described by the Hamiltonian (1) is itself extremely idealized. In real layered semiconductors and semimetals, the probability of an electron transition from one layer to another differs from zero; this leads to the dependence of the energies of the electrons and holes on the quasimomentum components p_z perpendicular to the planes of the layers. If the transition probability is sufficiently small, this dependence is of the form $W\cos(p_z c)$, where W is proportional to the overlap integral of the wave functions of the electrons on neighboring layers. The consideration given above, which does not take into account the dependence of $\varepsilon_{e,h}$ on p_z, is therefore valid only so long as all the energies entering into the calculations significantly exceed the width W of transverse-motion band, i.e., $W \ll c^{-1/2}$. In the case of opposite sign of this inequality, the motion of the electrons is essentially three-dimensional and for the description of the EHL we must use the approach based on the strong anisotropy of the effective masses for motion in the layer (m_t) and perpendicular to it (m_l).[1] The role of the small parameter of the theory is played in this case by $m_l/m_t \ll 1$ instead of c.

[1)]The coefficient A_{II} of Ref. 1, which determines the correlation contribution to the total energy, differs by the factor $\frac{4}{5}$ from the A coefficient used in the present work in the self-energy parts of the electrons and holes. Moreover, in formulas (10) from Ref. 1, there are errors. The correct form of these formulas agrees with formulas (30) and (31) of this paper.

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Translated by R. T. Beyer

Phase transitions in the Potts three-component model

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(Submitted April 19, 1977)
Zh. Eksp. Teor. Fiz. **73**, 1174–1179 (September 1977)

It is shown by the renormalization-group method that the Potts three-component model has a second-order phase transition. The dependence of the critical exponent of the susceptibility on the dimensionality of space is determined.

PACS numbers: 05.70.Fh

A model introduced by Potts back in 1952[1] has recently become the subject of intensive study, with particular attention paid to the question of the order of the transition. The point is that the Hamiltonian of this model contains invariants of third order, and calculation within the framework of the self-consistent field[2] yields a first-order transition at a number $q \geq 3$ of the components of the order parameter of the model. On the other hand, an exact calculation of the two-dimensional q-component Potts model[3] yields at $q \leq 4$ a second-order phase transition.

We investigate here the three-component Potts model. We show that if the dimensionality d is less than the critical d_c, a second-order phase transition takes place. The obtained critical dimensionality $d_c \approx 4.58$ greatly exceeds 3, and in all probability a more exact calculation will not lead to $d_c < 3$.

The Potts model is a generalization of the Ising model, in which each site can be in one of the q states, the interaction energy assuming only two values, depending on whether the nearest sites are in identical or different states. The Potts model with various q describes different real systems: electric circuits at $q=0$,[3] percolation at $q=1$,[4] and nematic liquid crystals[5] or crystals with cubic symmetry[6–8] at $q=3$. Of course, at $q=2$ we obtain the usual Ising model.

In the q-component Potts model, a phase transition is described by a tensor order parameter Q_{ij}.[4] The tensor Q_{ij} has no trace, is symmetric, and has the dimensionality q. In particular, in the phase transition from a nematic liquid crystal into an isotropic liquid, the role of the order parameter can be played by the anisotropic part of the dielectric tensor.[5] The Hamiltonian of the model is of the form

$$H = AQ^2 + BQ^3 + CQ^4, \tag{1}$$

where Q^2, Q^3, and Q^4 denote respectively invariants of second, third and fourth order in Q.

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We consider a continuous variant of the three-component Potts model. By a change of variables, we can reduce the Hamiltonian (1) in this case to the form[9]

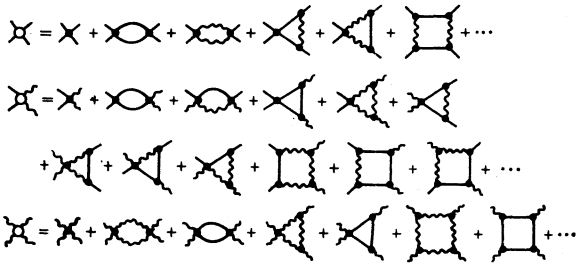
$$H=\int d^dk\Big\{{}^1/_2(\tau+k^2)[\varphi_1^2(k)+\varphi_2^2(k)]+[u_1\varphi_1(k_1)\varphi_1(k_2)\varphi_1(k_3)\varphi_1(k_4)$$
$$+u_2\varphi_1(k_1)\varphi_1(k_2)\varphi_2(k_3)\varphi_2(k_4)+u_3\varphi_2(k_1)\varphi_2(k_2)\varphi_2(k_3)\varphi_2(k_4)]\delta\Big(\sum_{i=1}^{4}k_i\Big)$$
$$+[\lambda_1\varphi_1(k_1)\varphi_1(k_2)\varphi_2(k_3)+\lambda_2\varphi_2(k_1)\varphi_2(k_2)\varphi_2(k_3)]\delta\Big(\sum_{i=1}^{3}k_i\Big)\Big\}, \qquad (2)$$

