One can expect to observe this phenomenon in materials that are perfect in the sense that the resonances of the polarization oscillations is of high Q, and the transition is not masked by the domain structure (i.e., the strong decrease of the speed of sound in the vicinity of the transition is well pronounced).

[1)] In addition, in the analysis it was implicitly assumed that $h < h_p$, where h_p is the field amplitude at which parametric oscillations are produced in the system. At $\mathbf{h} \perp \mathbf{m}_0$ and $\Omega = \Omega_0$ the threshold of the parametric excitation of the spin waves is according to[13] $h_p \sim (\Delta H/4\pi m)^{1/2}\Delta H$, $\Delta H = \Gamma/g$. At $h = h_p$ we obtain from (11) $\Delta s/s_t \sim \Delta H/4\pi m$.

[2)] It is assumed for the sake of argument that the system is in a collinear phase, $\mathbf{L} \parallel \mathbf{n} \parallel 1$. For a noncollinear phase, choosing as before the axis 1 in the direction of the equilibrium value of $\mathbf{L}$, we have (14), where β_1 must be replaced by $\beta_2 = (H^2 - H_2^2)/mH_E$. All the calculations that follow must be correspondingly modified.

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Diffraction of electromagnetic waves by a domain wall in a ferroelectric

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An analytic solution is obtained of the problem of diffraction of an electromagnetic wave by a domain wall in a ferroelectric. It is shown that the picture of the interference fringes is observed only in diffraction of light of sufficiently small wavelength. The distance between the interference fringes is determined not only by the geometrical dimensions of the wall, but also by the difference between the values of the refractive index inside the wall and far from it.

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1. INTRODUCTION

We solve here the problem of diffraction of an electromagnetic wave by a ferroelectric domain wall. This problem arises in connection with the possibility of using the diffraction of light for a direct measurement of the domain-wall thickness. In addition, the solution of this problem is also of independent interest, since analytic solutions of diffraction problems encounter as a rule great mathematical difficulties.

The use of optical methods to measure domain-wall thicknesses is particularly pressing because of the substantial discrepancies that exist between the *a priori* theoretical estimates and the data obtained from x-ray scattering. Theoretical estimates lead as a rule to a domain-wall thickness on the order of 10^{-6}–10^{-7} cm. Yet measurements made on sodium nitrate[4] and triglycine sulfate[2] yield values larger than 10^{-5} cm.

The general solution of the diffraction problem poses no fundamental difficulties. The formulas for the diffracted-wave amplitudes in quadratures are derived in Sec. 2. An investigation of these formulas, however, for the purpose of deriving expressions useful to experimenters, entails great technical difficulties. This investigation is the subject of an appreciable part of the paper. In the last section we discuss the form of the diffraction pattern in various cases and the possibility of extracting from it information on the structure of the domain wall.

We consider a plane 180° domain wall in a cubic uniaxial ferroelectric, or one belonging to a rhombic system, and exhibiting no piezoelectric effect in the paraphase. The wall thickness is assumed to be much larger than the lattice structure, so that its structure is described by a phenomenological theory. The length of the electromagnetic wave is also assumed to be much larger

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than the lattice constant, so that the optical properties of the crystal can be described by the dielectric tensor ε_{ik}. In the polarized phase, ε_{ik} acquires an increment proportional to the square of the spontaneous polarization. This increment depends on the coordinates in a direction normal to the domain wall. It is the dependence of ε_{ik} on the coordinates which leads to the appearance of the diffraction pattern.

The dependence of the optical permittivity on the spontaneous polarization in ferroelectrics is usually quite weak. Therefore the change of the permittivity inside the domain wall is relatively small. Consequently only small diffraction angles turn out to be significant, and we shall solve the problem by the parabolic-equation method. The variables in this equation are separable, and the amplitude of the diffracted wave can be expressed in quadratures. These quadratures are calculated only in the most interesting limiting cases, principally by contour-integration methods.

