Perturbation theory for solitons

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We consider the evolution equations for non-linear waves which differ by small terms from equations soluble by the inverse scattering method (Korteweg–de Vries equation, non-linear Schrödinger equation, and so on). We formulate a perturbation theory scheme which is based upon the inverse scattering method. Using this scheme we give an analytical description of the evolution of a soliton, including the deformation of its shape and tail formation as the effect of the perturbation in first approximation. We show that the found solution satisfies the whole infinite set of polynomial conservation laws, modified for the perturbation case.

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1. INTRODUCTION

The present paper is devoted to the problem of perturbation theory for non-linear waves described by evolution equations of the form

$$u_t=S[u]+\varepsilon R[u], \tag{1.1}$$

where S and R are non-linear operators acting on $u(x, t)$, ε is a small parameter, and Eq. (1.1) at $\varepsilon=0$ (unperturbed equation) can be solved by the inverse scattering method.[1–3] 1) This means, first of all, that the unperturbed equation can be put in the operator form[2]

$$i\partial\hat{L}/\partial t+[\hat{L}, \hat{A}]=0, \tag{1.2}$$

where $\hat{L}(u)$ and $\hat{A}(u)$ are linear operators depending on $u(x, t)$ and acting in a ψ-function space. The Korteweg-de Vries (KdV) equation

$$u_t-6uu_x+u_{xxx}=0, \tag{1.3}$$

belongs, for instance, to this class of equations.[1,2] In that case

$$\hat{L}(u)=-\partial^2/\partial x^2+u, \tag{1.4}$$

$$\hat{A}(u)=-4i\partial^3/\partial x^3+3i(\partial/\partial x)u+3iu\partial/\partial x. \tag{1.5}$$

The operators (1.4), (1.5) are Hermitean (u is real). This condition is, however, not obligatory. For instance, in the case of the non-linear Schrödinger equation (NLS) which describes the one-dimensional self-modulation or self-focusing of a plane-parallel beam,

$$iu_t+{}^1/_2u_{xx}+|u|^2u=0 \tag{1.6}$$

and of the modified Korteweg–de Vries (MKdV) equation, which plays an important role in plasma physics,

$$u_t+6|u|^2u_x+u_{xxx}=0 \tag{1.7}$$

(the operator form was found, respectively, in Refs. 3, 4) the operators $\hat{L}$ and $\hat{A}$ are non-Hermitean and matrix operators. They can be written in the form

$$\hat{L}=iP\partial/\partial x+Q(u), \quad \hat{A}=M\hat{D}+\hat{C}(u), \tag{1.8}$$

where

$$P=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad Q=\begin{pmatrix}0 & u^*\\ -u & 0\end{pmatrix}, \quad M=\begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}. \tag{1.9}$$

In the case (1.6)

$$m_1=-m_2=1, \quad \hat{D}=\partial^2/\partial x^2; \tag{1.10}$$

for (1.7)

$$m_1=m_2=4, \quad \hat{D}=-i\partial^3/\partial x^3. \tag{1.11}$$

For the sake of simplicity we do not give the form of the matrix operator $\hat{C}(u)$ which in both cases vanishes with the potential $u(x, t)$ as $|x|\to\infty$, since we do not need it.2) We shall not discuss here other examples of evolution equations which can be reduced to the form (1.2) (at the present time there are rather many of those known), referring to the paper by Zakharov and Shabat,[5] where a general study of this problem is made.

The perturbation theory developed in the present paper is, like the solution of the equations which are reduced to the form (1.2), based on the inverse scattering method. It turns out that for Eq. (1.1) with $\varepsilon\neq 0$ we can also introduce Jost functions and coefficients a and b which couple them. The wave field is established in terms of these coefficients and other parameters of the scattering matrix by means of a linear integral equation which has the same form as the equation obtained when we neglect the perturbation. The difference is that the perturbation completely changes the time dependence of the parameters of the scattering matrix. One of the basic results of the present paper consists in finding for the elements of the scattering matrix equations that determine their time dependence for a given form of $R[u]$ (Sec. 2). These equations can then be solved by the "adiabatic perturbation theory" method which is developed in the present paper using the example of the case when at the initial time the wave field consists of a single soliton. In the "adiabatic" approximation considered in Sec. 3 we obtain general expressions describing the change in the soliton parameters caused by the perturbation $\varepsilon R[u]$. In the adiabatic approximation we neglect the distortion of the shape of the soliton and tail forma-

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tion. These effects are described in the next, first, approximation (Secs. 4 and 5). In Secs. 4 and 5 we develop a method for finding "averaged" coefficients a and b in the first approximation (in ε) which, after substitution in the Gel'fand–Levitan equation and solution of the latter, lead to the general picture of "tail" formation and distortion of the soliton shape as the effect of the perturbation. It is interesting that in the asymptotic limit these processes can be completely described analytically for quite arbitrary $R[u]$, without having to invoke numerical calculations. Finally, in Sec. 6 we show that the results obtained by us are in complete agreement with the whole set of conservation laws, taking perturbations into account. The averaged "tails" which were obtained in the framework of our method are thus completely equivalent to real tails from the point of view of their contribution to the whole set of polynomial conservation laws.

In conclusion we note that the method can be generalized also to many-soliton states. To avoid unnecessary complications we expound our theory using as examples the three above-mentioned equations: KdV, NLS, and MKdV. It is, however, clear from what follows that this does not restrict the general nature of the method.

2. BASIC EQUATIONS

If we write Eq. (1.1) for $\varepsilon=0$ in the form (1.2), we can write it for $\varepsilon\neq 0$ in the form

$$i\partial\hat{L}/\partial t+[\hat{L},\hat{A}]=i\varepsilon\hat{R}, \tag{2.1}$$

where $\hat{R}$ is an operator acting in the ψ-function space. In the case of the KdV equation it is simply equal to the operator of multiplying by $R[u]$. For the NSE and MKdV, we have, according to (1.8) to (1.11)

$$\hat{R}=\begin{pmatrix}0 & R^*[u]\\ -R[u] & 0\end{pmatrix}. \tag{2.2}$$

We now consider the problem of the eigenvalues of the operator $\hat{L}$ which satisfies Eq. (2.1):

$$\hat{L}\psi(x,t)=\lambda(t)\psi(x,t). \tag{2.3}$$

We assume that there exists also a solution of the equation $\hat{L}^+\tilde{\psi}=\lambda^*\tilde{\psi}$, where $\hat{L}^+$ is the Hermitean conjugate operator with respect to the scalar product $(\psi,\varphi)=\int_{-\infty}^{\infty}\psi^*\varphi\,dx$ (in the matrix cases $\psi\varphi=\sum_r\psi_r\varphi_r$). We now differentiate (2.3) with respect to the time. Using (2.1) we get

$$(\hat{L}-\lambda)(\psi_t+i\hat{A}\psi)=-\varepsilon\hat{R}\psi+\lambda_t\psi. \tag{2.4}$$

For the continuous spectrum $\lambda_t=0$. For the discrete spectrum we take the scalar product of (2.4) with $\tilde{\psi}$ (from the left). In that case

$$d\lambda/dt=\varepsilon(\tilde{\psi},\hat{R}[u]\psi)/(\tilde{\psi},\psi). \tag{2.5}$$

(We assume that $(\tilde{\psi},\psi)\neq 0$.) In particular, we get $\lambda_t=0$ when $\varepsilon=0$—a well known result which occurs when the evolution equation can be written in the form (1.2). In the latter case the eigenvalues of the discrete spectrum determine the amplitudes of the solitons which are formed from the initial wave pulse.

Equation (2.5) gives only part of the information about the solution. It can be completed by other equations which in principle allow us to reconstruct $u(x,t)$ to any approximation in the framework of the inverse scattering method. We consider these equations firstly using the example of the perturbed KdV equation.

We introduce the Jost functions corresponding to Eq. (2.3), assuming that $\hat{L}$ has the form (1.4). In that case (2.3) is the Schrödinger equation. A positive $\lambda=k^2$ corresponds to the continuous spectrum where each eigenvalue is two-fold degenerate. The Jost functions in that case are the eigenfunctions f and g of the operator $\hat{L}$ which have the asymptotic form

$$f(x,k)\to e^{ikx},\ x\to\infty;\ g(x,k)\to e^{-ikx},\ x\to-\infty. \tag{2.6}$$

We list the basic properties of the Jost functions, which are important for what follows (for proofs and details see Ref. 6). We can write the function f in the form[3)]

$$f(x,k)=e^{ikx}+\int_x^{\infty}K(x,y)e^{iky}dy, \tag{2.7}$$

where K is some real kernel, while

$$-2\frac{dK(x,x)}{dx}=u(x). \tag{2.8}$$

Equation (2.3) has, apart from f and g, the eigenfunctions $f^*(x,k)=f(x,-k)$ and $g^*(x,k)=g(x,-k)$ (it is important here that k is real) and if $k\neq 0$ then f^* and g^* are linearly independent of f and g, respectively. In that case

$$g(x,k)=a(k)f^*(x,k)+b(k)f(x,k), \tag{2.9}$$

$$f(x,k)=a(k)g^*(x,k)-b^*(k)g(x,k);$$
$$|a(k)|^2=1+|b(k)|^2,\quad a^*(k)=a(-k),\quad b^*(k)=b(-k), \tag{2.10}$$

where $a(k)$ and $b(k)$ are coefficients which play a fundamental role in what follows. We note finally that

$$a(k)=\frac{i}{2k}\left[f(x,k)\frac{\partial g(x,k)}{\partial x}-g(x,k)\frac{\partial f(x,k)}{\partial x}\right]. \tag{2.11}$$

