Static approximation in calculating the bremsstrahlung of electrons on atoms and ions

V. P. Zhdanov

I. V. Kurchatov Atomic Energy Institute

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The use of the dipole moment, momentum, and nuclear force operators in calculating the intensity the bremsstrahlung of electrons on atoms and ions in the static approximation is analyzed. All three operators give the same value for the transition matrix element only if the disturbance of the bound electrons in the atom or ion by the incident electron is taken into account in the first order of perturbation theory. It is shown that in calculating the bremsstrahlung intensity in the static approximation the best results are obtained by using the dipole moment or momentum operator. The formal use of the nuclear force operator in static-approximation calculations does not give correct results for the emission of low-frequency bremsstrahlung photons. The limits of validity of the static approximation are discussed.

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1. INTRODUCTION

The bremsstrahlung (BS) emitted by an electron colliding with an atom corresponds to a transition of the electron from one continuum state to another. Hence the calculation of the intensity of the BS is a fairly complicated problem, even when no account is taken of the disturbance of the atomic electrons by the incident one. It is just this static approximation that is ordinarily used[1–8] to calculate the intensity of the BS emitted by high-energy ($\gtrsim 100$ eV) electrons colliding with neutral atoms and by electrons of any energy colliding with multiply charged ions. The basis of the static approximation is the weakness of the field associated with the perturbation of the atomic electrons as compared with the static field of the atom at the distances at which the BS is mainly emitted. That the field is only slightly disturbed at such distances is due to the tight binding of the electrons in the atom or the ion. Hence the static approximation is certainly not suitable for calculating the BS emitted by low-energy ($\lesssim 100$ eV) electrons in collisions with neutral or weakly ionized atoms, nor for calculating the emission of low-frequency BS photons by electrons of any energy in collisions with neutral atoms. This has previously been shown in a number of studies[9–11] and has recently been pointed out again by Amus'ya *et al.*[12]

Here we discuss the limits of validity of the static approximation, as well as the possibility of using the dipole-moment, momentum, and nuclear-force operators in calculating BS intensities in the static approximation.

2. BASIC FORMULAS OF THE STATIC APPROXIMATION

In the nonrelativistic case, when $v/c \ll 1$ (c is the velocity of light), the intensity of the dipole emission of photons of frequency $\omega = (v_1^2 - v_2^2)/2$ (we are using atomic units) incident to the scattering of an electron into the solid angle element $d\theta_{v2}$ is given by[1]

$$\frac{dW}{d\omega} = \frac{4\omega^4 v_2}{3(2\pi c)^3 v_1} \int \left| \langle \Psi_2(r_1,\dots,r_{N+1}) | \sum_{i=1}^{N+1} \mathbf{r}_i | \Psi_1(r_1,\dots,r_{N+1}) \rangle \right|^2 d\theta_{v2}, \qquad (1)$$

where Ψ_1 is the wave function (normalized to a plane wave) of the N bound electrons and the incident ($N+1$-st) electron before radiation, v_1 is the velocity of the incident electron before radiation, and v_2 and Ψ_2 are the velocity and wave function after radiation incident to scattering into the solid angle element $d\theta_{v2}$.

In calculating the transition matrix element one can replace the dipole moment operator by either the momentum operator or the nuclear-force operator[13]:

$$\left\langle \sum_{i=1}^{N+1} \mathbf{r}_i \right\rangle_{2,1} = -\frac{i}{\omega} \left\langle \sum_{i=1}^{N+1} \mathbf{P}_i \right\rangle_{2,1} \qquad (2a)$$

$$= \frac{1}{\omega^2} \left\langle \sum_{i=1}^{N+1} \frac{Z\mathbf{r}_i}{r_i^3} \right\rangle_{2,1}, \qquad (2b)$$

where $\omega = E_1 - E_2$ and Z is the nuclear charge. If exact wave functions are used in the calculation, all three ways of writing the matrix element are equally correct and lead to the same result. If approximate wave functions are used, however, the results may turn out to be different.