It turns out that the form of the diffraction pattern depends substantially on the wavelength of the diffracted light and on the character of the variation of the permittivity inside the domain wall. In the case of long waves, the diffraction pattern constitutes a single narrow maximum. In the case of short wave, the diffraction pattern consists of interference fringes. The distance between fringes depends not only on the geometrical dimensions of the wall (as is the case, e.g., in diffraction by a slit in a flat screen), but also on the extent to which the permittivity inside the wall differs from the permittivity far from it. With increasing diffraction angle ϑ, the intensity of the diffraction fringes falls off exponentially if ε_{ik} inside the wall is smaller than far from it, and like ϑ^{-2} in the opposite case. The power-law decrease of the intensity is due to the existence of electromagnetic-wave modes that can propagate inside the domain wall as in a waveguide.

2. GENERAL SOLUTION OF THE PROBLEM

We consider the simplest experimental situation, wherein a monochromatic electromagnetic wave is normally incident on a plane-parallel ferroelectric plate of thickness L, cut in such a way that one of the crystallographic axes parallel to the domain wall is perpendicular to the faces of the plane. We direct the z axis perpendicular to the faces of the plate, in the direction of the incident wave, and the x axis along the normal to the domain wall. The wave electric-field intensity in the plate can then be written in the form

$$\mathbf{E}=\mathbf{F}^{+}(x, z)e^{ikz}+\mathbf{F}^{-}(x, z)e^{ik(L-z)}, \tag{1}$$

where k is the wave vector of the wave propagating in the homogeneous sample along the z axis.

In view of the smallness of the alternating part of the permittivity, the diffraction angles are small, and the dependences of $\mathbf{F}^{+}$ and $\mathbf{F}^{-}$ on z can be regarded as small. Discarding all the small terms in the equation $\operatorname{div}\mathbf{D}=0$ ($\mathbf{D}$ is the electric induction vector), we easily obtain the relation

$$F_z^{\pm}=\pm\frac{i}{k}\frac{\varepsilon_{xx}^{0}}{\varepsilon_{zz}^{0}}\frac{\partial F_x^{\pm}}{\partial x}, \tag{2}$$

where ε_{ik}^{0} is the dielectric tensor in the paraphase. Discarding second-order quantities in the wave equation and using (2), we obtain the following equations for F_x and F_y:

$$\pm 2ik\frac{\partial F_x^{\pm}}{\partial z}+\frac{\varepsilon_{xx}^{0}}{\varepsilon_{zz}^{0}}\frac{\partial^2 F_x^{\pm}}{\partial x^2}+\left(\frac{\omega^2}{c^2}\varepsilon_{xx}-k^2\right)F_x^{\pm}+\frac{\omega^2}{c^2}\varepsilon_{xy}F_y^{\pm}=0, \tag{3}$$

$$\pm 2ik\frac{\partial F_y^{\pm}}{\partial z}+\frac{\partial^2 F_y^{\pm}}{\partial x^2}+\left(\frac{\omega^2}{c^2}\varepsilon_{yy}-k^2\right)F_y^{\pm}+\frac{\omega^2}{c^2}\varepsilon_{yx}F_x^{\pm}=0, \tag{4}$$

where ω is the frequency and c is the speed of light in vacuum.

In cubic and uniaxial ferroelectrics belonging to the rhombic system and having no piezo-effect in the paraphase, the off-diagonal components of ε_{ik} are equal to zero. Using the results of the phenomenological theory,[3] we can express the diagonal elements in the form

$$\varepsilon_{xx}=\varepsilon_{xx}^{0}(1+\alpha_x)\left[1-\frac{\alpha_x}{\operatorname{ch}^2(x/d)}\right],\qquad \varepsilon_{yy}=\varepsilon_{yy}^{0}(1+\alpha_y)\left[1-\frac{\alpha_y}{\operatorname{ch}^2(x/d)}\right], \tag{5}$$

where $|\alpha_x|$, $|\alpha_y|\ll 1$. Putting now $k^2=\omega^2(1+\alpha_x)\varepsilon_{xx}^{0}/c^2$ for a wave polarized along x and $k^2=\omega^2(1+\alpha_y)\varepsilon_{yy}^{0}/c^2$ for a wave polarized along y, we obtain equations of the same type for the waves with both polarizations:

$$\pm 2ik\frac{\partial F_x^{\pm}}{\partial z}+\frac{\varepsilon_{xx}^{0}}{\varepsilon_{zz}^{0}}\frac{\partial^2 F_x^{\pm}}{\partial x^2}-\frac{\alpha_x k^2}{\operatorname{ch}^2(x/d)}F_x^{\pm}=0, \tag{6}$$

$$\pm 2ik\frac{\partial F_y^{\pm}}{\partial z}+\frac{\partial^2 F_y^{\pm}}{\partial x^2}-\frac{\alpha_y k^2}{\operatorname{ch}^2(x/d)}F_y^{\pm}=0. \tag{7}$$