So far we have considered the functions f and g for real k. We can analytically continue them into the upper half-plane where they have no singularities while

$$f(x,k)\to e^{ikx},\quad g(x,k)\to e^{-ikx}\quad(|k|\to\infty,\ \mathrm{Im}\,k\geqslant 0). \tag{2.12}$$

It then follows from Eq. (2.11) that $a(k)$ is an analytical function in the upper half-plane. In that case

$$a(k)\to 1\quad(|k|\to\infty,\ \mathrm{Im}\,k\geqslant 0), \tag{2.13}$$

so that $a(k)$ can have only a finite number of zeroes for $\mathrm{Im}k>0$. (There are no zeroes for $\mathrm{Im}k=0$, as is clear from (2.10).) The zeroes of $a(k)$ are on the imaginary axis and they correspond to the eigenvalues of the discrete spectrum $\lambda_r = k_r^2 > 0$. We shall denote them in what follows by $k_r = i\varkappa_r$ ($\varkappa_r > 0$, $r = 1, 2, \ldots, n$). It follows from (2.11) that for $k = i\varkappa_r$ the functions f and g become linearly dependent (the quantity in the square brackets in (2.11) is the Wronskian), i.e.,

$$g(x, k_r) = \rho_r f(x, k_r), \tag{2.14}$$

where the ρ_r are constant factors while

$$a_r' \equiv \frac{da}{dk}\bigg|_{k=k_r} = -i\int_{-\infty}^{\infty} f_r(x) g_r(x)\,dx = -i\rho_r \int_{-\infty}^{\infty} f_r^2(x)\,dx, \tag{2.15}$$

where $f_r(x) \equiv f(x, k_r)$. It follows from this last relation, in particular, that $a_r' \neq 0$, i.e., all zeroes of $a(k)$ are simple ones.

It is shown in scattering theory that the coefficients of the scattering matrix $a(k)$ and $b(k)$ for real k, and the quantities characterizing the discrete spectrum, i.e., $\lambda_r = -\varkappa_r^2$, a_r', ρ_r, uniquely determine the kernel $K(x, y)$ and, hence, the potential $u(x)$ from Eq. (2.8). In fact, the kernel K satisfies the Gel'fand–Levitan equation[7,8,1]:

$$K(x, y; t) + F(x+y; t) + \int_x^{\infty} K(x, y'; t) F(y+y'; t)\,dy' = 0, \tag{2.16}$$

where $y > x$, and

$$F(x; t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(k,t)}{a(k,t)} \exp(ikx)\,dk - i\sum_{r=1}^{n} \frac{\rho_r(t)}{a_r'(t)} \exp[-\varkappa_r(t) x]. \tag{2.17}$$

In the cases of the other $\hat{L}$ and $\hat{A}$ operators, the inverse scattering technique is in principle similar, but the non-Hermiticity and matrix nature of the latter may lead to some peculiarities which must be borne in mind. As typical examples which have an important physical value we consider the NLS and MKdV. As before, the basic equations are here (2.3) and (2.4), where ψ is a two-dimensional eigenvector. However, the eigenvalues λ are no longer necessarily real because $\hat{L}$ is non-Hermitean. For each eigenvector $\psi = (\psi_1, \psi_2)$ of Eq. (2.3) we can define two vectors

$$\tilde{\psi} = (\psi_2^*, -\psi_1^*), \quad \bar{\psi} = (\psi_2^*, \psi_1^*), \tag{2.18}$$

satisfying the equations

$$\hat{L}\tilde{\psi} = \lambda^*\tilde{\psi}, \quad \hat{L}^+\bar{\psi} = \lambda^*\bar{\psi}, \tag{2.19}$$

as one can easily check, using (1.8) and (1.9). It is also important that real λ belong to the continuous spectrum, and complex ones to the discrete one. The Jost functions are for real λ defined as follows[3,4]:

$$f(x, \lambda) = (f_1, f_2) \to (0, 1)e^{i\lambda x}, \quad x \to \infty; \tag{2.20}$$

$$g(x, \lambda) = (g_1, g_2) \to (1, 0)e^{-i\lambda x}, \quad x \to -\infty, \tag{2.21}$$

while, if $\mathrm{Im}\lambda = 0$, we have

$$g(x, \lambda) = a(\lambda)\tilde{f}(x, \lambda) + b(\lambda) f(x, \lambda), \tag{2.22}$$

$$f(x, \lambda) = -a(\lambda)\tilde{g}(x, \lambda) + b^*(\lambda) g(x, \lambda),$$

$$|a(\lambda)|^2 + |b(\lambda)|^2 = 1, \tag{2.23}$$

$$a(\lambda) = g_1(x, \lambda) f_2(x, \lambda) - g_2(x, \lambda) f_1(x, \lambda). \tag{2.24}$$

The latter relations differ slightly from the corresponding formulae for the Schrödinger equation and this must be borne in mind in what follows.

It was shown in Refs. 3 and 4 that $f(x, \lambda)$ and $g(x, \lambda)$ can be analytically continued in the upper λ half-plane where they have no singularities. The eigenfunctions of the discrete spectrum in the upper half-plane are the analytical continuations of $f(x, \lambda)$ into the upper λ half-plane, while the eigenvalues $\lambda = \zeta_r$ are the roots of the function $a(\lambda)$ which can also be continued into the upper half-plane by using (2.24). If $a(\zeta_r) = 0$ ($r = 1, 2, \ldots$) we then have

$$g(x, \zeta_r) = \rho_r f(x, \zeta_r), \quad a_r' \equiv \frac{da}{d\lambda}\bigg|_{\lambda=\zeta_r} = -i\rho_r \int_{-\infty}^{\infty} \tilde{f}^*(x, \zeta_r) f(x, \zeta_r)\,dx. \tag{2.25}$$

Similarly, one can obtain the eigenfunctions of the discrete spectrum for $\mathrm{Im}\lambda < 0$ by analytically continuing $\tilde{f}(x, \lambda)$ into the lower half-plane, while the eigenvalues are the roots of the function which is the analytical continuation of $a^*(\lambda)$ into the lower half-plane.

We can write the vector function $f(x, \lambda)$ in the form

$$f(x, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda x} + \int_x^{\infty} \begin{pmatrix} K_1(x, y) \\ K_2(x, y) \end{pmatrix} e^{i\lambda y}\,dy, \tag{2.26}$$

while

$$u(x) = -2iK_1^*(x, x). \tag{2.27}$$

The analogue of the Gel'fand–Levitan equation now has the form ($y > x$)[3,4]

$$K_1(x, y; t) = F^*(x+y; t) - \int_x^{\infty} K_1(x, y''; t) \int_x^{\infty} F^*(y+y'; t) F(y'+y''; t)\,dy'\,dy'', \tag{2.28}$$

where

$$F(x; t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\lambda, t)}{a(\lambda, t)} \exp(i\lambda x)\,d\lambda - i\sum_r \frac{\rho_r(t)}{a_r'(t)} \exp(i\zeta_r(t) x). \tag{2.29}$$

From the relations given here it is clear that if the ratio of the Jost coefficients b/a as function of the time and also $\zeta_r(t)$ and the functions $a_r'(t)$ and $\rho_r(t)$ are known, the solution of the Gel'fand–Levitan equations allow us in principle to reconstruct the evolution of the wave.

In the case of the KdV equation one can start from (2.3) and (2.4) and use the analytical properties of the Jost functions to show that the following equations hold for the parameters which occur in the Gel'fand–Levitan Eq. (2.16)[9]:

$$\frac{\partial a(k,t)}{\partial t} = \frac{i\varepsilon}{2k}[a(k,t)\alpha(k,k;t) + b(k,t)\alpha(-k,k;t)], \tag{2.30}$$

$$\frac{\partial b(k,t)}{\partial t} = 8ik^3 b(k,t) - \frac{i\varepsilon}{2k}[a(k,t)\alpha(k,-k;t) + b(k,t)\alpha(k,k;t)], \tag{2.31}$$

$$\frac{d\rho_r}{dt} = 8\varkappa_r^3\rho_r + \frac{\varepsilon\rho_r}{2\varkappa_r a_r'}\frac{d}{dk}[\rho_r\alpha(-k, i\varkappa_r) - \beta(-k, i\varkappa_r)]_{k=i\varkappa_r}, \tag{2.32}$$

where, we remind ourselves, the $\varkappa_r$ are the eigenvalues of the discrete spectrum,

$$\alpha(k,k';t)=\int_{-\infty}^{\infty} f^*(x,k;t)R[u(x,t)]f(x,k';t)\,dx, \qquad (2.33)$$

$$\beta(k,k';t)=\int_{-\infty}^{\infty} g^*(x,k;t)R[u(x,t)]f(x,k';t)\,dx. \qquad (2.34)$$

