observed by Abraham *et al.*[21] In Ref. 21 it was assumed that the observed non-linearity was connected with a concentration dependence of the effective mass of the impurity excitations m^* which is caused by the Fermi-liquid interaction. However, Eq. (4.9) shows that the parameters characterizing the interaction between impurity quasi-particles do not enter at all in the expression for the first-sound speed, which is therefore the same as the first-sound speed in a solution of an ideal Fermi-gas with the spectrum (2.1) in superfluid ^{4}He.

Figure 6 illustrates the agreement between Eq. (4.9) and the experimental data,[21] and the remaining undetermined parameters γ and η turn out to equal $\gamma = -2.26$; $\eta = -0.89$. (In the case when the quantity $\xi = 0$ [9] the parameter η has the value $\eta = -1.15$.)

We emphasize that the first-sound speed was evaluated up to terms of order $(p_0/p_c)^6$, whereas the second-sound speed was evaluated only up to terms of order $(p_0/p_c)^5$. This is explained by the fact that retardation effects, which we neglected, gave a contribution of order $(p_0/p_c)^5$, to the second-sound speed, while the corresponding terms do not occur at all in the expression for the first-sound speed.

By analogy with pure He II, the so-called fourth sound[22] can also propagate in a degenerate solution. This occurs in the case when the solution fills capillaries that are so narrow that their diameter turns out to be less than the penetration depth of a viscous wave or the mean free path of the excitations. Oscillations will then propagate only in the superfluid component of the fluid, while its normal part turns out to be stationary. The fourth sound speed is determined from the set of Eqs. (4.1) to (4.3), (4.5) in which we must put $\mathbf{v}_n = 0$. Neglecting again all terms of order T/T_0 we find immediately

$$m_4\dot N_4 + \rho^{(s)}\,\mathrm{div}\,\mathbf{v}_s = 0,\quad \dot{\mathbf{v}}_s + \nabla\mu_4/m_4 = 0. \qquad (4.11)$$

Eliminating $\mathbf{v}_s$ and substituting μ_4 from (3.3) we find the fourth sound speed in a degenerate solution:

$$u_4^2 = \frac{\rho^{(s)}}{m_4^2}\frac{\partial\mu_4}{\partial N_4} \qquad (4.12)$$

or finally

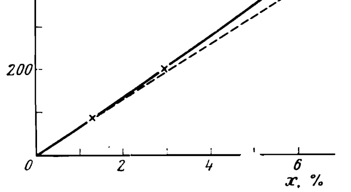

FIG. 6. Deviation of the concentration dependence of the first-sound speed in a solution from the linear dependence (indicated by a dashed line); the crosses are experimental data of Ref. 21.

$$u_4^2=s_0^2+s_0^2\left(\gamma-\frac{\delta m}{m_4}\right)c+\left(\frac{p_0}{M}\right)^2\frac{M}{m_4}\left[\frac{3}{10}\eta-2\xi\left(\frac{M}{m_4}\right)^2\right]c. \tag{4.13}$$

5. HIGH-FREQUENCY OSCILLATIONS

The complete set of equations which describes the propagation of high-frequency oscillations in a solution consists of the linearized equations of continuity, the equation for the superfluid motion, and the collisionless kinetic equation:

$$\frac{\partial}{\partial t}\Big[m_4N_4+\sum_{\mathbf{p}\sigma}m_3n_\sigma(\mathbf{p})\Big]+\operatorname{div}\Big[m_4N_4\mathbf{v}_s+\sum_{\mathbf{p}\sigma}\mathbf{p}n_\sigma(\mathbf{p})\Big]=0, \tag{5.1}$$

$$\frac{\partial\mathbf{v}_s}{\partial t}+\frac{1}{m_4}\nabla\frac{\partial}{\partial N_4}\Big[E_0+\sum_{\mathbf{p}\sigma}\varepsilon_\sigma(\mathbf{p})\,\delta n_\sigma(\mathbf{p})\Big]=0, \tag{5.2}$$

$$\frac{\partial n_\sigma(\mathbf{p})}{\partial t}+\nabla n_\sigma(\mathbf{p})\frac{\partial\varepsilon_\sigma(\mathbf{p})}{\partial\mathbf{p}}-\frac{\partial n_\sigma(\mathbf{p})}{\partial\mathbf{p}}\nabla\varepsilon_\sigma(\mathbf{p})=0. \tag{5.3}$$

Here E_0 is the total equilibrium energy of the system.

