Effect of a particle source on the non-linear evolution of a monochromatic wave

A. B. Mikhaĭlovskiĭ and A. I. Pyatak

I. V. Kurchatov Institute of Atomic Energy
(Submitted May 12, 1977)
Zh. Eksp. Teor. Fiz. **73**, 1370–1378 (October 1977)

We study the effect of a source of resonant particles on a non-linear monochromatic wave in a plasma. We consider the case of a Langmuir wave when there is no magnetic field present and that of an Alfvén wave under conditions of a toroidal geometry of the magnetic field. We show that a systematic change in the amplitude of the non-linear wave occurs when there is a source of particles present. We suggest that this effect may be important for the case of injection of fast neutral atoms into magnetic traps and also in problems of fast α-particles produced in a plasma in thermonuclear reactions and in problems of fast particle fluxes in the Earth's magnetosphere.

PACS numbers: 52.35.Mw, 52.40.Mj

1. INTRODUCTION

Started by the work by Mazitov[1] and O'Neil[2] and systematized in the survey by Galeev and Sagdeev[3] the non-linear theory of the interaction between a monochromatic wave and resonant particles predicts the vanishing of the Landau damping with time, owing to the "mixing" of these particles in phase space. The number of resonant particles was there assumed to be constant. However, if there is a particle source, the number of resonant particles is not constant. One meets the problem with a particle source, for instance, in the case of injection of fast neutral atoms into magnetic traps.[4] Another example of a particle source is provided by the production of α-particles in thermonuclear reactions.[5] Under cosmic conditions fluxes of fast particles impinging upon the Earth's magnetosphere may play the role of particle sources.[6] In this connection it is of interest to consider the non-linear evolution of a monochromatic wave when there is a particle source present. This is the aim of the present paper. We dwell first of all upon the standard example of a Langmuir wave, discussed in Refs. 1 to 3, and then analyze the effect which interests us for the case of an Alfvén wave in a tokamak with injection of fast neutral atoms.[7] There appears then a non-vanishing non-linear Landau damping.

2. CANONICAL FORM OF THE KINETIC EQUATION WITH A SOURCE

We start from the kinetic equation for the distribution function for electrons which interact with a monochromatic Langmuir wave:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{e}{m}E(x,t)\frac{\partial f}{\partial v}=S(v,t). \quad (1)$$

Here $E(x,t)=E_0\sin(kx-\omega t)$; $S(v,t)$ is a spatially uniform particle source, and the remainder of the notation is standard. For $S=0$, Eq. (1) is the same as the one studied in Refs. 1 to 3.

We introduce instead of $t, x, v, f,$ and S dimensionless variables $\tau=\omega t$, $u=v/v_{\rm ph}-1$, $\theta=kx-\omega t$, $\mathcal{F}=fv_{\rm ph}/n_0$, $I(u,\tau)=S(v,t)(n_0k)^{-1}$, where $v_{\rm ph}=\omega/k$ is the phase velocity of the wave and n_0 the density of the main component of the plasma (the non-resonant particles). Equation (1) can then be written in canonical form

$$\frac{\partial\mathcal{F}}{\partial\tau}+u\frac{\partial\mathcal{F}}{\partial\theta}-\frac{\sin\theta}{\tau_0^2}\frac{\partial\mathcal{F}}{\partial u}=I(u,\tau). \quad (2)$$

Here $\tau_0^2=\omega^2\tau_b^2$, $\tau_b^2=m/ekE_0$ is the square of the period of the oscillations of particles trapped by the wave field—the bounce period.