The complete derivation of Eqs. (2.30) to (2.34) is given in Ref. 9b. The method of the derivation is rather general and can be applied also to other forms of Eq. (1.1) which can be solved in the framework of the inverse scattering method when there are no perturbations. In particular, it was shown in Ref. 9b that the following equations which are similar to (2.30) to (2.34) are true for NLS and MKdV:

$$\partial a(\lambda,t)/\partial t=i\varepsilon[a(\lambda,t)\bar{\alpha}(\lambda,\lambda;t)+b(\lambda,t)\alpha(\lambda,\lambda;t)], \qquad (2.35)$$

$$\partial b(\lambda,t)/\partial t=ih(\lambda)b(\lambda,t)+i\varepsilon[a(\lambda,t)\alpha^*(\lambda,\lambda;t)-b(\lambda,t)\bar{\alpha}(\lambda,\lambda;t)], \qquad (2.36)$$

where

$$\alpha(\lambda,\lambda';t)=\int_{-\infty}^{\infty} f^*(x,\lambda;t)\hat{R}[u(x,t)]f(x,\lambda';t)\,dx, \qquad (2.37)$$

$$\bar{\alpha}(\lambda,\lambda';t)=\int_{-\infty}^{\infty} \bar{f}^*(x,\lambda;t)\hat{R}[u(x,t)]f(x,\lambda';t)\,dx \qquad (2.38)$$

and

$$h(\lambda)=-2\lambda^2, \quad h(\lambda)=8\lambda^3 \qquad (2.39)$$

respectively for NLS and MKdV. The analogue of Eq. (2.32) has the form

$$\frac{d\rho_r}{dt}=ih(\zeta_r)\rho_r+\frac{i\varepsilon\rho_r}{a_r'}\frac{d}{d\lambda}[\rho_r\alpha(\lambda,\zeta_r)-\beta(\lambda,\zeta_r)]_{\lambda=\zeta_r},$$
$$\beta(\lambda,\lambda';t)=\int_{-\infty}^{\infty} \bar{g}^*(x,\lambda;t)\hat{R}[u(x,t)]f(x,\lambda';t)\,dx. \qquad (2.40)$$

We draw attention to the fact that Eqs. (2.30), (2.31) and (2.35), (2.36) are the same as the unitarity conditions (2.10) and (2.23).

Equations (2.5), like (2.30) to (2.32) or, respectively, (2.35), (2.36), and (2.40), are exact and they are the basis of the perturbation theory developed here. In principle they allow us to find the soliton amplitudes and the Jost coefficients $a(\lambda, t)$ and $b(\lambda, t)$ to a given approximation in the matrix elements calculated to a lower approximation. Afterwards through a and b one calculates the kernels occurring in the Gel'fand–Levitan equation, and after solving that equation one evaluates the wave field $u(x, t)$. We demonstrate the general nature and the efficiency of this method by two approximations of the perturbation theory which we shall call the adiabatic and the first approximation. In the present paper these approximations will be applied to a study of the evolution of a single soliton acted upon by the perturbation.

3. ADIABATIC APPROXIMATION

We start for the sake of simplicity with the perturbed KdV equation. Let there be at some time t a wave pulse which has the shape of a soliton

$$u=u_s(x,t)=-2\varkappa^2\,\mathrm{sech}^2 z, \quad z=\varkappa(t)[x-\xi(t)]. \qquad (3.1)$$

When $\varkappa=\text{const}$, $\xi=4\varkappa^2t+\xi_0$, (3.1) describes a soliton which satisfies the KdV Eq. (1.3). In the adiabatic approximation considered here we substitute (3.1) and the Jost functions corresponding to it with the so far undetermined parameters $\varkappa=\varkappa(t)$ and $\xi=\xi(t)$ into the matrix elements (2.33) and (2.34). We can then obtain from (2.5) and (2.32) several equations for $\varkappa(t)$ and $\xi(t)$. To do this we first of all solve Eq. (2.3) for $\hat{L}$ in the form (1.4) and $u=u_s(x, t)$ and find the Jost functions

$$f(x,k)=e^{ikx}\{k+i\varkappa\,\mathrm{th}[\varkappa(x-\xi)]\}(k+i\varkappa)^{-1},$$
$$g(x,k)=e^{-ikx}\{k-i\varkappa\,\mathrm{th}[\varkappa(x-\xi)]\}(k+i\varkappa)^{-1}. \qquad (3.2)$$

Hence we find

$$a(k)=(k-i\varkappa)(k+i\varkappa)^{-1}, \quad b(k)=0 \qquad (3.3)$$

(the last equation corresponds to the fact that the potential (3.1) belongs to the class of "non-reflecting" potentials). The discrete spectrum consists here of a single eigenvalue which is the same as $-\varkappa^2$ ($\varkappa_r=\varkappa$) and satisfies the equation $a(k)=0$. The eigenfunctions $f(x, i\varkappa)$ and $g(x, i\varkappa)$ corresponding to it are linearly dependent:

$$g(x,i\varkappa)=\rho f(x,i\varkappa), \quad \rho=\exp(2\varkappa\xi), \quad f(x,i\varkappa)={}^1/_2\exp(-\varkappa\xi)\,\mathrm{sech}\,z. \qquad (3.4)$$

Substituting $\lambda=-\varkappa^2$ into (2.5) and $\rho_r=\rho$, $\varkappa_r=\varkappa$ into (2.32) we get

$$\frac{d\varkappa}{dt}=-\frac{\varepsilon}{4\varkappa}\int_{-\infty}^{\infty}R[u_s(z)]\,\mathrm{sech}^2 z\,dz, \qquad (3.5)$$

$$\frac{d\xi}{dt}=4\varkappa^2-\frac{\varepsilon}{4\varkappa^3}\int_{-\infty}^{\infty}R[u_s(z)]\left(z+\frac{1}{2}\mathrm{sh}\,2z\right)\mathrm{sech}^2 z\,dz. \qquad (3.6)$$

For $\varepsilon=0$ the well known results of the zeroth approximation follow from this. We note also that if $R[u_s(z)]$ is an odd function of z, the perturbation does in first approximation not affect the soliton amplitudes, but it does affect its velocity; if, however, $R[u_s(z)]$ is an even function of z the soliton amplitude is changed and its velocity depends on the amplitude in the same way as for $\varepsilon=0$.

To illustrate this we consider some examples.

1. Let $\varepsilon R[u]=\gamma u$ so that γ has the meaning of a field instability growth rate. In that case

$$\varkappa(t)=\varkappa_0\exp(2\gamma t/3), \quad \xi(t)=(3/\gamma)[\varkappa^2(t)-\varkappa_0^2]+\xi_0. \qquad (3.7)$$

2. Let $R[u]=\partial^2u/\partial x^2$, i.e., we have the Korteweg–de Vries–Burgers equation (see, e.g., Ref. 10). In that case

$$\varkappa(t)=(15/16\varepsilon)^{1/2}(t+t_0)^{-1/2},$$
$$\xi(t)=(15/4\varepsilon)\ln(1+t/t_0)+\xi_0, \quad t_0=15/16\varepsilon\varkappa_0^2. \qquad (3.8)$$

The corresponding expressions for $\varkappa(t)$ were obtained earlier in Refs. 11, 12 by other means. In our case they are the result of a simple approximation.

We finally evaluate with a view to further approximations the matrix elements $\alpha(k, \pm k)$ in the adiabatic approximation for a single soliton. Substituting (3.2) into (2.33) we find

$$\alpha(k,k)=\frac{1}{\varkappa(k^2+\varkappa^2)}\int_{-\infty}^{\infty}(k^2+\varkappa^2\operatorname{th}^2 z)R[u_s(z)]dz, \tag{3.9}$$

$$\alpha(k,-k)=\frac{\exp(-2ik\xi)}{\varkappa(k-i\varkappa)^2}\int_{-\infty}^{\infty}(k-i\varkappa\operatorname{th} z)^2R[u_s(z)]e^{-2ikz/\varkappa}dz. \tag{3.10}$$

One can obtain similar results for NLS and MKdV. We look here for the soliton pulse in the adiabatic approximation in the form

$$u_s(x,t)=2\nu\operatorname{sech} z\exp(i\mu z/\nu+i\delta),\quad z=2\nu(x-\xi), \tag{3.11}$$

where $\mu(t)$, $\nu(t)$, $\xi(t)$, and $\delta(t)$ must be determined. Substituting (3.11) and (1.8) into (2.3) we find the Jost functions and coefficients:

$$f(x,\lambda)=\frac{\exp(i\lambda z/2\nu+i\lambda\xi)}{\lambda-\mu+i\nu}\begin{pmatrix}\nu\operatorname{sech} z\exp(-i\mu z/\nu-i\delta)\\ \lambda-\mu+i\nu\operatorname{th} z\end{pmatrix}, \tag{3.12}$$

$$g(x,\lambda)=a(\lambda)\bar f(x,\lambda),\quad a(\lambda)=(\lambda-\mu-i\nu)/(\lambda-\mu+i\nu),\quad b(\lambda)=0. \tag{3.13}$$