We consider oscillations of the solution at $T=0$. We write the perturbed distribution function in the following form [see (1.5)]

$$n_\sigma(\mathbf{p})=n^{(0)}+\frac{\partial n^{(0)}}{\partial\varepsilon}\left[\left(1+\frac{F_1}{3}\right)^{-1}-\frac{m_3}{m^*}\right]\mathbf{p}\mathbf{v}_s+\frac{\partial n^{(0)}}{\partial\varepsilon}[\nu(\mathbf{p})+\psi(\mathbf{p})\boldsymbol{\sigma}], \tag{5.4}$$

where $n^{(0)}$ is the "globally" equilibrium distribution function when there is no superfluid motion (we consider here the case without an external magnetic field) and the small deviations $\nu(\mathbf{p})$, $\psi(\mathbf{p})$, δN_4, and $\mathbf{v}_s$ are periodic functions of the coordinates and the time and proportional to $e^{i\omega t-\mathbf{k}\cdot\mathbf{r}}$. One verifies easily that under the substitution

$$-\frac{p_0m^*}{\pi^2\hbar^3}\nu_0=\delta N_3^{(1)},\quad -\frac{p_0m^*}{\pi^2\hbar^3}p_0\frac{\nu_1}{3}=\rho^{(n)}v_n,$$

where ν_k are the harmonics in the expansion of the function $\nu(\theta)$ in Legendre polynomials, Eqs. (5.1) and (5.2) change into the corresponding equations of the two-velocity hydrodynamics (4.1) and (4.5) while we get by standard methods Eqs. (4.2) and (4.4) from the kinetic Eq. (5.3).

We solve Eqs. (5.1) and (5.2) for $\mathbf{v}_s$ and δN_4 and substitute the expressions obtained into the kinetic Eq. (5.3) with the distribution function (5.4) and the Hamiltonian (1.4) with $\mathbf{H}=0$. After straightforward calculations we get

$$\begin{aligned}(\omega-\mathbf{k}\mathbf{v}_0)[\nu(\theta,\varphi)+\psi(\theta,\varphi)\boldsymbol{\sigma}]-\mathbf{k}\mathbf{v}_0\langle[F(\theta,\theta')+Q(u)+R(u)\cos\theta']\\ \times\nu(\theta',\varphi')-\mathbf{k}\mathbf{v}_0\langle{}^1\!/_4Z(\theta,\theta')\psi(\theta,\varphi)\rangle\boldsymbol{\sigma}=0.\end{aligned} \tag{5.5}$$

In Equation (5.5) $\mathbf{v}_0=\mathbf{p}_0/m^*$; $\nu(\theta)$ and $\psi(\theta,\varphi)$ are the values of the functions $\nu(\mathbf{p})$ and $\psi(\mathbf{p})$ on the Fermi surface, $\langle\ldots\rangle$ indicates averaging over solid angles, $u=\omega/k$, and

$$\begin{aligned}Q(u)=\frac{3}{p_0^2}\frac{m^*}{m_4}\frac{1}{u^2-u_4^2}\Big[\frac{\rho^{(s)}}{m_4}N_3\left(\frac{\partial\mu_3}{\partial N_4}\right)^2-u^2(\rho^{(n)}-m_3N_3)\frac{m_3}{m_4}\frac{\partial\mu_4}{\partial N_4}\\ -u^2(2m_3N_3-\rho^{(n)})\frac{\partial\mu_3}{\partial N_4}\Big],\end{aligned} \tag{5.6}$$

$$R(u)=\frac{3}{p_0}\frac{m^*}{m_4}\frac{u}{u^2-u_4^2}\Big[(\rho^{(n)}-m_3N_3)\frac{1}{m_4}\frac{\partial\mu_4}{\partial N_4}+N_3\frac{\partial\mu_3}{\partial N_4}\Big]. \tag{5.7}$$