3. SOLUTION OF THE KINETIC EQUATION

One can find the distribution function when there is a source present by solving the kinetic Eq. (2) by the method of integrating along the particle trajectories in the wave field. We assume that at the moment when the source is switched on, i.e., at $\tau=0$, there were no resonant particles $\mathcal{F}(0,\theta,u)=0$. We then have for $\tau>0$

$$\mathcal{F}(\tau,\theta,u)=\int_0^\tau I[\tau',u(\tau')]d\tau'. \quad (3)$$

As we are interested in velocities close to the phase velocity of the wave, the integrand in the resonance region is approximately equal to

$$I(\tau,u)=I(\tau,0)+u(\partial I/\partial u)_{u=0}. \quad (4)$$

Recognizing that according to the equation of motion $u=d\theta/d\tau$ and assuming the time-dependence to be weak compared to the time-dependence of the coordinate θ we get from (3)

$$\mathcal{F}=\int_0^\tau I(\tau',0)d\tau'+(\theta-\theta_0)\frac{\partial I(\tau)}{\partial u_0}-\int_0^\tau(\theta-\theta_0)\frac{\partial}{\partial u_0}\frac{\partial I(\tau')}{\partial\tau'}d\tau', \quad (5)$$

where θ_0 is the coordinate at time $\tau=0$ of a particle which at time τ is at the point θ; $\partial I(\tau)/\partial u_0\equiv(\partial I/\partial u)_{u=u_0}$.

The connection between θ_0 and the running value θ of the coordinate can be found from the equations of motion which are the characteristics of Eq. (2). As the trapping frequency τ_b^{-1} is assumed to be large compared to the Landau damping rate γ_L, we neglect the slow change in the amplitude above the background of the fast process of the particle oscillations when we integrate the

 0038-5646/77/4604-0721$02.40

equations of motion. It then follows from (2) that

$$\frac{d\theta}{d\tau}=\frac{2\sigma}{\varkappa\tau_0}\left(1-\varkappa^2\sin^2\frac{\theta}{2}\right)^{1/2}, \tag{6}$$

where $\varkappa^2=4/C\tau_0^2$; C is the energy of an electron made dimensionless by dividing by $\frac{1}{2}mv_{ph}^2$; $\sigma=\pm 1$ determines the direction of particle motion relative to the wave.

In the case of untrapped particles ($\varkappa<1$) we get from (6)

$$\theta-\theta_0=\frac{2}{\varkappa\tau_0}\int_0 d\tau' dnF\left[\varkappa,\frac{\theta(\tau')}{2}\right], \tag{7}$$

where $dnF[\varkappa, \frac{1}{2}\theta(\tau')]$ is an elliptical Jacobi function, the argument of which is the elliptical integral of the first kind $F[\varkappa, \frac{1}{2}\theta(\tau')]$. Bearing in mind that according to (6)

$$F(\varkappa, {}^1\!/_2\theta)=F(\varkappa, {}^1\!/_2\theta_0)+\tau/\varkappa\tau_0, \tag{8}$$

and using the expansion of the function dnF in a trigonometric series[8] we evaluate the integral over τ' in (7):

$$\theta-\theta_0=\frac{\pi\tau}{\varkappa\tau_0 K(\varkappa)}+4\sum_{n=1}^{\infty}\frac{n^{-1}Q^n}{1+Q^{2n}}\left\{\sin\left[F\left(\varkappa,\frac{\theta_0}{2}\right)+\frac{\tau}{\varkappa\tau_0}\right]\frac{\pi n}{K} -\sin\frac{\pi n}{K}F\left(\varkappa,\frac{\theta_0}{2}\right)\right\}, \tag{9}$$

where $Q(\varkappa)=\exp(-\pi K'/K)$, $K'=K(\sqrt{1-\varkappa^2})$, $K(\varkappa)$ is a complete elliptical integral of the first kind.

Using (8) we can express θ_0 on the right-hand side of (9) in terms of θ and substitute the result into the second term of the right-hand side of (5). In the third term of the right-hand side of (5) we first integrate over the trajectory and then express θ_0 in terms of θ. After carrying out these steps we get an expression for the distribution function of the untrapped particles in the variables τ, θ, $\varkappa$

$$\mathscr{F}_{unt}(\varkappa, \theta, \tau)=A_{unt}(\varkappa, \tau)+B_{unt}(\varkappa, \theta, \tau)+C_{unt}(\varkappa, \theta, \tau), \tag{10}$$

where

$$A_{unt}(\varkappa, \tau)=\left(1+\frac{\pi}{\varkappa\tau_0 K}\frac{\partial}{\partial u_0}\right)\int_0^{\tau}I(\tau',0)\,d\tau', \tag{11}$$