The wave function ψ_{0v} in the static approximation has the form

$$\psi_{0v} = \Phi_0(r_1,\dots,r_N) F_v(r_{N+1}), \qquad (3)$$

where Φ_0 is the ground-state wave function of the ion (for definiteness we shall speak henceforth of collisions with an ion) without allowance for the perturbation by the incident electron and F_v is the wave function of the incident electron in the static field of the ion:

$$\left(-\frac{\Delta_{N+1}}{2} - \frac{Z}{r_{N+1}} + U_{00}(r_{N+1}) \right) F_v(r_{N+1}) = \frac{v^2}{2} F_v(r_{N+1}), \qquad (4)$$

$$U_{00}(r_{N+1}) = \langle \Phi_0(r_1,\dots,r_N) | \sum_{i=1}^{N} \frac{1}{r_{i,N+1}} | \Phi_0(r_1,\dots,r_N) \rangle. \qquad (5)$$

In what follows we shall assume that Φ_0, the ground-state wave function of the ion, is spherically symmetric, and consequently, that $U_{00}(r_{N+1})$, the potential for the interaction of the incident electron with the ionic electrons as calculated in the static approximation, is also spherically symmetric.

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Using the partial wave expansion of F_v and the addition theorem for Legendre polynomials, we transform Eq. (1) to the form

$$\frac{dW}{d\omega}=\frac{32v_2}{3c^3v_1}\sum_{l=0}^{\infty}(l+1)(M^2_{l,l+1}+M^2_{l+1,l}), \tag{6}$$

where the matrix element $M_{ll'}$ is given by

$$M_{ll'}=\int_0^{\infty}\varphi_l(v_1,r)\varphi_{l'}(v_2,r)\left[\frac{Z}{r^2}+\frac{dU_{00}(r)}{dr}\right]dr. \tag{7a}$$

if the dipole-moment or momentum operator (2a), together with the form of Eq. (4), is used in the calculations, and by

$$M_{ll'}=\int_0^{\infty}\varphi_l(v_1,r)\varphi_{l'}(v_2,r)\frac{Z}{r^2}dr, \tag{7b}$$

if the nuclear-force operator (2b) is used. The $\varphi_l(v,r)$ in Eqs. (7) are solutions of the radial wave equations in the static field of the ion, having the asymptotic form

$$\varphi_l(v,r)\underset{r\to\infty}{\to}\frac{1}{v}\sin\left(vr+\frac{Z_i}{v}\ln 2vr-\frac{\pi l}{2}+\delta_l\right),$$

where $Z_i=Z-N$ is the charge of the ion. The matrix elements (7a) and (7b) differ by the matrix element for the force on the incident electron due to its interaction with the bound electrons in the ion. In the case of an ion containing many electrons, therefore, matrix elements (7a) and (7b) may differ greatly from one another.

3. RESULTS OF THE CALCULATIONS

Figure 1 shows the results of calculating the intensity of the BS emitted by electrons colliding with quintuply charged molybdenum and tungsten ions, using formulas (6), (7a), and (7b). The matrix elements (7a) and (7b) were obtained by numerical integration of the radial Schrödinger equations, boundary conditions being imposed near the origin of coordinates by expanding the functions $\varphi_l(v,r)$ in powers of r. Exact Thomas-Fermi potentials for the ions concerned, obtained by numerical integration of the Thomas-Fermi equation, were used to take screening into account. The methods used in obtaining the Thomas-Fermi potentials and solving the radial Schrödinger equations have been discussed in more detail elsewhere.[6] Angular momenta up to $l\leqslant 7$ were taken into account in the calculations for $2\omega/v_1^2>0.25$, and up to $l\leqslant 9$ for $2\omega/v_1^2\leqslant 0.25$. The accuracy of the calculated curves was checked against numerical calculations[14] of the BS intensity for a pure Coulomb field, using Sommerfeld's formula; the curves were found to be accurate within 4% in the region $2\omega/v_1^2>0.25$. Many angular momenta, up to $l\lesssim v_1^2/2$, contribute to the emission of low-frequency BS photons,[6] and the summation in (6) converges slowly; in this case the accuracy of the calculations amounts to 10% when matrix elements (7a) are used and to 15% when matrix elements (7b) are used.

The results of relativistic numerical calculations[3] of the intensity of the BS from 1- and 2.5-keV electrons on neutral molybdenum and tungsten atoms, using the static approximation and operators corresponding in the nonrelativistic limit to formulas (6) and (7a), are also shown in Fig. 1 for comparison. At incident electron energies of ~1 keV the emission takes place within the atom at nuclear distances of $\sim 1/v_1$ where the static potentials of the quintuply charged ions are close to the potentials of the corresponding neutral atoms. Hence the BS intensity is nearly the same, whether the incident electron strikes a quintuply charged ion or the corresponding neutral atom (at these energies the relativistic corrections are still insignificant). The relativistic calculations[3] are accurate within ~10%.