We choose the direction of $\mathbf{k}$ as the polar axis. Multiplying both sides of Eq. (5.5) by $\boldsymbol{\sigma}$ and taking Tr_σ we get as the result

$$(u/v_0-\cos\theta)\psi(\theta,\varphi)-\cos\theta\langle{}^1\!/_4Z(\theta,\theta')\psi(\theta',\varphi')\rangle=0. \tag{5.8}$$

Landau[4] was the first to obtain the integral Eq. (5.8) which determines the propagation speed of spin waves. Following Landau, but taking into account that $\frac{1}{4}Z(\theta,\theta')\ll 1$ we find the speed of a symmetric ($m=0$) spin wave[7] (cf. Ref. 23, p. 231)

$$u_s/v_0=1+\exp\{1/\lambda-2\}. \tag{5.9}$$

An analysis of Eq. (5.8) shows that due to the smallness of the function $\frac{1}{4}Z(\theta,\theta')$ the propagation of undamped asymmetric spin modes ($m\neq0$) turns out to be impossible.

We now average Eq. (5.5) over the spins. We find

$$(u/v_0-\cos\theta)\nu(\theta,\varphi)-\cos\theta\langle[F(\theta,\theta')+Q(u)+R(u)\cos\theta']\nu(\theta',\varphi')\rangle=0. \tag{5.10}$$

Equation (5.10) describes zero-sound and high-frequency sound (see below) oscillations in the solution. In a closer consideration, similar to the one in Ref. 3 (p. 255) for a pure Fermi liquid it turns out, however, that due to the smallness of $F(\theta,\theta')$ Eq. (5.10) does not have real zero-sound roots $u\gtrsim v_0$. There is then one root $u\sim s_0$ which corresponds to the first-sound velocity close to the absolute zero when the relaxation time of the impurity excitations increases fast and may exceed the period of the sound oscillations.[24]

For a calculation of the speed of high-frequency first sound we note that the function $F(\theta,\theta')$ in Eq. (5.10) has a much smaller absolute magnitude than the functions $Q(u)$ and $R(u)$ and that it therefore can be neglected (this means, however, not the neglect of terms $\propto\lambda^2$ in the expressions for $Q(u)$ and $R(u)$). Using this fact the required dispersion equation takes the following form:

$$Q(u)+\frac{u}{v_0}R(u)=1/w(u/v_0), \tag{5.11}$$

where

$$w(x)=\frac{x}{2}\ln\frac{x+1}{x-1}-1.$$

When $u/v\gg1$ we have $w(u/v_0)\approx v_0^2/3u^2$. Then substituting Eqs. (5.6) and (5.7) into Eq. (5.11) we have

$$\begin{aligned}u^4-u^2\Big[u_4^2+\frac{1}{m_4^2}\frac{\partial\mu_4}{\partial N_4}(\rho^{(n)}-m_3N_3)\left(1-\frac{m_3}{m^*}\right)\\ +\frac{1}{m_4}\frac{\partial\mu_3}{\partial N_4}N_3\left(1-2\frac{m_3}{m^*}+\frac{\rho^{(n)}}{m^*N_3}\right)\Big]-\frac{1}{m_4^2}\rho^{(s)}\left(\frac{\partial\mu_3}{\partial N_4}\right)^2\frac{N_3}{m^*}=0.\end{aligned} \tag{5.12}$$

The solution $u_0\sim s_0$ of the dispersion Eq. (5.12) is the velocity of the high-frequency first sound in the solution:

$$\begin{aligned}u_0^2=u_4^2-\frac{p_0^2}{3M^2}\frac{m_4}{M}\left(\alpha+\frac{\delta m}{m_4}\right)\\ \times\left[\alpha+\frac{\delta m}{m_4}+\frac{8}{5}\left(s_0\frac{Ma}{\pi\hbar}\right)^2(7\ln2-1)\left(\alpha-\frac{m_3}{m_4}\right)\right]c.\end{aligned} \tag{5.13}$$

The first-sound speed of a given frequency thus turns out close to the absolute zero to be less than at higher temperatures (when we have the low-frequency hydrodynamic limit) by an amount of the order of $(p_0/p_c)^5$ given by Eq. (5.13). In other words, the temperature dependence of the first-sound speed must have a maximum.