$$B_{unt}(\varkappa,\theta,\tau)=8\frac{\partial I(\tau)}{\partial u_0}\sum_{n=1}^{\infty}\frac{n^{-1}Q^n}{1+Q^{2n}}\sin\frac{\pi n\tau}{2\varkappa\tau_0 K}\cos\frac{\pi n}{K}\left(F-\frac{\tau}{2\varkappa\tau_0}\right), \tag{12}$$

$$C_{unt}(\varkappa,\theta,\tau)=4\left(\frac{\partial I(\tau)}{\partial u_0}-\frac{\partial I(0)}{\partial u_0}\right)\sum_{n=1}^{\infty}\frac{n^{-1}Q^n}{1+Q^{2n}}\sin\frac{n\pi}{K}\left(F-\frac{\tau}{\varkappa\tau_0}\right) -\frac{8}{\pi}\frac{\partial}{\partial u_0}\frac{\partial I}{\partial\tau}\sum_{n=1}^{\infty}\frac{n^{-2}Q^n K\varkappa\tau_0}{1+Q^{2n}}\sin\frac{\pi n\tau}{2K\varkappa\tau_0}\sin\frac{\pi n}{K}\left(F-\frac{\tau}{2\varkappa\tau_0}\right). \tag{13}$$

Similarly we find the expression for the distribution function $\mathscr{F}_t$ of the trapped particles. Instead of (7) and (8) we then have

$$\theta-\theta_0=\frac{2\sigma}{\varkappa\tau_0}\int_0^{\tau}d\tau'\,\mathrm{cn}\,F\left\{\frac{1}{\varkappa},\varphi[\theta(\tau')]\right\}, \tag{14}$$

$$F(1/\varkappa, \varphi)=F(1/\varkappa, \varphi_0)+\sigma\tau/\tau_0, \tag{15}$$

so that, in analogy with (9)

$$\theta-\theta_0=8\sum_{n=1}^{\infty}\frac{(2n-1)^{-1}Q^{n-1/2}(1/\varkappa)}{1+Q^{2n-1}(1/\varkappa)}\left\{\sin\frac{(2n-1)\pi}{2K(1/\varkappa)}\times\left[F\left(\frac{1}{\varkappa},\varphi_0\right)+\frac{\sigma\tau}{\tau_0}\right]-\sin\frac{(2n-1)\pi F(1/\varkappa,\varphi_0)}{2K(1/\varkappa)}\right\}. \tag{16}$$

Here $\varphi(\theta)=\arcsin(\varkappa\sin\frac{1}{2}\theta)$; $\varphi_0=\varphi(\theta_0)$. Using (14) to (16) we find in analogy with (10)

$$\mathscr{F}_t(\varkappa, \theta, \tau)=A_t(\tau)+B_t(\varkappa, \theta, \tau)+C_t(\varkappa, \theta, \tau), \tag{17}$$

where

$$A_t(\tau)=\int_0^{\tau}I(\tau',0)\,d\tau', \tag{18}$$

$$B_t(\varkappa,\theta,\tau)=16\frac{\partial I(\tau)}{\partial u_0}\sum_{n=1}^{\infty}\frac{(2n-1)^{-1}Q^{n-1/2}}{1+Q^{2n-1}}\sin\frac{(2n-1)\pi\sigma\tau}{4K\tau_0}\times\cos\frac{(2n-1)\pi}{2K}\left[F\left(\frac{1}{\varkappa},\varphi\right)-\frac{\sigma\tau}{2\tau_0}\right], \tag{19}$$

$$C_t(\varkappa,\theta,\tau)=8\left(\frac{\partial I}{\partial u_0}-\frac{\partial I(0)}{\partial u_0}\right)\sum_{n=1}^{\infty}(2n-1)^{-1}\frac{Q^{n-1/2}}{1+Q^{2n-1}}\sin\frac{(2n-1)\pi}{2K}\times\left[F\left(\frac{1}{\varkappa},\varphi\right)-\frac{\sigma\tau}{\tau_0}\right]-\frac{\partial}{\partial u_0}\frac{\partial I}{\partial\tau}\frac{32}{\pi}\sum_{n=1}^{\infty}\frac{K\tau_0\sigma}{(2n-1)^2}\frac{Q^{n-1/2}}{1+Q^{2n-1}}\times\sin\frac{(2n-1)\pi\sigma\tau}{4K\tau_0}\sin\frac{(2n-1)\pi}{2K}\left[F\left(\frac{1}{\varkappa},\varphi\right)-\frac{\sigma\tau}{2\tau_0}\right]. \tag{20}$$