The boundary conditions for each of these waves are also of the same form. We put

$$E=E_0\exp(ik_0x)+E_1(x, z)\exp(-ik_0z)\quad\text{at}\quad z<0, \tag{8}$$

$$E=E_2(x, z)\exp(ik_0z)\quad\text{at}\quad z>L, \tag{9}$$

where $k_0^2=\omega^2/c^2$. Neglecting the small terms in the boundary conditions, we get

$$F^{+}+F^{-}e^{ikL}=E_0+E_1,\quad F^{+}-F^{-}e^{ikL}=\frac{k_0}{k}(E_0-E_1)\quad\text{at}\quad z=0, \tag{10}$$

$$F^{+}e^{ikL}+F^{-}=E_2,\quad F^{+}e^{ikL}-F^{-}=\frac{k_0}{k}E_2\quad\text{at}\quad z=L. \tag{11}$$

The intensity of the light propagating in a unit angle interval is given by

$$I(\vartheta)=2\pi k_0 d|A(k_0\vartheta)|^2 I_0,\qquad I_0=\frac{c}{4\pi}E_0^2 d, \tag{12}$$

where

$$A_1(q)=\frac{1}{2\pi E_0 d}\int E_1(x,0)e^{-iqx}dx,\qquad A_2(q)=\frac{1}{2\pi E_0 d}\int E_2(x,L)e^{-iqx}dx \tag{13}$$

are the dimensionless amplitudes of the reflected and transmitted light.

Since Eqs. (6) and (7) are obtained from each other by a scale transformation along the x axis, we consider only waves polarized along the y axis, and omit the subscript y for the sake of brevity.

We expand $F^{\pm}$ in terms of the eigenfunctions of the problem

$$\frac{d^2\varphi}{d\xi^2}+\left(\lambda-\frac{\alpha k^2d^2}{\operatorname{ch}^2\xi}\right)\varphi=0, \tag{14}$$

with $\varphi(\xi)$ bounded as $\xi\to\pm\infty$. The equations resulting from the expansion, which are of first order in z, can be easily solved for the suitable boundary conditions. The final expressions for $A_1(q)$ and $A_2(q)$ can be written in the form

$$A_1(q)=\frac{1}{2\pi}\int\varphi_\lambda(qd)R(\lambda)\varphi_\lambda^*(0)\,d\lambda, \tag{15}$$

$$A_2(q)=\frac{1}{2\pi}\int\varphi_\lambda(qd)T(\lambda)\varphi_\lambda^*(0)\,d\lambda, \tag{16}$$

where

$$R(\lambda)=\frac{i}{2}\left(\frac{k}{k_0}-\frac{k_0}{k}\right)\sin\left(\frac{\lambda L}{2kd^2}-kL\right)\left[\cos\left(\frac{\lambda L}{2kd^2}-kL\right)+\frac{i}{2}\left(\frac{k}{k_0}+\frac{k_0}{k}\right)\sin\left(\frac{\lambda L}{2kd^2}-kL\right)\right]^{-1}, \tag{17}$$

$$T(\lambda)=\left[\cos\left(\frac{\lambda L}{2kd^2}-kL\right)+\frac{i}{2}\left(\frac{k}{k_0}+\frac{k_0}{k}\right)\sin\left(\frac{\lambda L}{2kd^2}-kL\right)\right]^{-1} \tag{18}$$

are the partial reflection and transmission coefficients,

$$\varphi_\lambda(t)=\int e^{-it\xi}\varphi_\lambda(\xi)\,d\xi, \tag{19}$$

with the $\varphi_\lambda(\xi)$ normalized to a δ function of λ, while the integrals (15) and (16) are taken over the entire continuous spectrum.