The discrete spectrum consists here of a single eigenvalue

$$\lambda=\zeta\equiv\mu+i\nu. \tag{3.14}$$

One verifies easily that the functions $f(x,\lambda)$ and $g(x,\lambda)$ for $\lambda=\zeta$ become proportional to one another: $g(x,\zeta)=\rho f(x,\zeta)$,

$$\rho=i\exp(i\delta-2i\zeta\xi). \tag{3.15}$$

For $\zeta_r=\zeta$, $\rho_r=\rho$ Eqs. (2.5) and (2.40) give

$$\frac{d\mu}{dt}=\frac{\varepsilon}{2}\operatorname{Im}\int_{-\infty}^{\infty}\frac{\operatorname{th} z}{\operatorname{ch} z}R[u_s(z)]e^{-i\mu z/\nu-i\delta}dz, \tag{3.16}$$

$$\frac{d\nu}{dt}=\frac{\varepsilon}{2}\operatorname{Re}\int_{-\infty}^{\infty}\operatorname{sech} zR[u_s(z)]e^{-i\mu z/\nu-i\delta}dz, \tag{3.17}$$

$$\frac{d\xi}{dt}=-\frac{1}{2\nu}\operatorname{Im}h(\zeta)+\frac{\varepsilon}{4\nu^2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{z}{\operatorname{ch} z}R[u_s(z)]e^{-i\mu z/\nu-i\delta}dz, \tag{3.18}$$

$$\frac{d\delta}{dt}=2\mu\frac{d\xi}{dt}+\operatorname{Re}h(\zeta)+\frac{\varepsilon}{2\nu}\operatorname{Im}\int_{-\infty}^{\infty}\frac{1-z\operatorname{th} z}{\operatorname{ch} z}R[u_s(z)]e^{-i\mu z/\nu-i\delta}dz. \tag{3.19}$$

Of particular interest are perturbations for which $d\mu/dt\equiv 0$. For such perturbations $\mu\equiv 0$, if initially we had $\mu=0$. The condition which the quantity $R[u]$ must satisfy in order that we may assume that $\mu\equiv 0$ has the form

$$\operatorname{Im}\int_{-\infty}^{\infty}\frac{\operatorname{th} z}{\operatorname{ch} z}R[u_s(z)]e^{-i\delta}dz=0. \tag{3.20}$$

The physical meaning of the condition $\mu\equiv 0$ for the NLS consists in that then the soliton velocity $d\xi/dt\propto\varepsilon$ as one can easily check from (3.18) and (2.39).

For the MKdV equation condition (3.20) is, in particular, satisfied for real u, $R[u]$. If, therefore, initially u_s is a real quantity ($\mu=\delta=0$) and $\operatorname{Im}R[u]=0$, $\mu\equiv\delta\equiv 0$ for all t.

For the NLS and $\mu\equiv 0$, Eq. (3.17) was recently obtained in Ref. 13. This result was found by means of one of the conservation laws. However, the problem arises what happens to the remaining conservation laws, of which there is an infinite number (in particular, do they not lead to new equations for the soliton parameters?). The problem of the interrelationship between the perturbation theory considered here and the conservation laws is discussed in Sec. 6.

As an illustration we consider the example when $\varepsilon R[u]=\gamma u$ (γ is the growth rate). In that case condition (3.20) is satisfied. As a result we get for the NLS

$$\mu=\mu_0,\quad \nu=\nu_0\exp(2\gamma t),\quad \xi=2\mu_0 t+\xi_0,$$
$$\delta=\frac{\nu_0^2}{2\gamma}[\exp(4\gamma t)-1]+2\mu_0^2 t+2\mu_0\xi_0+\delta_0, \tag{3.21}$$

where $\nu_0>0$, μ_0, ξ_0, and δ_0 are arbitrary constants. For the MKdV we shall have

$$\mu=\mu_0,\quad \nu=\nu_0\exp(2\gamma t),\quad \xi=\frac{\nu_0^2}{\gamma}[\exp(4\gamma t)-1]-12\mu_0^2 t+\xi_0,$$
$$\delta=\frac{4\mu_0\nu_0^2}{\gamma}[1-\exp(4\gamma t)]-16\mu_0^3 t+2\mu_0\xi_0+\delta_0. \tag{3.22}$$

We finally give the expressions for the matrix elements $\alpha(\lambda,\lambda)$, $\bar\alpha(\lambda,\lambda)$ which occur in (2.35), (2.36) in the adiabatic approximation:

$$\alpha(\lambda,\lambda)=\frac{\exp(2i\lambda\xi-i\delta)}{2\nu(\lambda-\zeta^*)^2}\int_{-\infty}^{\infty}e^{i\lambda z/\nu}\Big[(\lambda-\mu+i\nu\operatorname{th} z)^2R^*[u_s(z)]e^{i\delta}$$
$$-\frac{\nu^2}{\operatorname{ch}^2 z}R[u_s(z)]e^{-2i\mu z/\nu-i\delta}\Big]dz, \tag{3.23}$$

$$\bar\alpha(\lambda,\lambda)=-\frac{1}{2|\lambda-\zeta|^2}\int_{-\infty}^{\infty}[(\lambda-\mu+i\nu\operatorname{th} z)R^*[u_s(z)]e^{i\mu z/\nu+i\delta}$$
$$+(\lambda-\mu-i\nu\operatorname{th} z)R[u_s(z)]e^{-i\mu z/\nu-i\delta}]\frac{dz}{\operatorname{ch} z}. \tag{3.24}$$

In the adiabatic approximation considered here we neglected the change in the shape of the soliton and the growth of the "tail" as the result of the perturbation. These effects are determined by the deviation of the coefficients $a(\lambda,t)$, $b(\lambda,t)$ from their "non-reflecting" values (3.3) or (3.13). As these deviations grow the wave increasingly differs from a soliton so that the validity of the adiabatic approximation is violated when time moves on. These effects will be analyzed in detail in the following sections where the next approximation is considered.

4. EVOLUTION OF KdV SOLITONS AS THE EFFECT OF PERTURBATIONS (FIRST PERTURBATION THEORY APPROXIMATION)

The first perturbation theory approximation which follows the adiabatic one allows us to describe the distortion of the shape of the soliton and the growth of the tail. We consider here this approximation for the perturbed KdV equation.

We look for a solution of the perturbed KdV equation in the form

$$u(x,t)=u_s(x,t)+\delta u(x,t)=-2\varkappa^2[\operatorname{sech}^2 z+w(z,t)], \qquad (4.1)$$

where z was defined in (3.1), $\varkappa(t)$ and $\xi(t)$ satisfy Eqs. (3.5) and (3.6), and $w(z, t)$ is a function which is not yet known, and to fix the ideas we assume that $w=0$ when $t=0$. Moreover, we write the functions K and F in the Gel'fand–Levitan Eq. (2.16) in the form

$$K(x,y)=K_s(x,y)+\delta K(x,y), \qquad (4.2)$$

$$F(x)=F_s(x)+\delta F(x), \qquad (4.3)$$

where K_s and F_s correspond to the soliton:

$$K_s(x,y)=-\frac{2\varkappa\exp[\varkappa(2\xi-x-y)]}{1+\exp[2\varkappa(\xi-x)]}, \qquad (4.4)$$

$$F_s(x)=2\varkappa\exp[\varkappa(2\xi-x)]. \qquad (4.5)$$

Unless it causes confusion, we drop here and henceforth the time t on which these functions depend as a parameter. Using the fact that u_s in (4.1) corresponds to Eq. (2.16) for $K=K_s$, $F=F_s$ and assuming that the functions w, δK, δF are first order quantities ($\propto\varepsilon$) we get in the first approximation

$$\delta K(x,y)+\int_x^\infty \delta K(x,y')F_s(y+y')\,dy'$$
$$=-\delta F(x+y)-\int_x^\infty K_s(x,y')\,\delta F(y+y')\,dy'\equiv\Phi(x,y). \qquad (4.6)$$

The function $\delta F(x)$ has then, according to (2.17) the form

$$\delta F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(k)}{a(k)}e^{ikx}\,dk, \qquad (4.7)$$

and $a(k, t)$ and $b(k, t)$ satisfy the set (2.30), (2.31).

The solution of Eq. (4.6) has the form

$$\delta K(x,y)=K_s(x,y)e^{\varkappa x}\int_x^\infty \Phi(x,y')e^{-\varkappa y'}\,dy'+\Phi(x,y), \qquad (4.8)$$

as one can easily verify through direct substitution. Substituting now (4.1), (4.2), (4.4), and (4.8) into (2.8) we get

$$w(z)=-\frac{1}{2\pi\varkappa}\frac{d}{dz}\int_{-\infty}^{\infty}\frac{b(k)}{a(k)}\left(\frac{k+i\varkappa\operatorname{th}z}{k+i\varkappa}\right)^2 e^{2ik\xi+2ikz/\varkappa}\,dk. \qquad (4.9)$$

For the calculation of $w(z)$ it is thus necessary to determine first of all the ratio b/a. As the quantity b/a, like w, is formally a first order quantity, we can determine it from the set (2.30) and (2.31), neglecting terms $\propto\varepsilon b$ on the right-hand sides. As a result we would get the expression

$$\frac{b(k)}{a(k)}\approx\frac{\varepsilon\alpha(k,-k)}{16k^2(k^2+\varkappa^2)}\{1-e^{8ik^3t+2ik[\xi(t)-\xi_0]}\}. \qquad (4.10)$$

However, this expression becomes invalid in the vicinity of $k=0$, as the small parameter ε occurs in the set (2.30) and (2.31) in the combination ε/k. The small k region is very important, as can be seen from what follows. We must therefore determine b/a in that region without using the fact that ε/k is small.