We note that in Eqs. (19), (20), as in (16), the functions Q and K depend on the argument $1/\varkappa$.

4. NON-LINEAR EVOLUTION OF THE WAVE

When studying the non-linear evolution of a monochromatic wave we start from the energy balance equation

$$\frac{\partial W}{\partial t}=-\overline{\mathbf{jE}}, \tag{21}$$

where W is the energy of the oscillations given by the relation

$$W=\frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2\varepsilon_{\alpha\beta})\frac{E_\alpha^* E_\beta}{16\pi}, \tag{22}$$

and $\mathbf{j}$ is the electric current of the resonant particles which is equal to

$$j=-e\int_{-\infty}^{+\infty}fv\,dv=-ev_\varphi n_0\int_{-\infty}^{\infty}\mathscr{F}du. \tag{23}$$

In the case of Langmuir oscillations which is of interest to us

$$\varepsilon_{\alpha\beta}=\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)\delta_{\alpha\beta}\equiv\varepsilon\delta_{\alpha\beta},\quad W=\frac{E^2}{16\pi}\omega\frac{\partial\varepsilon}{\partial\omega},\quad \omega\frac{\partial\varepsilon}{\partial\omega}=2.$$

Using the linearized kinetic Eq. (2) we get from (21) to (23) in the linear approximation

$$\frac{\partial\ln E}{\partial t}=\gamma_L, \tag{24}$$

where γ_L is the linear growth rate given by the relation

$$\gamma_L=\pi\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}\int_0^{\tau}\frac{\partial I(\tau')}{\partial u_0}d\tau'. \tag{25}$$

For the validity of the linear approximation it is necessary to assume that $\gamma_L\tau_b \gg 1$.[3] However, if $\gamma_L\tau_b \ll 1$, we have instead of (24)

$$\frac{\partial\varepsilon}{\partial\omega}\frac{1}{E}\frac{\partial E}{\partial t}=\frac{\tau_0^2}{\pi}\int_{-\infty}^{\infty}du\left\{\int_{-\pi}^{\pi}\mathcal{F}_{unt}\sin\theta\,d\theta+\int_{-\theta_m}^{\theta_m}\mathcal{F}_t\sin\theta d\theta\right\}, \quad (26)$$

where $\theta_m = 2\arcsin(1/\varkappa)$ while the functions $\mathcal{F}_{unt}$ and $\mathcal{F}_t$ are given by Eqs. (10) and (17).

We must substitute in Eq. (26) the quantities standing on the right-hand sides of Eqs. (11) to (13) and (18) to (20). It is convenient when integrating in (26) to change from the variables θ, u to the variables $F, \varkappa$ by using the definitions of the elliptical functions and Eq. (6). We get then

$$du\,d\theta=-\frac{4\sigma}{\tau_0\varkappa}dF\,d\varkappa, \quad (27)$$

$$\sin\theta=2\sin F(\varkappa,\theta/2)\,\mathrm{cn}\,F(\varkappa,\theta/2). \quad (28)$$

It follows from Eq. (26), written in the new variables, and Eqs. (11) to (13) and (18) to (20) that the integrals of A_{unt} and A_t vanish while the integrals with C_{unt} and C_t contain solely terms which oscillate with time with a period of the order τ_0:

$$\sin\frac{\pi n\tau}{\varkappa K\tau_0},\quad \cos\frac{\pi n\tau}{\varkappa K\tau_0}.$$

These terms describe the deviations of the amplitude of the electric field E_0 from its average value $\bar{E}_0$ which are damped in time.