The absorption coefficient of a high-frequency wave equals $1/\omega\tau$ (the imaginary part of the kinetic equation then identically vanishes). Here τ is the relaxation time which is, in general, different for spin and sound wave propagation processes. However, as to order of magnitude

$$\tau\approx\frac{1}{4\pi a^2N_3v_0}\left(\frac{T_0}{T}\right)^2.$$

The condition that the damping is small can then be expressed in the following form:

$$T\ll\frac{\hbar}{|a|}\left(\frac{\hbar\omega}{M}\right)^{1/2}.$$

Or, substituting for τ the experimental values, we get $T\ll 4.5\times10^{-6}\omega^{1/2}$. From the conditions $T>T_c$ and $\hbar\omega\ll T_0$ we find, on the other hand, a region of frequencies in which it is possible to observe high-frequency spin and sound waves:

$$x^{2/3}\gg\frac{\omega}{\omega_0}>(aN^{1/3})^2x^{4/3}\exp\left\{\left(\frac{\pi}{3Nx}\right)^{1/3}\frac{1}{a}\right\}.$$

Here $\omega_0=\hbar M^{-1}N^{2/3}$, $N=N_3+N_4$. Substitution of numerical values gives

$$x^{2/3}\gg1.43\cdot10^{-11}\omega>x^{4/3}\exp\{-2.36/x^{1/3}\}.$$

6. KINETIC COEFFICIENTS

The kinetic effects in a degenerate ^{3}He-He II solution are determined by fermion quasi-particle pair scattering processes corresponding to the transition

$$(P_1)+(P_2)\rightarrow(P_1')+(P_2'). \qquad (6.1)$$

We denote the 4-momentum by P_i, i.e., $P_i=(\varepsilon_i,\mathbf{p}_i)$. Brooker and Sykes[25] obtained exact expressions for the thermal conductivity, K, the viscosity, η, and the spin diffusion, D, coefficients by solving the kinetic equation; they have the following form:

$$KT=\frac{8}{3}\frac{\pi^2\hbar^3p_0^3}{m^{*4}}\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)\right\rangle^{-1}H(\lambda_K),$$
$$\eta T^2=\frac{64}{45}\frac{\hbar^3p_0^5}{m^{*4}}\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)^2\sin^2\varphi\right\rangle^{-1}C(\lambda_\eta), \qquad (6.2)$$
$$DT^2=\frac{32}{3}\frac{\pi^2\hbar^6p_0^2}{m^{*5}}\left(1+\frac{Z_0}{4}\right)\left\langle\frac{2W_D(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\varphi)(1-\cos\theta)\right\rangle^{-1}C(\lambda_D),$$

where θ is the angle between the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, φ the angle between the $(\mathbf{p}_1,\mathbf{p}_2)$ and $(\mathbf{p}_1',\mathbf{p}_2')$ planes, while the coefficients $H(\lambda_K)$, $C(\lambda_\eta)$, and $C(\lambda_D)$ are given by the following relations:

$$\lambda_K\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}\right\rangle=\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}(1+2\cos\theta)\right\rangle,$$
$$\lambda_\eta\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}\right\rangle=\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}\left[1-\frac{3}{4}(1-\cos\theta)^2\sin^2\varphi\right]\right\rangle,$$
$$(\lambda_D-1)\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}\right\rangle=-\left\langle\frac{W_D(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)(1-\cos\varphi)\right\rangle, \qquad (6.3)$$

$$H(\lambda_K)=\frac{3-\lambda_K}{4}\sum_{n=0}^{\infty}\frac{4n+5}{(n+1)(2n+3)[(n+1)(2n+3)-\lambda_K]},$$
$$C(\lambda)=\frac{1-\lambda}{4}\sum_{n=0}^{\infty}\frac{4n+3}{(n+1)(2n+1)[(n+1)(2n+1)-\lambda]}.$$