If the problem (14) admits of a discrete spectrum on top of the continuous one, the appropriate sums must be added to the integrals (15) and (16).

Equation (14) reduces to the equation of Legendre functions. The continuous spectrum occupies the region $\lambda>0$. The even eigenfunctions (the odd functions make no contribution to the integrals (15) and (16)) are given by

$$\varphi_\lambda(\xi)=C_\lambda[P_\nu^{i\sqrt{\lambda}}(\operatorname{th}\xi)+P_\nu^{-i\sqrt{\lambda}}(-\operatorname{th}\xi)], \tag{20}$$

where $\nu(\nu+1)=\alpha(kd)^2$, and

$$|C_\lambda|^2=\frac{1}{8}\frac{\operatorname{sh}\pi\sqrt{\lambda}}{\operatorname{sh}^2\pi\sqrt{\lambda}+\sin^2\pi\nu}. \tag{21}$$

At $\alpha<0$ there exists also a discrete spectrum, and

$$\lambda_n=-(\nu-n)^2,\quad \varphi_n(\xi)=C_nP_\nu^{n-\nu}(\operatorname{th}\xi), \tag{22}$$

where n is an integer in the interval $0\leq n\leq\nu$, and

$$|C_n|^2=(\nu-n)\Gamma(1-n-2\nu)/n!. \tag{23}$$

The functions $\varphi_n(\xi)$ tend to zero as $\xi\to\pm\infty$, and correspond to electromagnetic waves propagating in the domain wall as if it were a waveguide:

$$F^{\pm}\sim\exp\left(\mp i\frac{\lambda_n z}{2kd^2}\right)\varphi_n\left(\frac{x}{d}\right). \tag{24}$$

To obtain the diffraction picture it remains now to calculate the integrals (19), (15), and (16). These integrals are so complicated, that we shall hereafter calculate only the amplitude of the transmitted light, and confine ourselves to an investigation of the simplest limiting cases.

We assume first that the permittivity ε_0 satisfies the condition

$$((\varepsilon_0^{1/2}-1)/(\varepsilon_0^{1/2}+1))^2\ll1. \tag{25}$$

This inequality is satisfied in known ferroelectrics with "thick" domain walls, namely in triglycine sulfate and sodium nitrate we have $\varepsilon_0^{1/2}\sim1.5$.[4,5]

Condition (25) allows us to disregard entirely, when the transmitted wave is considered, the reflections from the plate boundaries. When the reflected wave is considered, we need take only single reflection into account. As a result we get

$$T(\lambda)=\exp\left[-i\left(\frac{\lambda L}{2kd^2}-kL\right)\right], \tag{26}$$

$$R(\lambda)=-\frac{\varepsilon_0^{1/2}-1}{\varepsilon_0^{1/2}+1}\left\{\exp\left[-i\left(\frac{\lambda L}{2kd^2}-2kL\right)\right]-1\right\}. \tag{27}$$

The amplitude $A_2(q)$ can be expressed in terms of the Fourier representation of the Green's function of the problem (14):

$$A_2(q)=-\frac{e^{ikL}}{2\pi}\frac{1}{2\pi i}\int_C\exp\left(-i\frac{\lambda L}{2kd^2}\right)\bar G(qd,0;\lambda)\,d\lambda, \tag{28}$$

$$\bar G(t,0;\lambda)=\int\frac{\varphi_{\lambda'}(t)\varphi_{\lambda'}^*(0)}{\lambda-\lambda'}d\lambda'+\sum_n\frac{\varphi_n(t)\varphi_n^*(0)}{\lambda-\lambda_n}. \tag{29}$$

The integration contour in (28) follows the real axis and circles from above all the singularities of the integrand. Expression (29) for the Green's function contains an integral over the continuous spectrum and a sum over the discrete spectrum.

A second important condition assumed by us to be satisfied is that the investigated plate has a large thickness, i.e.,

$$L\gg kd^2. \tag{30}$$

For plates of thickness of the order of a millimeter this inequality seems to be satisfied with a large margin.