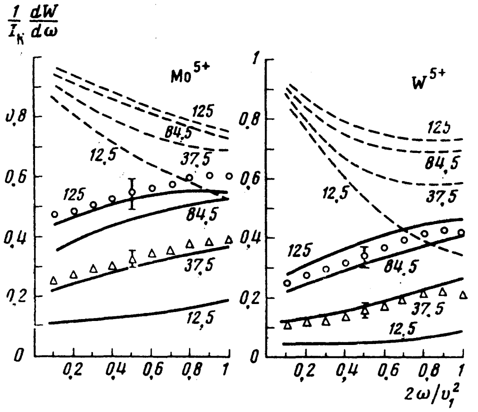

FIG. 1. Ratio of the spectral intensity of the bremsstrahlung from electrons on quintuply charged molybdenum or tungsten ions to the corresponding intensity $I_K=16\pi Z^2/3\sqrt{3}c^3v_1^2$ on the bare nucleus in Kramer's approximation.[13] The electron energies are given in atomic units near the corresponding curves. The full curves were calculated with formulas (6) and (7a) using the dipole moment or momentum operator; the dashed curves were calculated with formulas (6) and (7b) using the nuclear force operator. The points represent calculations[3] of the spectral intensity of the bremsstrahlung from electrons on neutral molybdenum or tungsten atoms for incident-electron energies of 92 (circles) and 36.8 (triangles) atomic units.

It will be seen that the results of the calculations using either the dipole moment operator or the momentum operator differ considerably from the results obtained using the nuclear force operator. The greatest difference is seen for low photon frequencies $2\omega/v_1^2<0.5$. As was noted above, many angular momenta contribute to the emission of low-frequency photons. For large angular momenta satisfying the condition

$$l\gg\max(2Z_i/v_1,\ R_iv_1),$$

where R_i is the radius of the ion, the probability for finding the incident electron at distances $r_{N+1}<l/v_1$ from the nucleus is exponentially small, so the contributions to the matrix elements (7a) and (7b) come from large distances $r_{N+1}\gtrsim l/v_1\gg R_i$. At such distances the static field of the ion is simply the Coulomb field of the ionic charge:

$$\frac{Z}{r^2_{N+1}}+\frac{dU_{00}(r_{N+1})}{dr_{N+1}}=\frac{Z_i}{r^2_{N+1}}.$$

For large angular momenta, therefore, matrix elements (7a) are smaller than matrix elements (7b) by a factor of $(Z_i/Z)^2$.

Let us examine more carefully whether the large difference between the results obtained with the dipole moment and nuclear force operators indicates that the static approximation is inadequate for calculating BS intensities, or that the use of one of the operators is less justified.

4. ALLOWANCE FOR PERTURBATION OF THE BOUND ELECTRONS

In the case we are interested in, i.e. in collisions of high-energy electrons with neutral atoms and of electrons of any energy with multiply charged ions, the field associated with the disturbance of the bound electrons in the ions is weak as compared with the static field and can be treated by perturbation theory:

$$H=H_0+V, \tag{8}$$

$$H_0=\sum_{i=1}^{N}\left(-\frac{\Delta_i}{2}-\frac{Z}{r_i}\right)+\sum_{i\neq j}^{N}\frac{1}{r_{ij}}-\frac{\Delta_{N+1}}{2}-\frac{Z}{r_{N+1}}+U_{00}(r_{N+1}), \tag{9}$$

$$V=\sum_{i=1}^{N}\frac{1}{r_{i.N+1}}-U_{00}(r_{N+1}), \tag{10}$$

where $U_{00}(r_{N+1})$ is the potential (5) for the interaction of the incident electron with the ionic electrons, calculated in the static approximation. The Hamiltonian H_0 is not symmetric with respect to the incident electron, so the use of this Hamiltonian in the zeroth approximation leads to an incorrect account of the exchange effects. However, the exchange effects are not important in the case of high-energy electrons or multiply charged ions.