where $\tau=(T-T_c)/T_c$ is the deviation from the critical temperature, k is the wave vector, and d is the dimensionality. The coupling constants are subject to the following relations:

$$u_1=u_3=u_2/2\equiv u,\ \lambda_1=-3\lambda_2\equiv\lambda. \qquad (3)$$

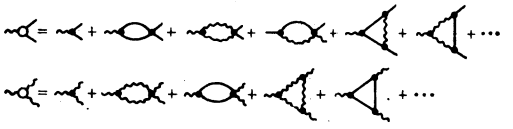
Relations (3) stem from the fact that from three-dimensional zero-trace tensors it is possible to make up only one third-order invariant, $\mathrm{Tr}\,Q^3$, and one fourth-order invariant $\mathrm{Tr}\,Q^4=\frac{1}{2}(\mathrm{Tr}\,Q^2)^2$.[10]

The equations for the total quadruple and triple vertices are of the form



(4a)

and



(4b)

In expressions (4), the vertices u_1, u_2, u_3, λ_1, and λ_2 are respectively designated as follows:

(5)

The straight and wavy lines represent the complete Green's functions of the fields φ_1 and φ_2.

As will be shown below, at $d=4$ there exist non-Gaussian fixed points (FP), so that the ε-expansion ($\varepsilon=4-d$) cannot be used. We therefore use a method proposed by S. Ginzburg.[11] Neglecting the critical exponent η, which characterizes the deviation of the correlation functions from the Ornstein-Zernike form, we differentiate the equations in (4) with respect to the square of the reciprocal correlation radius r_c^{-2}. Taking into account the relations (3) between the coupling constants, which remain in force also after the renormalization, we obtain equations for the complete quadruple (u) and triple ($\tilde{\lambda}$) vertices (the last two graphs in Eqs. (4b) cancel each other identically):

$$d\tilde{u}/dr_c^{-2}=80ar_c^{6-d}\tilde{u}^2-48br_c^{8-d}\tilde{u}\tilde{\lambda}^2+{}^8/_3cr_c^{10-d}\tilde{\lambda}^4,$$
$$d\tilde{\lambda}/dr_c^{-2}=48ar_c^{6-d}\tilde{u}\tilde{\lambda}. \qquad (6)$$

The constants a, b, and c depend on the dimensionality d[12]:

$$a=\frac{\Gamma(3-d/2)}{2^{d+1}\pi^{d/2}},\quad b=\frac{\Gamma(4-d/2)}{3\cdot2^{d+1}\pi^{d/2}},\quad c=\frac{\Gamma(5-d/2)}{3\cdot2^{d+3}\pi^{d/2}}.$$

Making the change of variables

$$x=ar_c^{4-d}\tilde{u},\quad y=ar_c^{6-d}\tilde{\lambda}^2,\quad t=\ln r_c^{-2}, \qquad (7)$$

we obtain

$$dx/dt=\psi_1(x,y),\ dy/dt=\psi_2(x,y), \qquad (8)$$

where

$$\psi_1(x,y)={}^1/_2(d-4)x+80x^2-8(6-d)xy+{}^1/_{18}(8-d)(6-d)y^2,$$
$$\psi_2(x,y)={}^1/_2(d-6)y+96xy. \qquad (9)$$

The series for $\psi_i(x,y)$, as all other power-law series of field theory, are asymptotic. In the model with the φ^4 interaction, the best approximation is that of $\psi(x)$ by the first two terms of its expansion.[13] We have therefore confined ourselves in (9) to the terms quadratic in x and y. That the choice of this approximation is reasonable is confirmed also by the fact that we obtain for the critical exponent of the susceptibility values (see expression (15)) that are close to those obtained by other methods.

Solving the system of equations $\psi_1=\psi_2=0$, we obtain four FP:

$$x_1=0,\quad y_1=0, \qquad (10a)$$

$$x_2=\frac{4-d}{160},\quad y_2=0, \qquad (10b)$$

$$x_3=\frac{6-d}{192},\quad y_3=\frac{6(6-d)+[2(19d^2-218d+600)]^{1/2}}{16(8-d)}, \qquad (10c)$$

$$x_4=\frac{6-d}{192},\quad y_4=\frac{6(6-d)-[2(19d^2-218d+600)]^{1/2}}{16(8-d)}. \qquad (10d)$$