Expression (28) turns out to be exceedingly convenient in the limiting case (30). The Green's function (29) has singularities of two types. First, poles on the real axis, corresponding to the discrete spectrum. Second, a branch point $\lambda=0$ corresponding to the end point of the continuum. Third, a pole at the point $\lambda=(qd)^2$. By shifting the integration contour in (28) into the lower half-plane we obtain, first, the sum over the poles, and second, the integral along a contour drawn along the imaginary axis and bypassing the point $\lambda=0$. By virtue of the condition (30), the main contribution to the integral is made by the pole at the point $\lambda=0$.

3. CASE OF LONG WAVES $|\alpha|(kd)^2\ll1$

In this case we can put

$$\nu=-\alpha(kd)^2,\quad |\nu|\ll1. \tag{31}$$

The last inequality makes it possible to simplify considerably the expressions for the eigenfunctions. We use the expression for the Legendre function in terms of the hypergeometric function

$$P_\nu^\mu(x)=\frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2}F\left(-\nu,\nu+1;1-\mu;\frac{1-x}{2}\right) \tag{32}$$

([6], Vol. I) and note that in each term of the hypergeometric series we can put

$$\Gamma(-\nu+n)\Gamma(\nu+1+n)=\Gamma(n)\Gamma(n+1)[1+O(\nu)],$$
$$\Gamma(-\nu)\Gamma(\nu+1)=-\nu^{-1}[1+O(\nu)].$$

Then

$$P_\nu^{i\sqrt{\lambda}}(\operatorname{th}\xi)=\frac{e^{i\sqrt{\lambda}\xi}}{\Gamma(1-i\sqrt{\lambda})}\left[1-\frac{\nu e^{-2\xi}}{1-i\sqrt{\lambda}}\right.$$
$$\left.\times F(1,1-i\sqrt{\lambda};\ 2-i\sqrt{\lambda};\ -e^{-2\xi})\right]. \tag{33}$$

In the derivation of this expression we used an elementary transformation of the hypergeometric function contained in the square brackets ([6], Vol. 1). The calculation of $\bar{\varphi}_\lambda(t)$ reduces now to a Mellin transformation of the hypergeometric function ([6], Vol. I). As a result we have

$$\bar{\varphi}_\lambda(t)=\frac{C_\lambda}{\Gamma(1-i\sqrt{\lambda})}\left\{2\pi\delta(\sqrt{\lambda}+t)+2\pi\delta(\sqrt{\lambda}-t)\right.$$
$$-\frac{2\nu}{(\sqrt{\lambda}-t-i\delta)(\sqrt{\lambda}+t+i\delta)}\frac{\pi(\sqrt{\lambda}-t)/2}{\operatorname{sh}[\pi(\sqrt{\lambda}-t)/2]}$$
$$\left.-\frac{2\nu}{(\sqrt{\lambda}+t-i\delta)(\sqrt{\lambda}-t+i\delta)}\frac{\pi(\sqrt{\lambda}+t)/2}{\operatorname{sh}[\pi(\sqrt{\lambda}+t)/2]}\right\}. \tag{34}$$

Here δ is an infinitesimally small positive parameter that arises when the Fourier transformation is regularized.

At $\alpha<0$ there is only one discrete-spectrum function:

$$\varphi_0(\xi)=\nu^{1/2}\exp(-\nu|\xi|),\quad \bar{\varphi}_0(t)=2\nu^{3/2}/(\nu^2+t^2). \tag{35}$$

To calculate the contribution of the continuum to $A_2(q)$ we use formula (16). After simple transformations with (30) and (31) taken into account, we get

$$A_2^c(q)=e^{-ikL}\left[\delta(qd)-\frac{2\pi\nu}{\pi^2\nu^2+\operatorname{sh}^2(\pi qd)}\frac{\pi qd/2}{\operatorname{sh}(\pi qd/2)}\exp\left(i\frac{q^2L}{2k}\right)\right.$$
$$\left.-\frac{\nu_2}{2\pi qd\operatorname{sh}(\pi qd/2)}\int_{-\infty}^{+\infty}\exp\left(i\frac{Lu^2}{2kd^2}\right)\frac{qd}{u-qd}\frac{du}{u^2+\nu^2}\right]. \tag{36}$$

The remaining integral (which does not contain a dependence on q) can be expressed in terms of the probability integral.