The eigenfunctions of the Hamiltonian H_0 have the form

$$\psi_{nv}=\Phi_n(r_1.\dots.r_N)F_v(r_{N+1}), \tag{11}$$

where the $\Phi_n(r_1, \dots, r_N)$ are the exact wave functions for the states of the ion and satisfy the equation

$$\left[\sum_{i=1}^{N}\left(-\frac{\Delta_i}{2}-\frac{Z}{r_i}\right)+\sum_{i\neq j}^{N}\frac{1}{r_{ij}}\right]\Phi_n(r_1.\dots.r_N)=\varepsilon_n\Phi_n(r_1.\dots.r_N), \tag{12}$$

while the $F_v(r_{N+1})$ are the continuum wave functions for the electron in the static field and satisfy Eq. (4). The perturbation matrix elements have the form

$$V_{nv_2.0v_1}=\langle F_{v_2}(r_{N+1})\,|U_{n0}(r_{N+1})\,|F_{v_1}(r_{N+1})\rangle.\ n\neq0,$$

$$U_{n0}(r_{N+1})=\langle\Phi_n(r_1.\dots.r_N)|\sum_{i=1}^{N}\frac{1}{r_{i,N+1}}|\Phi_0(r_1.\dots.r_N)\rangle, \tag{13}$$

$$V_{0v_2,0v_1}=0.$$

In the first order perturbation theory for a continuous spectrum[15] we obtain

$$\Psi_{0v}=\psi_{0v}+\sum_{n\neq0}\int\frac{V_{nv',0v}}{v^2/2-v'^2/2-\varepsilon_n}\psi_{nv'}d\frac{v'^2}{2}. \tag{14}$$

The summation over the excited states of the ion includes an integration over the continuous spectrum. The energies of the excited states of the ion are reckoned from the ground-state energy. Here the continuous-spectrum wave functions $F_v(r_{N+1})$ are normalized on the energy scale to a delta function.

Using wave functions (14), we obtain the dipole transition matrix element (similar formulas will be found in papers by Buĭmistrov and Trakhtenberg[11] and Amus'ya *et al.*[12]):

$$\left\langle\Psi_{0v_2}\left|\sum_{i=1}^{N+1}\mathbf{r}_i\right|\Psi_{0v_1}\right\rangle=\langle F_{v_2}(r_{N+1})\,|\mathbf{r}_{N+1}+\mathbf{D}(r_{N+1})\,|F_{v_1}(r_{N+1})\rangle, \tag{15}$$

$$\mathbf{D}(r_{N+1})=\sum_{n\neq0}\frac{2\varepsilon_n}{\omega^2-\varepsilon_n^{\,2}}U_{n0}(r_{N+1})\left\langle\Phi_n(r_1.\dots.r_N)\left|\sum_{i=1}^{N}\mathbf{r}_i\right|\Phi_0(r_1.\dots,r_N)\right\rangle. \tag{16}$$

Here $\mathbf{D}(r_{N+1})$ is the dipole moment vector resulting from the perturbation of the ionic electrons by the incident electron. As we have assumed the ground state of the ion to be spherically symmetric, the direction of $\mathbf{D}$ is the same as that of $\mathbf{r}_{N+1}$; we can therefore use the partial wave expansion of the wave functions $F_v(r_{N+1})$ to obtain formulas, analogous to formula (6), for the BS intensity with allowance for the perturbation of the ionic electrons. At large distances, when $r_{N+1}\gg R_i$, $\mathbf{D}$ is simply the polarization dipole moment of the ion: $\mathbf{D}=-\alpha(\omega)\mathbf{r}_{N+1}/r^3_{N+1}$, where $\alpha(\omega)$ is the dynamic polarizability of the ion. Of course formula (16) is valid only when the deviation $|\omega-\varepsilon_n|$ from resonance considerably exceeds the natural line width.

The matrix element of the momentum operator has the form

$$\left\langle\Psi_{0v_2}\left|\sum_{i=1}^{N+1}\mathbf{P}_i\right|\Psi_{0v_1}\right\rangle=\langle F_{v_2}(r_{N+1})\,|\mathbf{P}_{N+1}+\mathbf{P}(r_{N+1})\,|F_{v_1}(r_{N+1})\rangle. \tag{17}$$

$$\mathbf{P}(r_{N+1})=-i\sum_{n\neq0}\frac{2\omega\varepsilon_n}{\omega^2-\varepsilon_n^{\,2}}U_{n0}(r_{N+1})\left\langle\Phi_n(r_1.\dots.r_N)\left|\sum_{i=1}^{N}\mathbf{r}_i\right|\Phi_0(r_1.\dots,r_N)\right\rangle$$

Matrix elements (15) and (17) obviously satisfy relation (2a).