The problem of a monochromatic wave reduces, when there is no particle source present, to evaluating merely such kind of oscillating terms (Mazitov–O'Neil type of problem[1,2]); in that case $\partial\bar{E}_0/\partial t = 0$. However, when there is a particle source present, it follows from (26) that there appears, apart from the damped field oscillations which are unimportant for $\tau \gtrsim \tau_0$, also an evolution of the average value of the field $\bar{E}_0$ determined by the terms with B_{unt} and B_t (see (12) and (19)). Neglecting the oscillations, it follows from Eq. (26) that

$$\frac{\partial\ln\bar{E}_0}{\partial\tau}=G\tau_b(\bar{E}_0)\pi\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}\frac{\partial I(\tau)}{\partial u_0}. \quad (29)$$

Here

$$G=128\sum_{n=1}^{\infty}\int_0^1\frac{d\varkappa}{\varkappa^4 K}\left\{\frac{Q^{2n}}{(1+Q^{2n})^2}+\varkappa^5\frac{Q^{2n-1}}{(1+Q^{2n-1})^2}\right\}, \quad (30)$$

$$K=K(\varkappa),\quad Q=Q(\varkappa).$$

We note that if we take (25) into account Eq. (29) becomes

$$\frac{\partial\ln\bar{E}_0}{\partial\tau}=G\tau_b(\bar{E}_0)\frac{\partial\gamma_L}{\partial\tau}. \quad (31)$$

Hence we find that the amplitude $\bar{E}_0$ changes according to the relation

$$\bar{E}_0(\tau)=\bar{E}_0(0)\left[1+\frac{G}{2}\frac{\gamma_L(\tau)}{\omega_b(0)}\right]^2, \quad (32)$$

so that the non-linear growth rate made dimensionless by dividing by ω is equal to

$$\gamma\equiv\frac{\partial\ln\bar{E}_0}{\partial\tau}=2\frac{\partial\ln\gamma_L}{\partial\tau}\left[1+\frac{2\omega_b(0)}{G\gamma_L(\tau)}\right]^{-1}. \quad (33)$$

The results (31) and (33) allow the following interpretation.[1)] As the resonant particles effectively interact with the wave only during a time of the order τ_b we suppose that the evolution of the average value $\bar{E}_0$ at time τ is caused by the interaction between the wave and the particles which are produced from the source during the period from $\tau-\tau_b$ to τ (as to order of magnitude). The number of such particles is of the order of

$$\Delta n(\tau)\approx\tau_b(\tau)\,dn/d\tau. \quad (34)$$

They lead to a growth (damping) of the wave with growth (damping) rate of the order of

$$\gamma=\gamma_L\Delta n(\tau)/n(\tau)\approx\tau_b\partial\gamma_L/\partial\tau. \quad (35)$$

Bearing also in mind that $\gamma \equiv \partial\ln\bar{E}_0/\partial\tau$ we find an equation which is qualitatively the same as (31).

In the case of a supply of particles with $\partial F/\partial u_0 < 0$ it follows from (32) that the amplitude of the wave must systematically decrease. The damping rate (33) then increases "explosively." After the lapse of a time τ^*, defined by the relation $\gamma_L(\tau^*) \approx \omega_b(0)$ the wave amplitude may have decreased to such small values that the further change in the field will be determined by the usual Landau damping.

If, however, particles are injected with $\partial F/\partial u_0 > 0$, after some time the non-linearity of the wave will only be amplified (i.e., the product $\gamma(\tau)\tau_b(\tau)$ decreases with increasing time). This effect becomes important for $\tau \gtrsim \tau_1$, where τ_1 is given by the approximate relation

$$\int_0^{\tau_1}\gamma(\tau)d\tau\approx 1. \quad (36)$$

Using (31) this means that

$$\gamma_L(\tau_1)\tau_b(0)\approx 1. \quad (37)$$

For $\tau > \tau_1$ it follows from Eq. (32) that the wave amplitude increases according to the relation

$$\bar{E}_0(\tau)=\bar{E}_0(0)\left[{}^1\!/_2 G\gamma_L(\tau)/\omega_b(0)\right]^2. \quad (38)$$