To study that problem, we consider the relation

$$b(k)=\delta b(k)=\int_{-\infty}^{\infty}\left[\frac{\delta b(k)}{\delta u(x)}\right]_{u=u_s}\delta u(x)\,dx, \qquad (4.11)$$

where $\delta b(k)/\delta u(x)$ is the variational derivative of the coefficient $b(k)$, considered as a functional of the potential $u(x)$. According to Ref. 14

$$\frac{\delta b(k)}{\delta u(x)}=\frac{1}{2ik}f^*(x,k)\,g(x,k),$$

where f and g are Jost functions. Substituting this expression into (4.11) and using the fact that for $u=u_s$ we have Eqs. (3.2), (3.3) we get

$$b(k,t)=\exp[-2ik\xi(t)]\,\bar b(k,t), \qquad (4.12)$$

$$\bar b(k,t)=\frac{i\varkappa}{k(k^2+\varkappa^2)}\int_{-\infty}^{\infty}(k-i\varkappa\operatorname{th}z)^2 w(z)\,e^{-2ikz/\varkappa}\,dz. \qquad (4.13)$$

We shall now assume that $\bar b(k, t)$ is a quantity which varies slowly with time and we define

$$\sigma(k,t)\equiv\frac{1}{8\varkappa^3}\frac{\partial\ln\bar b(k,t)}{\partial t},\quad |\sigma|\ll1. \qquad (4.14)$$

Differentiating (4.12) with respect to the time we get

$$\partial b(k,t)/\partial t=-8i\varkappa^2[k+i\varkappa\sigma(k,t)]\,b(k,t)+O(\varepsilon^2), \qquad (4.15)$$

where we used (4.14), (3.6) and neglected terms of order ε^2. Now substituting (4.15) into (2.31) we get

$$\frac{b(k)}{a(k)}=\frac{\varepsilon\alpha(k,-k)}{16k[(k^2+\varkappa^2)k+i\varkappa^3\sigma(k)]-\varepsilon\alpha(k,k)}. \qquad (4.16)$$

For sufficiently large k we can neglect terms containing σ and ε in the denominator and (4.16) changes to the non-oscillating term in (4.10) (since b/a occurs in (4.7) under the integral sign in the integral over k, the contribution from the oscillating terms is small for not too small k). On the other hand, for small k the terms with σ and ε in the denominator in (4.16) are very important and here (4.16) differs from the expression (4.10).

In particular, it follows from (4.16) that

$$\lim_{k\to0}(b(k)/a(k))=-1. \qquad (4.17)$$

This result is in correspondence with the general properties of the Jost coefficients for the Schrödinger equation and, for instance, when $b(k)$ has a singularity for $k=0$, $a(k)$ has the same singularity while their ratio is given by Eq. (4.17).[6]

From what we have said earlier it follows that as $\varepsilon\alpha$ and σ are small and play an important role only when $k\to0$, we can make the substitution in (4.16)

$$\sigma(k,t)\to\sigma(0,t)\equiv\sigma(t),$$
$$\varepsilon\alpha(k,k;t)\to\varepsilon\alpha(0,0;t)\equiv4\varepsilon\varkappa^4(t)q(t), \qquad (4.18)$$

where according to (3.9)

$$q=(1/4\varkappa^5)\int_{-\infty}^{\infty}R[u_s(z)]\,\mathrm{th}^2 z\,dz. \tag{4.19}$$

Performing this substitution and substituting (4.16) into (4.9) we get the following expression for the correction to the soliton:

$$w(z)=-\frac{\varepsilon}{32\pi\varkappa^5}\frac{d}{dz}v(z), \tag{4.20}$$

$$v(z)=\int_{-\infty}^{\infty}dz'\,R[u_s(z')]\int_{-\infty}^{\infty}\frac{(p+i\,\mathrm{th}\,z)^2(p-i\,\mathrm{th}\,z')^2}{(p^2+1)^3(p-p_1)(p-p_2)}\exp[2ip(z-z')]\,dp$$

$$(p=k/\varkappa), \tag{4.21}$$

where we have also substituted the expression for the matrix element (3.10). When $|\varepsilon q/\sigma^2|$, $|\sigma|\ll 1$ we have[4)]

$$p_1=-\frac{i\varepsilon q}{2\sigma}(1+\sqrt{1-\varepsilon q/\sigma^2})^{-1}\approx-\frac{i\varepsilon q}{4\sigma},$$
$$p_2=-\frac{i\sigma}{2}(1+\sqrt{1-\varepsilon q/\sigma^2})\approx-i\sigma. \tag{4.22}$$

After integrating over p in (4.21) we must put $p_1\to 0$, as in (4.20) the small parameter ε is already contained as a factor and it would correspond to excessive accuracy to assume that $p_1\neq 0$ after integration (the term $\varepsilon\alpha$ in the denominator in (4.16) thus serves solely to determine the correct way of going around the pole as $k\to 0$).

It follows from (4.21) and what has been said earlier that the asymptotic form of $v(z)$ has the form

$$v(z)\to 8\pi\varkappa^5 q\begin{cases}\dfrac{1}{\sigma}\left[\theta\left(-\dfrac{\varepsilon q}{\sigma}\right)-\theta(-\sigma)e^{2\sigma z}\right]+O(z^2e^{-2z}), & z\to\infty\\ \dfrac{1}{\sigma}\left[\theta(\sigma)e^{2\sigma z}-\theta\left(\dfrac{\varepsilon q}{\sigma}\right)\right]+O(z^2e^{2z}), & z\to-\infty\end{cases} \tag{4.23}$$

where $\theta(x)=1$ $(x>0)$, $\theta(x)=0$ $(x<0)$. We find from (4.20) and (4.23) that whatever the signs of σ and εq

$$\int_{-\infty}^{\infty}w(z)\,dz=-\frac{\varepsilon q}{4\sigma}. \tag{4.24}$$

In particular, using (4.24) we can determine the quantity σ which was earlier introduced purely phenomenologically. To do this we note that it follows from (4.13) for small k that

$$\bar{b}(k,t)\approx-\frac{i\varkappa}{k}\int_{-\infty}^{\infty}\mathrm{th}^2 z\,w(z)\,dz=-\frac{i\varkappa}{k}\int_{-\infty}^{\infty}w(z)\,dz, \tag{4.25}$$

where we have used the relation

$$\int_{-\infty}^{\infty}w(z)\,\mathrm{sech}^2 z\,dz=0, \tag{4.26}$$

which can be proved as follows. Rewriting Eq. (4.9) in the form

$$w(z)=\frac{d}{dz}\int_{-\infty}^{\infty}\varphi(p)\,(p+i\,\mathrm{th}\,z)^2\exp(2ipz)\,dp,$$
$$\varphi(p)=-\frac{b(k)}{a(k)}\frac{\exp(2ip\varkappa\xi)}{2\pi(p+i)^2},\quad k=\varkappa p,$$

one checks easily, after integrating by parts, that

$$\int_{-\infty}^{\infty}w(z)\,\mathrm{sech}^2 z\,dz=\int_{-\infty}^{\infty}\varphi(p)I(p)\,dp,$$
$$I(p)=-\int_{-\infty}^{\infty}(p+i\,\mathrm{th}\,z)^2\frac{d\,\mathrm{sech}^2 z}{dz}\exp(2ipz)\,dz.$$

Integrating $I(p)$ by parts we get zero.

It follows from (4.24) and (4.25) that $\bar{b}\to i\varepsilon\varkappa q/4\sigma k$ as $k\to 0$. Substituting this into (4.14) and neglecting terms $\propto\varepsilon^2$, we get $d\sigma/dt=-8\varkappa^3\sigma^2$ which leads to

$$1/\sigma=8\int_{t_1}^{t}\varkappa^3(t')\,dt', \tag{4.27}$$

where t_1 is an arbitrary integration constant. Bearing in mind that $\varkappa(t)$ and $q(t)$ are slowly varying functions of the time we can write

$$1/\sigma(t)\approx 8\varkappa^3(t-t_1)\sim 8\varkappa^3 t \tag{4.28}$$

(the above-made assumption $|\sigma|\ll 1$ (see (4.14)) is correct when $t\gg t_1$). Another restriction is connected with the condition $|\varepsilon q/\sigma^2|\ll 1$ (see footnote[4)]). The above obtained results are thus applicable in the region

$$t_s\ll t\ll t_p(t_s/t_p)^{1/3}, \tag{4.29}$$

where we have introduced two characteristic time scales

$$t_s=(2\varkappa)^{-3},\quad t_p=|\varepsilon q\varkappa^3|^{-1},\quad t_s/t_p\ll 1. \tag{4.30}$$

We can call the quantity t_s the characteristic soliton time scale; after this time the soliton is displaced over a distance of the order of its length, t_p is a time scale caused by the perturbation (it is clear from (3.5) that after a time t_p the amplitude of the soliton is significantly changed due to the action of the perturbation).

We have thus established that σ decreases with time as t_s/t and the area of the deviation of the wave pulse from the soliton increases, according to (4.24), proportional to the time (in the region (4.29)).

We now turn to Eq. (4.20). Evaluation of the integral (4.21) leads to the following result:

$$w(z)=w_1(z)+w_2(z)+w_3(z); \tag{4.31}$$

$$w_1(z)=-\frac{\varepsilon\,\mathrm{th}^2 z}{8\varkappa^5}e^{2\sigma z}\int_z^{\infty}R[u_s(z')]\,\mathrm{th}^2 z'\,dz',$$
$$w_2(z)=-\frac{\varepsilon\theta(-\varepsilon q)}{8\varkappa^5\sigma}\frac{\mathrm{th}\,z}{\mathrm{ch}^2 z}\int_{-\infty}^{\infty}R[u_s(z')]\,\mathrm{th}^2 z'\,dz'. \tag{4.32}$$

As far as $w_3(z)$ is concerned, this function has a rather complicated form so that in the interest of succinctness we restrict ourselves in listing its most important characteristics.