The transition matrix element calculated with wave functions (14) using the nuclear-force operator has the form

$$\left\langle\Psi_{0v_2}\left|\sum_{i=1}^{N+1}\frac{Z\mathbf{r}_i}{r_i^{\,3}}\right|\Psi_{0v_1}\right\rangle=\left\langle F_{v_2}(r_{N+1})\left|\frac{Z\mathbf{r}_{N+1}}{r^3_{N+1}}+\mathbf{f}(r_{N+1})\right|F_{v_1}(r_{N+1})\right\rangle. \tag{18}$$

$$\mathbf{f}(r_{N+1})=\sum_{n\neq0}\frac{2\varepsilon_n^{\,3}}{\omega^2-\varepsilon^{\ 2}}U_{n0}(r_{N+1})\left\langle\Phi_n(r_1.\dots.r_N)\left|\sum_{i=1}^{N}\mathbf{r}_i\right|\Phi_0(r_1.\dots,r_N)\right\rangle \tag{19a}$$

$$=-\sum_{n\neq0}2\varepsilon_nU_{n0}(r_{N+1})\left\langle\Phi_n(r_1,\dots,r_N)\left|\sum_{i=1}^{N}\mathbf{r}_i\right|\Phi_0(r_1,\dots,r_N)\right\rangle+\omega^2\mathbf{D}(r_{N+1}) \tag{19b}$$

Now we shall show that the transition matrix elements (15) for the dipole moment operator and (18) for the nuclear force operator as calculated with the perturbed wave functions (14) give the same value for the transition probability, i.e., satisfy relation (2). Taking Eq. (4) into account, we obtain the following expression for the matrix element (15) of the dipole moment operator:

$$\omega^2\langle\mathbf{r}_{N+1}+\mathbf{D}(r_{N+1})\rangle_{2.1}=\left\langle\frac{Z\mathbf{r}_{N+1}}{r^3_{N+1}}+\nabla_{N+1}U_{00}(r_{N+1})+\omega^2\mathbf{D}(r_{N+1})\right\rangle_{2.1}. \tag{20}$$

On comparing Eq. (20) with Eqs. (18) and (19b), we see that for the proof it will be necessary to show that the sum in (19b) is equal to $-\nabla_{N+1}U_{00}(r_{N+1})$, i.e., to the force on the incident electron due to its interaction with the ionic electrons, as calculated in the static approximation.

Let us calculate the sum in expression (19b) by a method analogous to that used in proving the sum rule for the oscillator strengths.[13] We have the following relation between the momentum operator of an ionic electron and the operator for the interaction of that ionic electron with the incident electron:

$$i\left(\mathbf{P}_i\frac{1}{r_{i,N+1}}-\frac{1}{r_{i,N+1}}\mathbf{P}_i\right)=-\frac{\mathbf{r}_{i,N+1}}{r^3_{i,N+1}},\quad \mathbf{r}_{i,N+1}=\mathbf{r}_i-\mathbf{r}_{N+1},\quad i\neq N+1. \tag{21}$$

The coordinates and momenta of different electrons commute, so from (21) we obtain

$$i\left(\sum_{i=1}^{N}\mathbf{P}_i\right)\left(\sum_{i=1}^{N}\frac{1}{r_{i,N+1}}\right)-i\left(\sum_{i=1}^{N}\frac{1}{r_{,N+1}}\right)\left(\sum_{i=1}^{N}\mathbf{P}_i\right)=-\sum_{i=1}^{N}\frac{\mathbf{r}_{i,N+1}}{r^3_{i,N+1}}. \tag{22}$$

On calculating the diagonal matrix elements of the left- and right-hand sides of Eq. (22) and making use of the completeness of the set of eigenfunctions of the ion, which satisfy Eq. (12), we find that

$$-2i\sum_{n\neq 0}\left\langle \Phi_n(r_1,\dots,r_N)\left|\sum_{i=1}^{N}\mathbf{P}_i\right|\Phi_0(r_1,\dots,r_N)\right\rangle$$

$$\times U_{n0}(r_{N+1})=-\nabla_{N+1}U_{00}(r_{N+1}). \tag{23}$$

From this we obtain

$$\sum_{n\neq 0}2\varepsilon_n\left\langle \Phi_n(r_1,\dots,r_N)\left|\sum_{i=1}^{N}\mathbf{r}_i\right|\Phi_0(r_1,\dots,r_N)\right\rangle U_{n0}(r_{N+1})=-\nabla_{N+1}U_{00}(r_{N+1}) \tag{24}$$