In particular, for a constant rate of particle injection $\gamma_L(\tau) \sim \tau$ so that

$$\bar{E}_0(\tau)\sim\tau^2. \quad (39)$$

Since the width of the resonance region increases together with the increase in the amplitude of the field, in the case of a particle source with a small spread in velocities (small compared to the resonance velocity) all particles coming out of the source will turn out to be resonant particles after some time. The further evolution of the field and of the particle distribution must therefore be studied in the framework of a hydrodynamical description. Such a transition to the hydrodynamic stage begins at $\tau \approx \tau_2$, where τ_2 satisfies the approximate equation

$$\gamma_L(\tau_2)/\omega \approx \Delta v/v, \tag{40}$$

Δv is the width of the velocity distribution of the particles in the source. We note that the analogous criterion for the transition to the hydrodynamic regime occurs also in the linear approximation when γ_L plays the role of the true growth rate.

5. EVOLUTION OF A NON-LINEAR ALFVÉN WAVE IN A TOKAMAK WITH INJECTIONS OF NEUTRAL ATOMS

One actual example of a particle source is the injection of fast neutral atoms in a tokamak with the aim of obtaining a plasma with thermonuclear parameters. The beam of fast ions which appears as a result of ionization can, according to Ref. 9, excite Alfvén oscillations. The problem then arises of the relaxation of the ion beam when it interacts with the Alfvén oscillations and the problem of the evolution of such oscillations. In the case where a wide oscillation spectrum is excited one can consider this problem in the quasi-linear approximation, as was done by Kulygin *et al.*[10] The study of the excitation of an ion beam by a monochromatic Alfvén wave is also of interest, since the fact that the wavenumbers in a tokamak are discrete may make it impossible to excite several waves at the same time. The evolution of an Alfvén wave when there is no source present was studied in Ref. 7. Using the general results obtained in the preceding sections of the present paper we consider the process, which is most important in practice, of the evolution of such a wave under the conditions of a continuous injection of neutral atoms.

We start the study of an Alfvén wave with the drift equations of motion and the kinetic equation for the ion injected along the lines of force of the magnetic field of the tokamak from a continuously acting source (cf. Ref. 7)

$$\frac{\partial f}{\partial t}+\frac{d\mathbf{r}}{dt}\nabla f+\frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel}=S(v_\parallel, t), \tag{41}$$

$$\frac{d\mathbf{r}}{dt}=v_\parallel\frac{\mathbf{B}_0}{B_0}+\frac{v_\parallel^2}{\omega_B B_0^2}[\mathbf{B}_0\times\nabla B_0], \tag{42}$$

$$\frac{dv_\parallel}{dt}=c\frac{v_\parallel}{B_0^3}\mathbf{E}[\mathbf{B}_0\times\nabla B_0]. \tag{43}$$

Here $S(t, v_\parallel)$ is the particle source and the remainder of the notation is standard. The main toroidal magnetic field of the tokamak has in the normally used coordinates a, Θ, ζ the form (see, e.g., Ref. 11)

$$B_0=B_s[1+(a/R)\cos\Theta]. \tag{44}$$

Here R is the large radius of the tokamak, B_s the magnetic field on the axis of the torus. We assume for the sake of simplicity that the cross section of the magnetic surfaces of the tokamak is a circle.

Instead of $t, \Theta, v_\parallel, f$ we introduce the dimensionless quantities

$$\tau=\frac{t}{t_*},\quad u=\frac{v_\parallel-v_{\parallel\Phi}}{v_0},\quad \theta=\Theta-\omega t+\pi,\quad F=f\frac{v_0}{n_0},\quad I=S\frac{v_0\tau_*}{n_0},$$

where $\tau_* \equiv qR/v_0=1/\omega$ (see Ref. 9), v_0 is the average velocity of the injected ions, $v_{\parallel ph}$ the phase velocity of the wave, n_0 the density of the main component of the plasma in the tokamak, and q the safety factor of the tokamak. The kinetic Eq. (41) then takes the form of Eq. (2). The role of τ_0 in (2) will now be played by the quantity $\tau_0^2=\tau_b^2/\tau_*^2$, where

$$\tau_b^2=2B_0qR^2/cv_0E_a(\tau). \tag{45}$$

Here $E_a(\tau)$ is the covariant component (number a) of the electric field of the wave.