The functions $W(\theta,\varphi)$ and $W_D(\theta,\varphi)$ in Eqs. (6.2) and (6.3) are phenomenological probabilities for quasi-particle pair collisions near the Fermi surface for the appropriate processes. We emphasize that because the particles are indistinguishable the angle φ varies between the limits of 0 and π.

One can easily obtain the collision integral in the quasiclassical approximation by using the Liouville equation for the single-particle density matrix including exchange effects (see Ref. 23, p. 220):

$$I(n_1)=-\int\frac{d\mathbf{p}_2}{(2\pi\hbar)^3}\frac{d\mathbf{p}_1'}{(2\pi\hbar)^3}d\mathbf{p}_2'\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\cdot$$
$$\times\frac{2\pi}{\hbar}\left[\frac{1}{4}|\Gamma_{\uparrow\downarrow}(\theta,\varphi)|^2+\frac{3}{4}|\Gamma_{\uparrow\uparrow}(\theta,\varphi)|^2\right][n_1'n_2'(1-n_1)(1-n_2)$$
$$-n_1n_2(1-n_1')(1-n_2')]. \qquad (6.4)$$

Here $\Gamma_{\uparrow\downarrow}(\theta,\varphi)$ and $\Gamma_{\uparrow\uparrow}(\theta,\varphi)$ are the scattering amplitudes for quasi-particles in the singlet and triplet states, respectively.

Comparing Eq. (6.4) with the collision integral used in Ref. 25 we find that

$$W(\theta,\varphi)=\frac{\pi}{8\hbar}(|\Gamma_{\uparrow\downarrow}(\theta,\varphi)|^2+3|\Gamma_{\uparrow\uparrow}(\theta,\varphi)|^2). \qquad (6.5)$$

On the other hand, the probability $W_D(\theta,\varphi)$ is determined by those terms in (6.4) which correspond to the scattering of quasi-particles with opposite spins. Finally we get

$$2W_D(\theta,\varphi)=\frac{\pi}{16\hbar}|\Gamma_{\uparrow\downarrow}(\theta,\varphi)|^2. \qquad (6.6)$$

Galitskiĭ[26] obtained the equation which enables us to determine the vertex part $\Gamma(P_1',P_2';P_1,P_2)$ in terms of the scattering amplitude of two particles in vacuum $f(\mathbf{p}_1',\mathbf{p}_2';\mathbf{p}_1,\mathbf{p}_2)$ in the case of a non-ideal low-density Fermi gas:

$$\Gamma(\mathbf{p}',\mathbf{p},G)=-\frac{2\pi\hbar^2}{M}f(\mathbf{p}',\mathbf{p})+\frac{16\pi^2\hbar^4}{M^2}\int\frac{d\mathbf{k}}{(2\pi\hbar)^3}f(\mathbf{p}',\mathbf{k})f^*(\mathbf{p},\mathbf{k})$$
$$\times\left[\frac{N(\mathbf{k})}{g_0-g^2/4M-k^2/M+i\delta N(\mathbf{k})}+\frac{1}{k^2/M-p^2/M-i\delta}\right]. \qquad (6.7)$$

We have changed in Eq. (6.7) to relative momenta P and P' and the center of mass momentum G:

$$P'={}^1/_2(P_1'-P_2'),\quad P={}^1/_2(P_1-P_2),\quad G=P_1+P_2=P_1'+P_2'=(g_0,\mathbf{g}). \qquad (6.8)$$

The factor $N(\mathbf{k})$ takes into account prohibitions following from the Pauli principle:

$$N(\mathbf{k})=1-n({}^1/_2\mathbf{g}+\mathbf{k})-n({}^1/_2\mathbf{g}-\mathbf{k}). \qquad (6.9)$$