Consequently, the matrix elements (15) of the dipole operator and (17) of the nuclear force operator satisfy relation (2). In calculating the sum (24) we have used only the explicit forms of the operators and the Hamiltonian for the ionic electrons, together with the theorem that the eigenfunctions form a complete set; hence Eq. (14), like the sum rule for the oscillator strengths, is exact. Consequently, calculations in the first order of perturbation theory also lead exactly to the correct relation between the matrix elements of the dipole moment, momentum, and nuclear force operators.

Another important consequence of formulas (18), (19), and (24) is the fact that when we take the perturbation of the bound electrons into account and use the nuclear force operator we again arrive at the force due to the static potential. This is most clearly evident in the case of low-frequency BS (with $\omega\ll\varepsilon_1$, where ϵ_1 is the energy of the first excited state of the ion) emitted by an electron colliding with an ion, for in this case the quantity $\mathbf{f}$ in formula (18) reduces at once to the force on the incident electron due to its interaction with the ionic electrons as calculated in the static approximation (cf. (19a) and (24)). As we have said before, low-frequency photons are primarily emitted at large distances, and Z_i/r^2_{N+1}, the force due to the static field in formula (19b) in this case, is considerably larger than the quantity $\omega^2 D(r_{N+1})=\omega^2\alpha(0)/r^2_{N+1}$ ($\alpha(0)$ is the polarizability of the ion) at frequencies satisfying the condition $\omega^2\ll Z_i/\alpha(0)$ (this condition is actually equivalent to the condition $\omega\ll\varepsilon_1$). Again, therefore, we obtain formula (7a), not (7b). Thus, the formal use of the nuclear force operator within the limitations of the static approximation in accordance with formula (7b) leads to entirely incorrect results for the emission of low-frequency photons.

It is evident from formulas (15)–(19) that allowance for the perturbation of the bound electrons leads to the appearance of resonant frequencies in the BS process. The BS intensity rises considerably at emission frequencies close to the energies of the excited states of the atom or ion. This has previously been repeatedly noted.[16,11,12] In the immediate vicinity of a resonance frequency, formulas (15)–(19) represent just the emission of a photon from a state of the atom excited by the incident electron. The intensity of the radiation near a resonance[16] is entirely determined by a single resonance term in the sum (16). Burmistrov and Trakhtenberg[11] calculated the BS cross section for an electron on a hydrogen atom in the Born approximation and showed that allowance for the polarizability leads to an appreciable change in the BS intensity at frequencies of the order of the energy of the excited states but far from a resonance.

At high incident-electron energies, the contribution of the atomic electrons to the BS intensity at frequencies of the order of the incident-electron energy will decrease with increasing nuclear charge of the target atom. The emission by the atomic electrons is associated with the possibility of exciting one of the atomic electrons and is primarily a two-electron process; hence the intensity of the emission from the bound electrons does not change greatly when the nuclear charge of the target atom increases. However, the intensity of the direct radiation (associated with the vector $\mathbf{r}_{N+1}$ in formula (15)) rises rapidly. Indeed, let us calculate the contribution of the n-th excited state of the atom to the emission at frequencies $\omega\gtrsim\varepsilon_n$. Taking only one term of the sum into account in formulas (15) and (16) and using formula (1), we obtain[1)]

$$\frac{dW_n}{d\omega}=\frac{8\omega^4 d_{n0}{}^2\varepsilon_n{}^2}{3\pi c^3(\omega^2-\varepsilon_n{}^2)^2}\sigma_{n0}, \tag{25}$$

$$d_{n0}=\left\langle \Phi_n(r_1,\dots,r_N)\left|\sum_{i=1}^{N}\mathbf{r}_i\right|\Phi_0(r_1,\dots,r_N)\right\rangle,$$

$$\sigma_{n0}=\frac{v_2}{4\pi^2 v_1}\int|\langle F_{v_2}(r_{N+1})|U_{n0}(r_{N+1})|F_{v_1}(r_{N+1})\rangle|^2 d\theta_{v_2}. \tag{26}$$