The contribution of the discrete spectrum of $A_2(q)$ (at $\alpha<0$) is

$$A_2^d(q)=\frac{2}{\pi}\frac{\nu}{\nu^2+(qd)^2}\exp\left(i\frac{\nu^2L}{2kd^2}+ikL\right). \tag{37}$$

The first term in the square brackets of (36) describes the non-diffracted wave. The diffraction pattern comprises a single maximum of width $\Delta\vartheta\sim|\nu|/k_0d\sim|\alpha|kd$.

4. CASE OF SHORT WAVES, $|\alpha|(kd)^2\gg1$, $\alpha>0$

In this case, at $\lambda\lesssim1$, the functions $\varphi_\lambda(\xi)$ are exponentially small everywhere except in the region $|\xi|\gg1$. But in this region it is simpler not to start with the exact solution, but to consider the simplified equation

$$d^2\varphi/d\xi^2+(\lambda-4|\nu|^2e^{-2|\xi|})\varphi=0 \tag{38}$$

The spectrum is double degenerate, and the eigenfunctions are expressed in terms of Macdonald functions:

$$\varphi_{\lambda1}(\xi)=C_\lambda K_{i\sqrt{\lambda}}(2|\nu|e^{-\xi}),\quad \varphi_{\lambda2}(\xi)=C_\lambda K_{i\sqrt{\lambda}}(2|\nu|e^{\xi}), \tag{39}$$
$$|C_\lambda|^2=\operatorname{sh}(\pi\sqrt{\lambda})/\pi^2. \tag{40}$$

It follows therefore that

$$\bar{\varphi}_{\lambda1}(t)=\frac{C_\lambda}{4}\Gamma\left(i\frac{t-\sqrt{\lambda}}{2}+\delta\right)\Gamma\left(i\frac{t+\sqrt{\lambda}}{2}+\delta\right),$$
$$\bar{\varphi}_{\lambda2}(t)=\bar{\varphi}_{\lambda1}(-t) \tag{41}$$

([6], Vol. II). It is now easy to verify that the Green's function can be written in the form of a contour integral:

$$\bar{G}(t,0;\lambda)=\frac{1}{2}\int_\infty^{(0-)}\frac{\bar{\varphi}_{\lambda'1}(t)+\bar{\varphi}_{\lambda'2}(t)}{\lambda-\lambda'}\bar{\varphi}^*_{\lambda'1}(0)d\lambda'. \tag{42}$$

The singularities of the contour integral are due to the fact that as $\lambda\to0$ and $\lambda\to t^2$ the pole of the integrand at the point $\lambda'=\lambda$ and the branch point $\lambda'=0$ or the pole at the point $\lambda'=t^2-i\varepsilon$ straddle the integration contour. To separate the singular part of the Green's function it suffices to replace the integration contour in (42) by another that circles around not only the point $\lambda'=0$ but also the point $\lambda'=\lambda$, adding the corresponding residue to the integral. It is possible to separate similarly the singularities that arise as $\delta\to0$. The result is

$$\bar{G}(t,0;\lambda)=-i\frac{\operatorname{ch}(\pi\sqrt{\lambda}/2)}{4\sqrt{\lambda}}\left[\Gamma\left(i\frac{t-\sqrt{\lambda}}{2}\right)\Gamma\left(i\frac{t+\sqrt{\lambda}}{2}\right)|\nu|^{-it}\right.$$
$$\left.+\Gamma\left(-i\frac{t+\sqrt{\lambda}}{2}\right)\Gamma\left(-i\frac{t-\sqrt{\lambda}}{2}\right)|\nu|^{it}\right]$$
$$+\text{ regular function} \tag{43}$$

Using now expression (28) and taking the inequality (30) into account, we obtain

$$A_2(q)=(\pi qd\operatorname{sh}\pi qd)^{-1/2}\operatorname{ch}(\pi qd/2)\cos[qd\ln|\nu|$$
$$-\gamma(qd)]\exp(ikL-iq^2L/2k), \tag{44}$$

where $\gamma(x)=\arg\Gamma(ix)$.

Thus, the diffraction pattern assumes the form of interference fringes, the distance between which is determined from the condition $qd\ln|\nu|\sim\pi$. With increasing qd, the intensity of these fringes attenuates like $(qd)^{-1}$.