In the case of s-wave scattering of slow particles

$$f(\mathbf{p}',\mathbf{p})=-a(1-ip'a/\hbar) \qquad (6.10)$$

and for collisions leaving the quasi-particles on the Fermi surface we find up to and including terms of or-

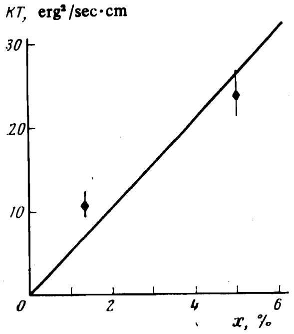

FIG. 7. Concentration dependence of the thermal conductivity coefficient; filled diamonds are experimental data of Ref. 27.

der a^2 from (6.7) that the vertex part $\Gamma(\mathbf{p}', \mathbf{p}, G)$ is independent of the angle φ and thereby becomes the same as the forward scattering amplitude $\Gamma^{(0)}(\theta)$ (the situation is analogous to the scattering of slow particles in vacuum). The real part of the forward scattering amplitude can for singlet and triplet states, respectively, be expressed as follows (Ref. 3, p. 286):

$$\Gamma_{\uparrow\downarrow}(\theta)=4\lambda\left[2+\lambda\left(\frac{1-5v^2}{v}\operatorname{arth}v+7\right)\right]\frac{\pi^2\hbar^3}{p_0M}, \tag{6.11}$$

$$\Gamma^{(0)}_{\uparrow\uparrow}(\theta)=4\lambda^2\left(\frac{1-v^2}{v}\operatorname{arth}v-1\right)\frac{\pi^2\hbar^3}{p_0M}. \tag{6.12}$$

Here $v=\sin(\frac{1}{2}\theta)$. It follows from Eqs. (6.11), (6.12) that up to terms corresponding to p-wave scattering of quasi-particles only those terms from $|\Gamma^{(0)}_{\uparrow\downarrow}(\theta)|^2$ occur in the transition probabilities $W(\theta, \varphi)$ and $W_D(\theta, \varphi)$ which are porportional to a^2 and a^3. Finally we have

$$\begin{gathered} W(\theta,\varphi)=(2\pi\hbar)^3\left(\frac{a}{M}\right)^2\left[1+\lambda\left(7+\frac{1-5v^2}{v}\operatorname{arth}v\right)\right], \\ 2W_D(\theta,\varphi)={}^1/_2W(\theta,\varphi). \end{gathered} \tag{6.13}$$

Substitution of Eqs. (6.13) into (6.2) leads after straightforward, but tedious calculations to the expressions

$$\begin{aligned} \lambda_K&={}^1/_3+4.76\lambda, & H(\lambda_K)&=0.52(1-0.26\lambda), \\ \lambda_\eta&={}^1/_5+2.23\lambda, & C(\lambda_\eta)&=0.81(1-0.18\lambda), \\ \lambda_D&={}^1/_3+1.19\lambda, & C(\lambda_D)&=0.80(1-0.10\lambda). \end{aligned} \tag{6.14}$$

The final results for the kinetic coefficients then look as follows:

$$\begin{gathered} KT=\frac{1}{8\pi}\frac{p_0^3}{(Ma)^2}0.52(1-0.34\lambda), \quad \eta T^2=\frac{1}{12\pi^3}\frac{p_0^5}{(Ma)^2}0.81(1+0.74\lambda), \\ DT^2=\frac{1}{\pi}\left(\frac{\hbar}{M}\right)^3\left(\frac{p_0}{a}\right)^2 0.80(1-2.18\lambda). \end{gathered} \tag{6.15}$$

The characteristic relaxation time in a degenerate Fermi system

$$\tau=\frac{8\pi^2\hbar^6}{m^{*}T^2}\left\langle\frac{W(\theta,\varphi)}{\cos(\theta/2)}\right\rangle^{-1}$$

thus has the following concentration dependence:

$$\tau=\frac{1}{2\pi}\frac{\hbar}{M}\left(\frac{\hbar}{a}\right)^2\frac{1}{T^2}(1-1.86\lambda). \tag{6.16}$$

A comparison of the calculated kinetic coefficients with experimental data[14,27] is made in Figs. 7 and 8. They reveal a satisfactory agreement between the theoretical and experimental results.