When studying the evolution of the Alfvén wave we start from an equation such as (21) which in this case means (cf. Ref. 7)

$$\frac{\partial W}{\partial t}=-\overline{j^1E_1}, \tag{46}$$

where $W=\varepsilon_{11}E_{10}^2/8\pi$; $\varepsilon_{11}=c^2/c_A^2$, c_A is the Alfvén speed, j^1 the contravariant component of the current which is connected with the distribution function through the relation

$$j^1=\frac{e\sin\Theta}{\omega_{Bi}R}\int_{-\infty}^{+\infty}v_\parallel^2 f dv_\parallel=\frac{Mcv_0^2n_0\sin\Theta}{B_sR}\int_{-\infty}^{+\infty}F\,du, \tag{47}$$

where M is the mass of an injected ion. Evaluating j^1 and substituting the result into (46) we find a relation similar to (26):

$$\varepsilon_{11}\frac{\partial E_{10}}{\partial t}\frac{1}{E_{10}}=\omega\frac{q^2\tau_0^2c^2n_0M}{2B_s^2}\int_{-\infty}^{+\infty}du\left[\int_{-\pi}^{+\pi}F_{unt}\sin\theta\,d\theta+\int_{-\theta_m}^{\theta_m}F_t\sin\theta d\theta\right]. \tag{48}$$

Using the standard expressions for F_{unt} and F_t (see (10) and (17)) and the expression for the non-linear growth rate, similar to (25),

$$\gamma_L(\tau)=\frac{1}{2}\left(\frac{\pi qc}{B_s}\right)^2\frac{n_0\omega}{\varepsilon_{11}}\int_0^\tau\frac{\partial I(\tau')}{\partial u_0}d\tau', \tag{49}$$

we reduce Eq. (48) to the form (31) where we must take $\overline{E}_{10}$ for $\overline{E}_0$. Moreover, for the case of an Alfvén wave the expressions for the field and the non-linear growth rate determined by Eqs. (32), (33) also remain valid. All results for the Langmuir wave which are written in canonical form therefore remain valid also for the case of an Alfvén wave.

Using this fact and bearing in mind the analysis of Eqs. (31) to (33) given above we can conclude that when there is a source of fast ions present the non-linear evolution of an Alfvén wave in a tokamak can be looked at as a sequence of the following three stages.

1) *Stage of an almost constant wave applitude.* This corresponds to the time interval $0<\tau<\tau_1$. Using (37), (45), (49) we find that in the case of an Alfvén wave and a constant rate of injection (I = const)

$$\tau_1=\left(\frac{\partial I}{\partial u_0}\right)^{-1}\left(\frac{v_E(0)}{v_0q^3}\right)^{1/2}, \tag{50}$$

where $v_E(0) = c\bar{E}_{10}(0)/B_s$ is the electric drift velocity at $\tau = 0$.

2) *Stage of a growing field in the kinetic regime corresponding to the time interval* $\tau_1 < \tau < \tau_2$. In the above mentioned case of a constant rate of injection

$$\tau_2 > \frac{\Delta v}{v_0 q^2}\left(\frac{\partial I}{\partial u_0}\right)^{-1}. \qquad (51)$$

In particular, for a distribution of the form $I = \bar{a}e^{-(v-1)^2/b^2}$ which was considered in Ref. 10, $\tau_2 \approx b^2/\bar{a}$; $\Delta v/v_0 \approx b$.

We note also that $\bar{E}_{10}(\tau_2) \approx B_0 v_0 b^2/c$. In such a field the electric drift velocity $v_E(\tau_2) \approx v_0 b^2$ so that an ion during a period of oscillations drifts along the small torus radius over a distance of the order of $\Delta a = Rb^2$.