5. CASE OF SHORT WAVES, $|\alpha|(kd)^2\gg1$, $\alpha<0$

In this case the total amplitude of the diffracted wave is a sum of the contributions $A_2^c(q)$ and $A_2^d(q)$ of the continuous and discrete sections of the spectrum:

$$A_2(q)=A_2^c(q)+A_2^d(q). \tag{45}$$

We consider first the continuum. At large values of ν, the Legendre functions take the asymptotic form

$$P_\nu^\mu(\cos\vartheta)\approx\nu^\mu\left(\frac{2}{\pi\nu\sin\vartheta}\right)^{1/2}\cos\left[\left(\nu+\frac{1}{2}\right)\vartheta+\frac{\pi\mu}{2}-\frac{\pi}{4}\right] \tag{46}$$

([6], Vol. I) which is suitable in the region $|\nu\sin\vartheta|\gg 1$. When $\bar{\varphi}_\lambda(t)$ is calculated, the integration with respect to ϑ is from zero to π. We break up the cosine in (46) into a sum of two exponentials and shift the integration contour of one of them to the upper half plane and of the other to the lower half-plane of the complex variable ϑ. It is then obvious that the integrals over the sections where $\mathrm{Re}\,\vartheta\sim 1$ and $\mathrm{Re}(\pi-\vartheta)\sim 1$ make an exponentially small contribution. The only substantial integration regions are those where $|\nu\sin\vartheta|\lesssim 1$. For the region $|\vartheta|\ll \nu^{-1/2}$ there exists another asymptotic formula that expresses the Legendre functions in term of Bessel functions:

$$P_\nu^\mu(\cos\vartheta)\approx\nu^\mu J_{-\mu}(\nu\vartheta) \quad (47)$$

([6], Vol. II). In the region $|\pi-\vartheta|\ll\nu^{-1/2}$ it is easy to obtain an analogous formula by using the relation

$$P_\nu^\mu(-\cos\vartheta)=P_\nu^\mu(\cos\vartheta)\cos[\pi(\nu+\mu)]-\frac{2}{\pi}Q_\nu^\mu(\cos\vartheta)\sin[\pi(\nu+\mu)] \quad (48)$$

([6], Vol. I) and the asymptotic form of Legendre functions of the second kind, analogous to (46):

$$Q_\nu^\mu(\cos\vartheta)\approx-\frac{\pi}{2}\nu^\mu Y_{-\mu}(\nu\vartheta) \quad (49)$$

([6], Vol. II). Combining these relations, we obtain

$$P_\nu^\mu(\cos\vartheta)+P_\nu^\mu(-\cos\vartheta)=\frac{2\nu^\mu}{\sin\pi\mu}\cos\left(\pi\frac{\nu+\mu}{2}\right)\left[J_\mu(\nu\vartheta)\sin\left(\pi\frac{\nu+\mu}{2}\right)-J_{-\mu}(\nu\vartheta)\sin\left(\pi\frac{\nu-\mu}{2}\right)\right] \quad (50)$$

in the region $|\vartheta|\ll\nu^{-1/2}$. In the region $\nu^{-1}\ll|\vartheta|\ll\nu^{-1/2}$, the asymptotic forms (50) and (46) are joined together.

Calculation of $\bar{\varphi}_\lambda(t)$ with the aid of (49) yields

$$\bar{\varphi}_\lambda(t)=-\frac{C_\lambda\nu^{i\sqrt{\lambda}}}{\pi}\cos\left(\pi\frac{\nu+i\sqrt{\lambda}}{2}\right)\left[\nu^{-it}\sin\left(\pi\frac{\nu-it}{2}\right)\Gamma\left(i\frac{t+\sqrt{\lambda}}{2}+\delta\right)\times\Gamma\left(i\frac{t-\sqrt{\lambda}}{2}+\delta\right)+\nu^{it}\sin\left(\pi\frac{\nu+it}{2}\right)\Gamma\left(-i\frac{t+\sqrt{\lambda}}{2}+\delta\right)\Gamma\left(-i\frac{t-\sqrt{\lambda}}{2}+\delta\right)\right] \quad (51)$$