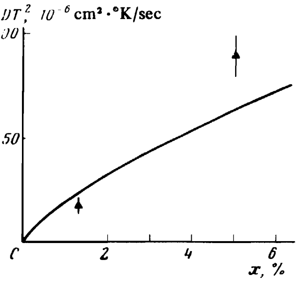

FIG. 8. Concentration dependence of the spin diffusion coefficient.

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[1] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. **156**, 207 (1967).
[2] C. Ebner and D. O. Edwards, Phys. Repts. **2**, No. 2 (1971).
[3] I. M. Khalatnikov, Teoriya sverkhtekuchesti (Theory of Superfluidity) Nauka, 1971.
[4] L. D. Landau, Zh. Eksp. Teor. Fiz. **30**, 1058 (1956); **32**, 59 (1957); **35**, 97 (1958) [Sov. Phys. JETP **3**, 920 (1957); **5**, 101 (1957); **8**, 70 (1958)].
[5] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).
[6] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957).
[7] E. P. Bashkin, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 3 (1977) [JETP Lett. **25**, 1 (1977)].
[8] N. R. Brubaker, D. O. Edwards, R. E. Sarwinski, P. Seligmann, and R. A. Sherlock, Phys. Rev. Lett. **25**, 715 (1970).
[9] V. I. Sobolev, Thesis, Physico-Tech. Inst. Low Temp., Khar'kov, 1975.
[10] B. N. Esel'son, V. N. Grigor'ev, V. G. Ivantsov, E. L. Rudavskii, D. G. Sanikidze, and I. A. Serbin, Rastvory kvantovykh zhidkosteĭ He^3–He^4 (Solutions of the quantum liquids ^{3}He and ^{4}He), Nauka, 1973.
[11] D. O. Edwards, D. F. Brewer, P. Seligmann, M. Skertic, and M. Yaqub, Phys. Rev. Lett. **15**, 773 (1965).
[12] P. Seligmann, D. O. Edwards, R. E. Sarwinski, and J. T. Tough, Phys. Rev. **181**, 415 (1969).
[13] J. Landau, J. T. Tough, N. R. Brubaker, and D. O. Edwards, Phys. Rev. Lett. **23**, 283 (1969); Phys. Rev. **A2**, 2472 (1970).
[14] A. C. Anderson, D. O. Edwards, W. R. Roach, R. E. Sarwinski, and J. C. Wheatley, Phys. Rev. Lett. **17**, 367 (1966).
[15] D. L. Husa, D. O. Edwards, and J. R. Gaines, Proc. LT-10, Moscow, **1**, 345 (1966); Phys. Lett. **21**, 28 (1966).
[16] A. I. Ahonen, M. A. Paalanen, R. C. Richardson, and Y. Takano, J. Low Temp. Phys. **25**, 733 (1976).
[17] L. P. Gor'kov and T. K. Melik-Barkhudarov, Zh. Eksp. Teor. Fiz. **40**, 1452 (1961) [Sov. Phys. JETP **13**, 1018 (1961)].
[18] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Vol. 1, Nauka, 1976 [English translation published by Pergamon Press, Oxford].
[19] G. Baym, Phys. Rev. Lett. **18**, 71 (1967).
[20] D. O. Edwards, E. M. Ifft, and R. E. Sarwinski, Phys. Rev. **177**, 380 (1969).
[21] B. M. Abraham, Y. Eckstein, and J. B. Ketterson, Phys. Rev. Lett. **21**, 422 (1968).
[22] D. G. Sanikidze and A. N. Shaanova, Pis'ma Zh. Eksp. Teor.