We shall assume that $R[u_s(z)]$ decreases sufficiently fast as $|z|\to\infty$. In that case $w_1(z)$ also decreases fast as $z\to\infty$, but decreases very slowly as $z\to-\infty$. The function $w_2(z)$ is localized approximately in the same region as the soliton, but it has a relatively large amplitude (of the order ε/σ). This amplitude increases proportional to t/t_s. It is interesting that this term is absent when $\varepsilon q>0$. Finally, $w_3(z)\propto\varepsilon$ and is characterized by the following asymptotic behavior as $|z|\to\infty$:

$$w_3(z)\to-\frac{\varepsilon\,\mathrm{sign}\,z}{8\varkappa^5}z^2e^{-2|z|}\int_{-\infty}^{\infty}R[u_s(z')]\frac{dz'}{\mathrm{ch}^2 z'}. \tag{4.33}$$

We see that while $w_2(z)$, $w_3(z)$ describe the distortion of the shape of the soliton proper, the function $w_1(z)$ characterizes the formation of a tail at the soliton, the length of which increases as $1/\sigma\propto t/t_s$. As a consequence of our method of solution we obtained an "averaged" tail. However, it can be seen from the results of Sec. 6 that due to its contribution to the conservation laws it is equivalent to a true tail which has an oscillating structure.

As illustration we give two examples which are of independent interest.

1. Let $\varepsilon R[u]=\gamma u$. In that case Eqs. (4.20) and (4.21) give

$$w(z)=\frac{\gamma}{12\varkappa^3}\left\{e^{2\sigma z}\,\mathrm{th}^2 z(1-\mathrm{th}^2 z)+\frac{1}{\mathrm{ch}^2 z}\left[\left(\frac{2\theta(\gamma)}{\sigma}+\frac{\pi^2}{12}-\frac{1}{2}\right)\mathrm{th}\,z+\frac{\mathrm{th}\,z}{\mathrm{ch}^2 z}-1+2z(\mathrm{th}\,z-1)+z^2\,\mathrm{th}\,z\right]\right\}. \tag{4.34}$$

2. $R[u]=u_{xx}$ (KdV–Burgers equation), $\varepsilon>0$. In that case

$$w(z)=\frac{\varepsilon}{4\varkappa}\left\{e^{2\sigma z}\,\mathrm{th}^2 z\left[\frac{6}{5}(1-\mathrm{th}^2 z)-\frac{2}{3}(1-\mathrm{th}^2 z)\right]+\frac{1}{\mathrm{ch}^2 z}\left[\left(\frac{16}{15\sigma}-\frac{\pi^2}{45}-\frac{28}{15}\right)\mathrm{th}\,z-\frac{8}{15}+\frac{44}{15}\frac{\mathrm{th}\,z}{\mathrm{ch}^2 z}-\frac{6}{5}\frac{\mathrm{th}\,z}{\mathrm{ch}^4 z}+\frac{8z}{15}(1+2\,\mathrm{th}\,z)-\frac{4z^2}{15}\mathrm{th}\,z\right]\right\}. \tag{4.35}$$

So far we have assumed that $q\neq0$. The case $q=0$ requires a separate consideration and will not be discussed here.

5. ACTION OF PERTURBATIONS ON NLS AND MKdV SOLITONS IN THE FIRST APPROXIMATION

We write the correction to the soliton in the adiabatic approximation (3.11) in the form

$$\delta u(x,t)=2\nu w(z,t)\exp[i(\delta+\mu z/\nu)],\quad z=2\nu(x-\xi), \tag{5.1}$$

while we write the functions K and F in the Gel'fand–Levitan Eq. (2.28) in the form

$$K_1(x,y)=K_{1s}(x,y)+\delta K_1(x,y),\quad F(x)=F_s(x)+\delta F(x), \tag{5.2}$$

where the index s distinguishes the appropriate quantities for the soliton:

$$K_{1s}(x,y)=\frac{2\nu\rho^*\exp[-i\zeta^*(x+y)]}{1+|\rho|^2\exp(-4\nu x)},\quad F_s(x)=2\nu\rho\exp(i\zeta x) \tag{5.3}$$

$$\delta F(x)=\int_{-\infty}^{\infty}\frac{b(\lambda)}{a(\lambda)}e^{i\lambda x}\frac{d\lambda}{2\pi}. \tag{5.4}$$

In that case $\delta K_1(x,y)$ satisfies the equation

$$\delta K_1(x,y)+\int_x^\infty\delta K_1(x,y'')\int_x^\infty F_s^*(y+y')F_s(y'+y'')\,dy'\,dy''=\Phi(x,y),$$

$$\Phi(x,y)=\delta F^*(x,y) \tag{5.5}$$

$$-\int_x^\infty K_{1s}(x,y'')\int_x^\infty[F_s^*(y+y')\delta F(y'+y'')+\delta F^*(y+y')F_s(y'+y'')]\,dy'\,dy''. \tag{5.6}$$

$\Phi(x,y)$ is considered as a known function. The solution of Eq. (5.5) has the form

$$\delta K_1(x,y)=\Phi(x,y)-\rho e^{i\zeta x}K_{1s}(x,y)\int_x^\infty\Phi(x,y')e^{i\zeta y'}dy', \tag{5.7}$$

whence, using (2.27), (5.1), (5.3), and (5.4) we find

$$w(z)=\frac{\exp[-i(\delta+\mu z/\nu)]}{2\pi i\nu}\int_{-\infty}^{\infty}\frac{b(\lambda)}{a(\lambda)}\left(\frac{\lambda-\mu+i\nu\,\mathrm{th}\,z}{\lambda-\mu+i\nu}\right)^2\exp\left[i\lambda\left(\frac{z}{\nu}+2\xi\right)\right]d\lambda$$
$$-\frac{\nu\exp[i(\delta+\mu z/\nu)]}{2\pi i\,\mathrm{ch}^2 z}\int_{-\infty}^{\infty}\frac{b^*(\lambda)}{a^*(\lambda)}\frac{\exp[-i\lambda(z/\nu+2\xi)]}{(\lambda-\mu-i\nu)^2}d\lambda, \tag{5.8}$$

where a and b satisfy the set (2.35) and (2.36). We further apply the method already employed in the preceding section, namely, we bear in mind that b is a first order quantity and write

$$b(\lambda)=\int_{-\infty}^{\infty}\{[\delta b(\lambda)/\delta u(x)]_s\delta u(x)+[\delta b(\lambda)/\delta u^*(x)]_s\delta u^*(x)\}dx.$$

The variational derivatives for this case are evaluated in the Appendix (see (A.2)). We take the components of the Jost functions from (3.12), (3.13), and δu from (5.1). As a result we get

$$b(\lambda,t)=\exp i[\delta(t)-2\lambda\xi(t)]\bar b(\lambda,t), \tag{5.9}$$

$$\bar b(\lambda,t)=\frac{i}{(\lambda-\mu)^2+\nu^2}\int_{-\infty}^{\infty}[(\lambda-\mu-i\nu\,\mathrm{th}\,z)^2w(z)-\nu^2\,\mathrm{sech}^2 z\,w^*(z)]\times\exp\left[i(\mu-\lambda)\frac{z}{\nu}\right]dz.$$

By analogy with (4.14) we introduce the quantity

$$\sigma(\lambda,t)=-ih^{-1}(\zeta)\partial\ln\bar b(\lambda,t)/\partial t, \tag{5.10}$$

and we shall assume in what follows that $|\sigma(\lambda,t)|\ll1$. Differentiating $b(\lambda,t)$ in (5.9) and using (5.10) we find

$$b_t=ib[\delta_t-2\lambda\xi_t+h(\zeta)\sigma(\lambda,t)]\quad(\zeta=\mu+i\nu). \tag{5.11}$$

Substituting this into (2.36) we get

$$\frac{b(\lambda)}{a(\lambda)}=\frac{\varepsilon\alpha^*(\lambda,\lambda)}{\delta_t-2\lambda\xi_t-h(\lambda)+h(\zeta)\sigma(\lambda,t)+\varepsilon\bar\alpha(\lambda,\lambda)}. \tag{5.12}$$

We now apply these relations to the NSE. In order to avoid cumbersome formulae we restrict ourselves now to the case when the perturbation satisfies condition (3.20) which enables us to assume $\mu\equiv0$. One can also reduce the more general case ($\mu\neq0$, R a linear operator) to the case $\mu=0$ in the first approximation by means of a transformation of variables, which means in actual fact to a transition to a frame of reference in which the soliton is "almost at rest."

Substituting ξ_t and δ_t from (3.18), (3.19) into (5.12) and $h(\lambda) = -2\lambda^2$ one can easily check that in contrast to the perturbed KdV equation the small terms with ε and σ in the denominator in (5.12) are unimportant as it does not vanish, even without them. As a result we get

$$b(\lambda)/a(\lambda) = \varepsilon\alpha^*(\lambda,\lambda)/2(\lambda^2+\nu^2). \tag{5.13}$$

We note that we could have obtained the same result directly by integrating Eq. (2.36) with the small term with b dropped from the right-hand side.[5] If after that we drop the terms which are fast oscillating for sufficiently large t and which give a small contribution when we integrate over λ, one obtains (5.13). One can thus say that (5.13) is obtained from the exact expressions after some averaging and that it is the asymptotic ratio, valid for sufficiently large t.