([6], Vol. II). The remaining calculations are made in the same manner as when $\alpha>0$. The contribution of the continuous spectrum to the amplitude is of the form

$$A_2^c(q)=\frac{2\sqrt{2}\sin(\pi\nu/2)}{(\pi qd\,\mathrm{sh}\,\pi qd)^{1/2}}\mathrm{ch}\frac{\pi qd}{2}\frac{(\mathrm{ch}\,\pi qd-\cos\pi\nu)^{1/2}}{\mathrm{ch}\,2\pi qd-\cos 2\pi\nu}\times\sin\left[qd\ln\nu+\mathrm{arctg}\left(\mathrm{th}\frac{\pi qd}{2}\mathrm{ctg}\frac{\pi\nu}{2}\right)-\gamma(qd)\right]\exp\left(ikL-i\frac{q^2L}{2k}\right). \quad (52)$$

Comparing this expression with (44), we can note that $A_2^c(q)$ can assume anomalously large values when $\pi qd\ll 1$ and $\cos(\pi\nu/2)\ll 1$. The physical reason for this is resonance with a discrete-spectrum level close to $\lambda=0$. In addition, the function $A_2^c(q)$ falls off exponentially with increasing q.

As to the contribution made to $A_2^d(q)$ by the discrete spectrum, it comes principally from values of n for which $\nu-n\ll\nu$. For these we can use the cited asymptotic expressions for the Legendre functions. Then

$$A_2^d(q)=\frac{1}{2\pi}\sum_{m=0}^{\infty}\exp\left[-i\frac{L}{2kd^2}(\nu_1+2m)^2-ikL\right]\times\left\{\frac{\Gamma[(\nu_1+iqd)/2+m]}{\Gamma[(\nu_1-iqd)/2+m+1]}\nu^{-iqd}+\frac{\Gamma[(\nu_1-iqd)/2+m]}{\Gamma[(\nu_1+iqd)/2+m+1]}\nu^{iqd}\right\}, \quad (53)$$

where $\nu_1=\nu-2[\nu/2]$. This rather complicated expression has two important properties. First, $A_2^d(q)$ is an oscillating function of q, with a period determined by the condition $qd\ln\nu\sim\pi$. Second, the function $A_2^d(q)$ falls off with increasing qd not exponentially but like $(qd)^{-1}$. It is of interest to note that this is precisely the fall-off typical of the amplitude of a wave diffracted by a slit in a flat screen.

6. DISCUSSION OF RESULTS

The presented calculation shows that the most informative, from the point of view of investigations of the structure of a domain wall, is the diffraction of short waves for which the condition $|\alpha|(kd)^2\gg 1$ is satisfied. Only in this case does the diffraction picture take the form of interference fringes. The distance between fringes depends both on the geometric dimensions of the wall and on the dielectric constant inside the wall. This means, in particular, that to investigate the wall structure it may not be enough to obtain the diffraction pattern for only one wavelength.

The character of the decay of the intensity of the interference fringence can tell us whether the domain wall has waveguide properties. The characteristic diffraction angle at which the intensity of the interference fringes is appreciable is determined by the condition $qd\sim 1$. Thus, the number of such interference fringes is of the order of $\ln|\alpha(kd)^2|$. This means that for a reliable observation of the diffraction pattern the necessary condition is not only $|\alpha(kd)^2|\gg 1$, but also a sufficiently large value of $\ln|\alpha(kd)^2|$.

It is obvious that the condition $|\alpha(kd)^2|\gg 1$ can be satisfied only in crystals with sufficiently thick domain walls. Among the ordinary ferroelectrics, such crystals seem to be triglycine sulfate and sodium nitrite, for which x-ray measurements yield $d\sim 5\times 10^{-5}$ cm.[1,2] We can therefore attain values $kd\sim 5$ for the violet edge of the spectrum. Only experiment can show whether these values are sufficient to obtain a distinct diffraction pattern. It is important, however, that the thickness of the domain wall can be measured also in the case $|\alpha(kd)^2|\ll 1$. The diffraction pattern then takes the form of a single maximum, and the entire information on the wall thickness is contained in the character of the intensity fall-off with increasing diffraction angle.

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