Here the function $F_v(r_{N+1})$ is again normalized to a plane wave. When $v_2^2=v_1^2-2\varepsilon_n$, expression (26) is simply the distorted-wave approximation to the cross section for exctiation of the n-th atomic state. Calculating expression (26) in the Born approximation,[15]

$$\sigma_{n0}\approx\frac{8\pi}{3v_1{}^2}d_{n0}{}^2\ln v_1,$$

we obtain, for frequencies[2)] $\omega\gtrsim\varepsilon_n$,

$$\frac{dW_n}{d\omega}=\frac{64 d_{n0}{}^4\varepsilon_n{}^2}{9c^3 v_1{}^2}\ln v_1. \tag{27}$$

In analogy with Kramer's approximation,[13] the intensity of the direct BS is

$$\frac{dW_0}{d\omega} \approx \frac{16\pi Z_{\text{eff}}^2}{3^{3/2} c^3 v_1^2} \tag{28}$$

where Z_{eff} is the effective charge of the atom at the distance that contributes most to the BS component of frequency ω.

The ratio of the contribution (27) of the n-th excited state to the BS intensity (28) as calculated in the static approximation is

$$\frac{dW_n}{d\omega} \Big/ \frac{dW_0}{d\omega} \approx \frac{d_{n0}^4 \varepsilon_n^2 \ln v_1}{Z_{\text{eff}}^2} . \tag{29}$$

It is evident from Eq. (29) that the emission from bound states cannot be neglected in calculating the BS intensity from an electron on a neutral atom ($d_{n0}^2 \lesssim 1$, $\varepsilon_n \lesssim 1$) at frequencies of the order of the ionization potential since in this case the emission comes from distances $r_{N+1} \gtrsim 1$ where $Z_{\text{eff}} \lesssim 1$ and (as has already been noted elsewhere[11,12]) the radiation from the atom is more intense than the direct radiation from the incident electron.

The emission from the bound electrons can be neglected at emission frequencies of the order of the incident-electron energy ($\omega \sim v_1^2/2$). According to the Pauli principle, the electrons in a many-electron atom are mostly in states of large principal quantum number, so the effective charge of the self-consistent field rises rapidly on approaching the nucleus. Even at incident-electron velocities $v_1 \approx Z^{1/3}$ the effective charge of the atom at distances $\sim 1/v_1$, which contribute most to the BS, is $Z_{\text{eff}} \approx Z/2$ in the Thomas-Fermi approximation.[15] Hence the direct emission at frequencies $\omega \sim v_1^2/2$ is considerably more intense than the atomic emission when $Z \gg 1$. Relation (29) becomes even more favorable for the static approximation in the case of ions, sinze Z_{eff} rises as the charge of the ion is increased with the nuclear charge held constant. The contribution to the BS intensity from the discrete excited states of the atom is taken into account in formulas (25)–(29), and taking the continuous spectrum of excited states of the atom into account will apparently leave the qualitative conclusions unchanged.

5. CONCLUSION

For arbitrary relations between the incident-electron energy, the emission frequency, the energies of the excited states of the ion, the nuclear charge, and the number of bound electrons, one must take the vector dipole moment (16) due to disturbance of the bound electrons into account in formula (25) in order to obtain accurate results in calculating the BS intensity. The static approximation does not reproduce the "fine structure" of the BS spectrum associated with resonance enhancement of the BS at frequencies close to the energies of the excited states of the target atom or ion; it therefore only gives the BS intensity on the average at frequencies of the order of the incident-electron energy for collisions of high-energy electrons with atoms and ions and of electrons of any energy with multiply charged ions.

Investigation of the emission of low-energy photons in collisions of electrons with ions shows that in calculating the BS intensity in the one-electron static approximation the best results will be obtained by using the dipole-moment or momentum operator, i.e., by using formulas (7a). The same situation obtains in calculating oscillator strengths or photoionization cross sections: when approximate wave functions are employed the best results will be obtained, as a rule,[13] when using the dipole-moment or momentum operator, while using the nuclear force operator usually leads to unsatisfactory results.

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1) We note that formula (25) reduces at once to the Lorentz line shape formula for the case when $|\omega - \varepsilon_n| \ll \varepsilon_n$.

2) Similar estimates can be obtained by using results found in Refs. 11 and 12.

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