Using now $\alpha(\lambda,\lambda)$ (see (3.23)) and substituting (5.13) into (5.8) we get after integrating over λ

$$w(z) = \frac{\varepsilon}{32i\nu^3\,\mathrm{ch}^2 z}\left\{\int\limits_{-\infty}^{z}\frac{dz'}{\mathrm{ch}^2 z'}[A(z,z')R[u_s(z')]e^{-i\delta}+B(z,z')R^*[u_s(z')]e^{i\delta}]\right. \tag{5.14}$$
$$\left.+\int\limits_{z}^{\infty}\frac{dz'}{\mathrm{ch}^2 z'}[A(-z,-z')R[u_s(z')]e^{-i\delta}+B(-z,-z')R^*[u_s(z')]e^{i\delta}]\right\},$$

where

$$\begin{aligned} A(z,z') &= \mathrm{ch}^2 z' e^{z+z'}+4\,\mathrm{ch}\,z\,\mathrm{ch}\,z'+\mathrm{ch}^2 z e^{-z-z'}-3\mathscr{P}_1(z-z')(e^z\,\mathrm{ch}\,z'\\ &\quad+e^{-z'}\,\mathrm{ch}\,z)+3\mathscr{P}_2(z-z')\,\mathrm{ch}\,(z-z'),\\ B(z,z') &= e^{z'-z}(\mathrm{ch}^2 z+\mathrm{ch}^2 z')-3\mathscr{P}_1(z-z')(e^{z'}\,\mathrm{ch}\,z+e^{-z}\,\mathrm{ch}\,z')\\ &\quad+3\mathscr{P}_2(z-z')\,\mathrm{ch}\,(z+z'),\\ \mathscr{P}_1(z) &= 1+{}^2/_3 z,\quad \mathscr{P}_2(z)=1+z+{}^1/_3 z^2. \end{aligned} \tag{5.15}$$

It has been assumed here, of course, that $R[u_s(z)]$ decreases sufficiently fast as $|z|\to\infty$ so that all integrals converge.

We get the asymptotic behavior of $w(z)$ from (5.14), (5.15):

$$w(z)\to\frac{\varepsilon z^2 e^{-|z|}}{16i\nu^3}\begin{cases}\displaystyle\int\limits_{-\infty}^{\infty}[R[u_s(z')]e^{-z'-i\delta}+R^*[u_s(z')]e^{z'+i\delta}]\frac{dz'}{\mathrm{ch}^2 z'}, & z\to\infty\\ \displaystyle\int\limits_{-\infty}^{\infty}[R[u_s(z')]e^{z'-i\delta}+R^*[u_s(z')]e^{-z'+i\delta}]\frac{dz'}{\mathrm{ch}^2 z'}, & z\to-\infty\end{cases} \tag{5.16}$$

We see that here no tail is formed and $w(z)\propto\varepsilon$.

As a simple, but important example we consider the case when $\varepsilon R[u]=\gamma u$ (i.e., γ is a growth rate). We then find from the relations obtained

$$w(z)=\gamma(4z^2+\pi^2/3)/16i\nu^2\,\mathrm{ch}\,z. \tag{5.17}$$

We now turn to the MKdV equation and restrict ourselves for the sake of simplicity to $R[u]$ so that we can assume that $\delta\equiv\mu\equiv0$. We note also that for $\sigma\neq0$ the denominator in (5.12) does not vanish for real λ. If we neglect terms $\propto\varepsilon$ and use the fact that now $h(\lambda)=8\lambda^3$, we can write

$$b(\lambda)/a(\lambda)=-\varepsilon\alpha^*(\lambda,\lambda)/8(\lambda^3+\lambda\nu^2+i\sigma\nu^3), \tag{5.18}$$

where, as for the KdV equation, we substituted $\sigma(\lambda,t)\to\sigma(0,t)\equiv\sigma$ (as, due to $\sigma(\lambda,t)$ being small this term is important only when $\lambda\to0$). Putting $\lambda\to0$ in (5.18) and $a(\lambda)\approx a_s(\lambda)=(\lambda-\zeta)/(\lambda-\zeta^*)\to-1$ and substituting this into (5.11) for $\lambda\to0$ we get an expression for σ:

$$\frac{1}{\sigma(t)}=8\int\limits_{t_1}^{t}\nu^3(t')\,dt'\sim\frac{t}{t_s},\quad t_s=\frac{1}{8\nu^3}, \tag{5.19}$$

valid for $t\gg t_s$. Our assumption that σ is small is thus justified for sufficiently large t.

Finally, substituting (5.18) into (5.8) we get after integrating over λ an explicit expression for $w(z)$. The latter, however, turns out to be rather complicated so that we here only give the asymptotic behavior following from it:

$$w(z)\to\frac{\varepsilon}{32\nu^4}\begin{cases}\displaystyle z^2e^{-z}\int\limits_{-\infty}^{\infty}R[u_s(z')]\frac{dz'}{\mathrm{ch}\,z'}, & z\to\infty\\ \displaystyle 2e^{\sigma z}\int\limits_{-\infty}^{\infty}R[u_s(z')]\,dz', & z\to-\infty\end{cases}. \tag{5.20}$$

Furthermore, we can obtain a simple expression for the complete integral of the perturbation. Putting $\mu=0$, $\lambda=0$ into (5.9) we get

$$\int_{-\infty}^{\infty}w(z,t)\,dz-2i\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{w(z,t)}{ch^2 z}\,dz=ib(0,t)e^{-i\delta(t)} \tag{5.21}$$

This relation is valid both for the MKdV and for the NLS. We can evaluate the quantity $b(0,t)$ respectively from (5.13) and (5.18):

$$b(0,t)=-\frac{\varepsilon e^{i\delta}}{4\nu^3}\int\limits_{-\infty}^{\infty}[\mathrm{th}^2 zR[u_s(z)]e^{-i\delta}+\mathrm{sech}^2 zR^*[u_s(z)]e^{i\delta}]\,dz\quad\text{(NLS)}, \tag{5.22}$$

$$b(0,t)=\frac{\varepsilon}{16i\sigma\nu^4}\int\limits_{-\infty}^{\infty}R[u_s(z)]\,dz\quad\text{(MKdV)}, \tag{5.23}$$

where we used Eq. (3.23). The integral (5.21) for the MKdV thus increases proportional to t (corresponding to the tail "growth") while it is constant for the NLS.

Finally, we consider the condition of the applicability of the results obtained. They are restricted by the condition $|b(\lambda,t)|\ll1$. Since, as can be seen from (5.9), $|b(\lambda,t)|\lesssim|b(0,t)|$ and using (5.22), (5.23) we find that the conditions $|b|\ll1$, $\sigma\ll1$ reduce for the MKdV to

$$t_s\ll t\ll t_p,\quad t_p^{-1}=\frac{\varepsilon}{\nu}\int\limits_{-\infty}^{\infty}R[u_s(z)]\,dz. \tag{5.24}$$

Here t_p is a characteristic time produced by the perturbation. For the NLS $|b(0,t)|\propto\varepsilon$ even for sufficiently large t.

6. SATISFYING THE CONSERVATION LAWS BY THE FIRST APPROXIMATION SOLUTIONS

For Eqs. (1.1) with $\varepsilon=0$ there is an infinite number of conserved quantities (invariants) of the form

$$I_n\{u,u^*\}=\int\limits_{-\infty}^{\infty}q_n[u,u^*]\,dx, \tag{6.1}$$

where the $q_n[u, u^*]$ are polynomials of the functions u and u^* and their spatial derivatives. For the KdV, NLS, and MKdV equations these invariants were found and studied in Refs. 15, 3, and 4. When $\varepsilon \neq 0$ these quantities are no longer conserved, but one can obtain a simple expression for their time derivatives, namely[16]:

$$\frac{dI_n}{dt} = \varepsilon \int_{-\infty}^{\infty} \left\{ \frac{\delta I_n}{\delta u(x)} R[u(x)] + \frac{\delta I_n}{\delta u^*(x)} R^*[u(x)] \right\} dx, \quad n=1,2,3\ldots \tag{6.2}$$

where, for instance, $\delta I_n\{u, u^*\}/\delta u(x)$ are the variational derivatives of the functional $I_n\{u, u^*\}$ with respect to u at the point x for fixed u^*. We shall call Eqs. (6.2) the conservation laws, modified for the case of perturbations or, briefly, the modified conservation laws.