3) *Stage of the hydrodynamical evolution starting for* $\tau > \tau_2$. For a study of this state of the evolution of a monochromatic Alfvén wave one must develop a nonlinear theory of the hydrodynamic Alfvén instability which was discussed in the linear approximation in Ref. 9.

According to what has been said in the foregoing, the presence of a source of resonant particles thus affects considerably the non-linear evolution of a monochromatic wave. The main effect displayed when there is a source present consists in a systematic change in the wave amplitude which is qualitatively different from the effect of damped oscillations of the amplitude when there is no source present.

We are grateful to A. B. Kitsenko, K. N. Stepanov, and V. D. Shapiro for a discussion of the results of the present paper which was very useful for us.

[1)] This interpretation is due to V. D. Shapiro.

[1] R. K. Mazitov, Zh. Prikl. Mekh. Fiz. No. 1, 27 (1965) [Translation in J. Appl. Mech. Tech. Phys.].
[2] T. O'Neil, Phys. Fluids **8**, 2255 (1965).
[3] A. A. Galeev and R. Z. Sagdeev, Voprosy teorii plazmy (M. A. Leontovich editor) 7, 3 (1973) [Translation in Rev. Plasma Phys., published by Consultants Bureau, New York].
[4] V. I. Pistunovich, Fiz. Plazmy **2**, 3 (1976) [Sov. J. Plasma Phys. **2**, 1 (1976)].
[5] J. D. Lawson, Proc. Phys. Soc. (London) **B70**, 6 (1957).
[6] H. I. West, Rev. Geophys. Space Phys. **13**, 943 (1975).
[7] A. B. Mikhaĭlovskiĭ, A. I. Pyatak, and A. M. Fridman, Fiz. Plazmy **2**, 922 (1976) [Sov. J. Plasma Phys. **2**, 511 (1976)].
[8] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of integrals, sums, series, and products) Fizmatgiz, 1962 [Translation published by Academic Press].
[9] A. B. Mikhaĭlovskiĭ, Fiz. Plazmy **1**, 72 (1975) [Sov. J. Plasma Phys. **1**, 38 (1975)].
[10] V. M. Kulygin, A. B. Mikhaĭlovskiĭ, A. I. Pyatak, A. M. Fridman, and E. S. Tsapelkin, Zh. Eksp. Teor. Fiz. **70**, 2152 (1976) [Sov. Phys. JETP **43**, 1123 (1976)].
[11] V. D. Shafranov and E. I. Yurchenko, Zh. Eksp. Teor. Fiz. **53**, 1157 (1967) [Sov. Phys. JETP **26**, 682 (1968)].

Translated by D. ter Haar

The phenomenon of parametric trapping of electromagnetic waves in an inhomogeneous plasma

V. P. Silin and A. N. Starodub

P. N. Lebedev Institute of Physics, USSR Academy of Sciences
(Submitted May 20, 1977)
Zh. Eksp. Teor. Fiz. **73**, 1379–1388 (October 1977)

A theory is developed of absolute parametric aperiodic instability in a spatially inhomogeneous plasma, when the electromagnetic waves generated in the plasma are trapped by the plasma near the peaks of the pumping-wave field.

PACS numbers: 52.35.Py, 52.35.Hr

1. The present paper is devoted to the theory of the phenomenon of electromagnetic-wave trapping by a plasma. The essence of such a phenomenon consists in the fact that the secondary electromagnetic waves parametrically excited under the action of a pumping electromagnetic wave do not get out of the plasma, but are trapped inside it. It may be expected that such a phenomenon is one of the causes of the reduction in reflection of electromagnetic waves by a plasma during some short interval of time.

As a specific example of the appearance of the trapping phenomenon, below we consider parametric instability in an inhomogeneous plasma, during which the pumping wave gets transformed into an electromagnetic wave and perturbations aperiodically growing in time.[1)] It is shown in the process that, after the intensity of the electric field of the pump exceeds some threshold value in the spatially inhomogeneous plasma, the development of absolute parametric instability becomes possible. The growing—in time—plasma perturbations are localized near the peaks of the electric field of the pumping wave. The regions of such localization are small compared to the characteristic dimension of the pump inhomogeneity

 0038-5646/77/4604-0725$02.40