The first two points were obtained earlier in calculations within the framework of the ε-expansion.[14-16] The last two points were not obtained in the cited papers, since the renormalized vertices are not small as $d\to4$. The only stable FP among these obtained is the fourth one at $d<d_c$, where

$$d_c=(109-\sqrt{481})/19\approx4.58.$$

It is known that the free energy can be represented as a series in powers of the quantity $M=\langle\varphi\rangle$. The averaging is over the distribution function with the total Hamiltonian of the system. The coefficients of the series are in this case the renormalized vertices. In particular, the terms quadratic and cubic in M take the forms $\chi^{-1}M^2$ and $\tilde{\lambda}M^3$, where χ is the susceptibility of the system. To ascertain whether the triple vertices influence the thermodynamics, let us compare these two terms. Recognizing that $M\sim\chi^{-(d-2+\eta)/2(2-\eta)}$ and defining the dimensionality of the triple vertex by the expression $\tilde{\lambda}\sim\chi^{-\Delta_3/(2-\eta)}$, we get

$$\tilde{\lambda}M^3/\chi^{-1}M^2\sim\chi^{[3(2-\eta)-2\Delta_3-d]/2(2-\eta)}=\chi^x.$$

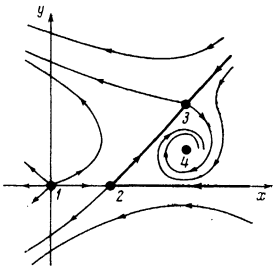


FIG. 1.

In accord with the work of Vigman, Larkin and Filev,[10] the triple vertices do not influence the thermodynamics at $z<0$, and the system undergoes a first-order phase transition at $z>0$. If $z=0$, the free-energy terms quadratic and cubic in M are of the same order, and the thermodynamic quantities depend on $\tilde{\lambda}$. However, allowance for $\tilde{\lambda}$ does not change the kind of the phase transition. It is easy to verify that for our stable FP the condition $z=0$ is satisfied, so that this point describes a second-order phase transition.

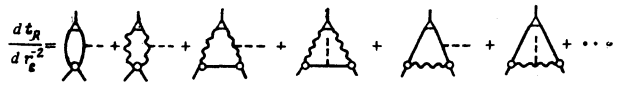
The phase plane of the system (8) for $d<4$ is shown in Fig. 1. The arrows on the trajectories show the directions of the motion as r_c is increased, i.e., τ is decreased. The thick lines are the separatrices that delimit the stability region of the fourth FP. The numbers at the FP correspond to those in (10). It is easily seen that only one quadrant of the phase plane need be considered, inasmuch as $y>0$ by definition (7), and the condition $x>0$ ($u>0$) ensures stability of the system. The trajectories that go outside the limits of this quadrant corresponds to a system that undergoes a first-order phase transition.

At $d=4$, the first and second FP ((10a) and (10b)) coincide. In this case the region of stability of the fourth point includes arbitrarily small values of the fourth quadruple constant. At $d\neq4$, even in the case of a small constant of the triple interaction, a second order transition takes place only if the value of the quadruple constant x exceeds a certain x_c. At $d<4$, x_c coincides with the value of the second FP (see the figure). Golner[17] investigated the three-component Potts constant at $d=3$ with the aid of Wilson's[18] approximate recurrence relations and found that the phase transition is of first order. This may be due to the choice of too small a bare quadruple constant ($u_0=0.1$). Our conclusion is confirmed by the results of Burkhardt *et al.*[19] who used Kadanoff's[20] renormalization transformations to obtain a stable FP at $d=2$ and $d=3$.

To calculate the critical exponent γ of the susceptibility, we consider, following Ginzburg,[11] the vertex t_R defined by the expression

$$dr_c^{-2}/d\tau=t_R. \tag{11}$$

The equation for the derivative dt_R/dr_c is of the form



(12)

where the triangle stands for t_R and the dashed line for differentiation of the G function. As a result we get

$$d\ln t_R/dt=\zeta(x,y),\quad \zeta(x,y)=32x-4(6-d)y. \tag{13}$$

The exponent γ is determined by the expression

$$\gamma^{-1}=1-\zeta(x_4,y_4), \tag{14}$$

from which we obtain, in particular

$$\gamma=1.47 \text{ at } d=2,\quad \gamma=1.10 \text{ at } d=3. \tag{15}$$

These values agree with those obtained either from a numerical solution of the renormalization-group equations[19] ($\gamma=1.451$ at $d=2$ and $\gamma=0.9761$ at $d=3$) or with the aid of power-law expansions ($\gamma=1.5\pm0.2$,[9] 1.45 ± 0.15,[21] 1.42 ± 0.05[22] at $d=2$; $\gamma=1.0\pm0.15$[23] at $d=3$).

Experimental investigations[6,24] of systems describable by the Potts model reveal a first order transitions, with the discontinuities of the thermodynamic quantities smaller than those predicted by the self-consistent-field theory. These results can be understood by assuming that the bare values of λ and u are not too large. Then the distance to the stability limit $x=0$ along the trajectory (see the figure) exceeds the shortest distance appreciably. Since a longer path corresponds to a larger r_c (and accordingly to a smaller τ), the phase transition will in fact take place at a value of τ smaller than in accord with self-consistent-field theory, and the discontinuities of the thermodynamic quantities will be correspondingly smaller.

The author thanks E. E. Gorodetskiĭ for useful discussions.

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Translated by J. G. Adashko