We apply Eqs. (6.2) to a study of the evolution of solitons under the action of perturbations to first order in ε. In that case we can put in (6.2)

$$\delta I_n/\delta u \to [\delta I_n/\delta u]_s, \quad \delta I_n/\delta u^* \to [\delta I_n/\delta u^*]_s,$$

where the index s indicates quantities evaluated for solitons. On the other hand, we can write the left-hand side of (6.2) in the form

$$\frac{d}{dt} I_n\{u_s + \delta u\} = \frac{dI_n\{u_s\}}{dt} + \frac{d}{dt}\int_{-\infty}^{\infty} \left\{ \left[\frac{\delta I_n}{\delta u(x)}\right]_s \delta u(x) + \left[\frac{\delta I_n}{\delta u^*(x)}\right]_s \delta u^*(x) \right\} dx. \tag{6.3}$$

We now show by the example of the KdV equation how one can evaluate the variational derivative "at the soliton." In this case we can use the following relations[14]:

$$\ln a(k) = -\sum_{n=1}^{\infty} I_n/(2ik)^{2n-1}, \tag{6.4}$$

$$\frac{\delta a(k)}{\delta u(x)} = \frac{i}{2k} f(x,k)\, g(x,k). \tag{6.5}$$

Substituting (3.2) into (6.5) and varying (6.4) with respect to $u(x)$ we find

$$\left[\frac{\delta I_n}{\delta u(x)}\right]_s = \begin{cases} 1, & n=1 \\ (2\varkappa)^{2n-2} \operatorname{sech}^2 z, & n>1. \end{cases} \tag{6.6}$$

We now substitute (6.3) and (6.6) into (6.2) and use the fact that

$$I_n\{u_s\} = -\frac{2^{2n}\varkappa^{2n-1}}{2n-1}, \quad n=1,2,\ldots \tag{6.7}$$

(One can obtain this relation from (6.4) if we substitute (3.3) into the left-hand side and afterwards expand it in powers of $\varkappa/k$. [6]) The derivatives $d\varkappa/dt$ occurring on the left-hand side of (6.2) can be taken from (3.5). Taking only terms $\propto \varepsilon$ into account we then find that the infinite set of Eqs. (6.2) leads in all to two equations for w:

$$\frac{d}{dt}\int_{-\infty}^{\infty} w(z,t)\,dz = -\frac{\varepsilon}{2\varkappa^2}\int_{-\infty}^{\infty} R[u_s(z)] \operatorname{th}^2 z\, dz, \tag{6.8}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} w(z,t) \operatorname{sech}^2 z\, dz = 0, \tag{6.9}$$

where (6.8) follows from (6.2) for $n=1$, while all remaining conservation laws ($n \geq 2$) lead to the single Eq. (6.9). Since, as we saw above, $w(z,t)$ satisfies condition (4.26), it also satisfies Eq. (6.9). Further, substituting (4.24) and (4.28) into (6.8) and restricting ourselves to terms $\propto \varepsilon$, we can satisfy ourselves that (6.8) is also satisfied. The solution of the perturbed KdV equation found by us thus satisfies the whole infinite set of modified conservation laws (6.2).

We now apply Eqs. (6.2) to the NLS and MKdV. In that case one has instead of (6.4) and (6.7)[3]

$$\ln a(\lambda) = -\sum_{n=1}^{\infty} (i/2\lambda)^n I_n, \tag{6.10}$$

$$I_n\{u_s\} = \frac{1}{n}\left(\frac{2}{i}\right)^n (\zeta^n - \zeta^{n*}), \tag{6.11}$$

and instead of (6.5) Eq. (A.1), obtained in the Appendix. Substituting into (A.1) instead of f_1 and g_1 their values for the soliton from (3.12), (3.13) for $\mu = 0$ and expanding in powers of ν/λ and after that comparing the expressions obtained with the appropriate variations of (6.10) we get

$$\left[\frac{\delta I_{2m-1}}{\delta u(x)}\right]_s = \left[\frac{\delta I_{2m-1}}{\delta u^*(x)}\right]_s^* = (2\nu)^{2m-1} \operatorname{sech} z\, e^{-i\delta}, \tag{6.12}$$

$$\left[\frac{\delta I_{2m}}{\delta u(x)}\right]_s = -\left[\frac{\delta I_{2m}}{\delta u^*(x)}\right]_s^* = -(2\nu)^{2m} \frac{\operatorname{th} z}{\operatorname{ch} z} e^{-i\delta}, \tag{6.13}$$

where $m = 1, 2, 3, \ldots$. Finally, we substitute (6.3), (6.11), to (6.13), and (5.1) into (6.2) and after that use (3.17). As a result we find that all conservation laws with odd n lead to the relation

$$\frac{d}{dt} \operatorname{Re} \int_{-\infty}^{\infty} \frac{w(z)}{\operatorname{ch} z} dz = 0, \tag{6.14}$$

and with even n to

$$\frac{d}{dt} \operatorname{Im} \int_{-\infty}^{\infty} \frac{\operatorname{th} z}{\operatorname{ch} z} w(z)\, dz = \frac{\varepsilon}{2\nu} \operatorname{Im} \int_{-\infty}^{\infty} \frac{\operatorname{th} z}{\operatorname{ch} z} R[u_s(z)] e^{-i\delta} dz = 0. \tag{6.15}$$

In (6.15) we have used the fact that in the case $\mu = 0$, which is considered here, (3.20) holds.

We now check that the solutions found by us satisfy the conditions (6.14), (6.15). Substituting (5.8) for $\mu = 0$ into the integral from (6.14) and integrating over z we get

$$\int_{-\infty}^{\infty} w(z) \operatorname{sech} z\, dz = \frac{ie^{-i\delta}}{4\nu} \int_{-\infty}^{\infty} \frac{b(\lambda)}{a(\lambda)} \frac{(\lambda^2+\nu^2)}{(\lambda+i\nu)^2} \frac{\exp(2i\lambda\xi)}{\operatorname{ch}(\pi\lambda/2\nu)} d\lambda - \text{c.c.}$$

The right-hand side of this equation is purely imaginary and, hence, its real part vanishes. Thus, (6.14) is satisfied. Similarly, we can check that (6.15) is satisfied. All these results are valid both for the NSE and for the MKdV. If in the latter we assume that $u(x, t)$ is a real function we are, by integrating the perturbed MKdV equation, to an additional conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = \varepsilon \int_{-\infty}^{\infty} R[u]\, dx \tag{6.16}$$

(this relation is not valid for the complex MKdV equation and it is, of course, not contained in the above discussed conservation laws). Putting $u=u_s+\delta u$ in (6.16) and using (5.1), where $\mu=\delta=0$, we get in the first approximation

$$\frac{d}{dt}\int_{-\infty}^{\infty} w(z,t)\,dz=\frac{\varepsilon}{2\nu}\int_{-\infty}^{\infty} R[u_s(z)]\,dz. \tag{6.17}$$

Substituting here (5.21), (5.23) and using (5.19) we verify that also this equation is satisfied by the solutions found by us.

Thus, notwithstanding that for our method of solution $w(z)$ is an averaged correction to the soliton, in all cases considered it satisfies all conservation laws with perturbations included.

APPENDIX: VARIATIONAL DERIVATIVES OF THE JOST COEFFICIENTS FOR THE NSE AND MKdV

Varying the equation $\hat{L}\psi=\lambda\psi$ for $\hat{L}$ from (1.8) we find

$$\left(iP\frac{\partial}{\partial x}+Q-\lambda\right)\frac{\delta f(x,\lambda)}{\delta u(x')}=\delta(x-x')\begin{pmatrix}0&0\\1&0\end{pmatrix}f(x',\lambda).$$

The solution of this equation with the boundary condition $\delta f(x,\lambda)/\delta u(x')\to 0$ $(x\to\infty)$ is of the form

$$\frac{\delta f(x,\lambda)}{\delta u(x')}=\frac{\theta(x'-x)f_1(x',\lambda)}{ia(\lambda)}[g_1(x',\lambda)f(x,\lambda)-f_1(x',\lambda)g(x,\lambda)].$$

Similarly one can find

$$\frac{\delta f(x,\lambda)}{\delta u^*(x')}=\frac{\theta(x'-x)f_2(x',\lambda)}{ia(\lambda)}[f_2(x',\lambda)g(x,\lambda)-g_2(x',\lambda)f(x,\lambda)].$$

Now taking the limit as $x\to-\infty$ and using (2.21) and (2.22) we get

$$\frac{\delta a(\lambda)}{\delta u(x)}=-if_1(x,\lambda)g_1(x,\lambda),\quad \frac{\delta a(\lambda)}{\delta u^*(x)}=if_2(x,\lambda)g_2(x,\lambda); \tag{A.1}$$

$$\frac{\delta b(\lambda)}{\delta u(x)}=if_2^*(x,\lambda)g_1(x,\lambda),\quad \frac{\delta b(\lambda)}{\delta u^*(x)}=if_1^*(x,\lambda)g_2(x,\lambda), \tag{A.2}$$

where f_j and g_j $(j=1,2)$ are the components of the Jost functions.

Note added in proof (June 29, 1977). The authors recently became aware of the fact that Eqs. (2.5), (2.35), (2.36), and (2.40) were for the operator $\hat{L}$ of the form (1.8) obtained by D. Kaup.[18]

1) At the present there is a large literature about the inverse scattering method. We restrict ourselves to referring solely to the most fundamental papers about that problem, the results of which are used in the present paper.

2) We note that we restrict ourselves here to a study of problems for which $u(x,t)\to 0$, $|x|\to\infty$.

3) We do not write out the independent variable t where this is inessential.

4) When $|\varepsilon q/\sigma^2|>1$ the physical nature of the solution (4.21) changes, in general, as compared to the one given below and will not be discussed here.

5) In contrast to (2.30) and (2.31), Eqs. (2.35) and (2.36) do not have singularities as $\lambda\to 0$ and we can therefore assume that $b\propto\varepsilon$ and $\delta a\propto\varepsilon$ for all λ.

6) The expression for $I_n\{u_s\}$ (for a somewhat different definition of I_n) was obtained in Ref. 17 even before the discovery of the inverse scattering method.

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