Fiz. 10, 482 (1969) [JETP Lett. 10, 310 (1969)].
[23] V. P. Silin, Vvedenie v kineticheskuyu teoriyu gazov (Introduction to the kinetic theory of gases) Nauka, 1971.
[24] I. M. Khalatnikov, Pis'ma Zh. Eksp. Teor. Fiz. 5, 288 (1967) [JETP Lett. 5, 235 (1967)].
[25] G. A. Brooker and J. Sykes, Phys. Rev. Lett. 21, 279(1968).
[26] V. M. Galitskiĭ, Zh. Eksp. Teor. Fiz. 34, 151 (1968) [Sov. Phys. JETP 7, 104 (1958).
[27] W. R. Able, R. T. Johnson, J. C. Wheatley, and W. Zimmermann, Phys. Rev. Lett. 18, 737 (1967).

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Evaporation of shock-compressed lead in release waves

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We determined experimentally the rates of expansion of lead samples in air following compression by strong shock waves of amplitude 0.4–3.4 Mbar. The strong increase of the expansion rate at pressures exceeding 1.32 Mbar attest to evaporate of the lead in the release wave. The results are used to determine the shock-compression entropy and to refine the equation of state of the liquid phase of lead at high pressures and temperatures.

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1. INTRODUCTION

The solution of many problems of applied physics calls for the study of high-temperature evaporation of metals under the action of pulses, and an important question is that of the thermodynamic equilibrium of the evaporation process.[1] Near the liquid–vapor equilibrium line and in the transcritical region, a rigorous theoretical calculation of the characteristics of metals is impossible because of the presence of a strong interparticle interaction of complex structure, while static experiments are limited by the simultaneous action of the high temperatures and pressures that are typical of metals. It is therefore of great importance to investigate the properties of metals near the evaporation line by dynamic methods.

Dynamic investigation methods with the aid of strong shock waves make it possible to obtain, in a wide range of parameters, the caloric characteristics of metals—the pressure P, the specific volume V, and the internal energy E.[2,3] This information was obtained for many metals in experiments on shock compression of solid and porous samples[3–6] and by recording the release isentropes.[7–9] By using these data we can, without resorting to models, construct a phenomenological equation of state in the form of the function $E(P, V)$. In this case the equation of state, however, does not contain such important material characteristics as the temperature and the entropy. Zel'dovich[10] was the first to call attention to the possibility of determining, in view of the adiabaticity of the release process, the entropy of shock compression from the final parameters of the expanding substance. If a metal expands in the state of an ideal gas or plasma, its entropy can be calculated from the pressure and density (or temperature) measured in the course of the dilatation.

The program proposed in[10] was not realized because of the great difficulty of obtaining, in shock compression, the high-energy states needed to reach the ideality region. Another difficulty lies in the exact registration of high expansion velocities on the order of dozens of km/sec. In the case of lower-intensity shock waves, the entropy of copper was determined[11,12] from the residual temperature of the solid and liquid metal. In[12], the entropy of shock compression of sodium, strontium, barium, and uranium was determined from spectroscopic measurements of the fraction of the evaporated metal after expansion in vacuum, and the object of the investigation was the states within the two-phase region.

We have developed a new variant of obtaining entropy information, based on an investigation of the evaporation process when metal samples, previously compressed by shock waves of various intensities, expand in air. The release pressure at which the evaporation started, yielded the point of intersection of one of the isentropes with the phase boundary, whose parameters, including the entropy, were assumed known. The start of the evaporation upon release can be observed if the relaxation time of the metastable states that are produced below the saturation curve is much shorter than the characteristic time of the evaporation process. Otherwise the metal will expand along the adiabats of the heated liquid or supercooled vapor. It is possible, however (see[14]) that owing to the presence of a large number of charged particles in the metal on the saturation line, the relaxation is of the order of 10^{-9} sec. At such times, the difference between the temperatures of the metastable and equilibrium states, i.e., the superheat of the liquid metal, does not exceed several degrees.

The question of evaporation of shock-compressed metals in release waves was investigated in experiment us-

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