The collective-excitation spectrum of the Wigner electron crystal on a liquid-helium surface

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A self-consistent system of equations is proposed which describes the effect of the deformation of the surface of liquid helium that accompanies the Wigner crystallization of the surface electrons on the collective-excitation spectrum of this two-dimensional crystal. The solution of the proposed system of equations in the harmonic approximation leads to the discovery of a threshold frequency ω_0 of deformational origin in the spectrum of the longitudinal and transverse vibrations of the two-dimensional crystal. The dependence of ω_0 on the parameters of the problem (the clamping field $E_\perp$, surface-electron density n_s and temperature) is studied.

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INTRODUCTION

The phenomenon of crystallization of electrons in various low-density systems1) with a compensating positive background was predicted long ago by Wigner.[1] Wigner's ideas subsequently received wide theroetical development. However, there is still no clear-cut experimental confirmation of the existence of Wigner crystallization. The basic experimental problem has been to create suitable systems possessing the aggregate of properties necessary for the onset of crystallization—a low electron concentration, uniformity of the mean density, the presence of a compensating background, the absence of strong random perturbations, etc.

As suitable objects for this purpose it is currently proposed to use the two-dimensional electron systems that arise in the inversion layers near the surface of a semiconductor placed in a strong electric field,[2] or those which can be created artificially on the liquid-vapor surface of liquid helium.[3,4] In the second variant the electrons are supplied to a free helium surface from an external source and are held near the liquid-vapor boundary, which is for them a potential barrier of height 1 eV, by electrostatic image forces and also, when necessary, by an external electric field. The neutralization of the electron gas is achieved by introducing an additional metallic plate positioned at a macroscopic depth in the liquid helium. If L is the length of the electron system along the surface, h is the depth at which the metallic backing is submerged and λ is the length-scale of localization of the electrons on the helium surface, the conditions that favor the observation of collective properties in the two-dimensional electron system are established by the inequalities $L\gg h\gg\lambda$, which will be assumed to be fulfilled in the following.

From a formal point of view the two-dimensional elec-electron systems in the inversion layers of semiconductors and on a planar liquid-helium surface are equivalent. In particular, the spectra of the collective excitations of these systems have, in the disordered state, the same branch of longitudinal vibrations:

$$\bar\omega_l^2\left(1-\frac{i}{\bar\omega_l\tau}\right)=\frac{2\pi e^2 n_s}{m}q,\quad \bar\omega_l\tau\gg1, \tag{1}$$

where q is the wave number of the vibrations, m and e are the free-electron mass and charge, n_s is the mean density of the surface electrons, and τ is a characteristic relaxation time.

In the crystalline state, in addition to the longitudinal branch $\bar\omega_l(q)$ there is one branch of transverse oscillations, $\bar\omega_t(q)$, which, in the long-wavelength limit, is sound-like:

$$\bar\omega_t(q)=sq,\quad s\approx0.82e(ma)^{-1/2}, \tag{1a}$$

where a is the lattice constant; $a^{-2}\approx n_s$. The results (1) and (1a) for two-dimensional electron systems on a planar support are contained, e.g., in papers by Chaplik[2] and Crandall.[4] 2)

Besides the great similarity of the systems indicated, the liquid surface of helium possesses one important quality—it is atomically smooth, whereas the solid surface of a semiconductor contains, as a rule, various defects perturbing the electron system strongly and in a random manner. As a result, the observable properties of the electrons on a helium surface admit a very much clearer interpretation than is the case for the inversion electrons in semiconductors. In particular, owing to the weak interaction between the surface electrons and the thermal vibrations of the free helium surface, longitudinal weakly damped plasma oscillations with the spectrum (1) are already observed in the system of surface electrons in the regime of comparatively low surface-electron densities $n_s\gtrsim10^7$ cm^{-2} and temperatures $T\lesssim0.5$ K.[6]

The observation, first made by Grimes and Adams,[6] of two-dimensional plasma oscillations with the spectrum (1) in a system of electrons on a liquid-helium surface permits us to hope that, with lowering of the temperature into the region $T\ll0.5$ K, transverse oscillations, whose presence is a characteristic indicator of crystallization of the electrons, should also arise in this system. One must only keep in mind that the real spectrum of the longitudinal and transverse vibrations of the Wigner electron crystal on a helium surface should differ from the expressions (1) and (1a).

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The point is that in the presence on the liquid-vapor boundary of a lattice of electrons clamped to the surface from the gas-phase side by the strong electric field $E_\perp$ necessary for good occupation of the ground surface level, the boundary does not remain planar but is self-consistently deformed under each of the electrons localized on the helium surface.[7] The shape of this deformation possesses great inertia and, in the problem of the electron-vibration spectrum, can be assumed to be static. As a result, electrons vibrating about their equilibrium positions experience the influence not only of the self-consistent Coulomb forces but also of the deformational forces tending to return each electron to the center of its deformational potential well. The corresponding calculation, carried out earlier[8] in the hydrodynamic approximation, showed that a local deformation of the helium surface substantially alters the longitudinal-vibration spectrum of the system of surface electrons:

$$\omega_l^2(q)=\omega_0^2+\bar{\omega}_l^2(q), \tag{2}$$

where $\bar{\omega}_l$ is given by (1) and ω_0 is the characteristic vibration frequency of an electron in the deformational well. In the limit that the electron concentration tends to zero, $\omega_0=(eE_\perp)^2/2\pi\hbar\alpha$, where α is the surface-tension coefficient.

The more detailed theory of the vibrations of the Wigner electron crystal on a helium surface described in the present work makes it possible to study also the influence of deformation of the helium surface on the transverse branches of the electron-lattice vibrations. In addition, the question of the dependence of the threshold frequency ω_0 on n_s and T is investigated in the paper.

Basic definitions

A. We shall consider a system of surface electrons arranged on a square or triangular lattice on the liquid-vapor boundary and clamped to the helium surface by a field $E_\perp$. The potential energy of an individual electron above a deformed surface is changed (in comparison with the planar problem) by the quantity $V_\xi(\mathbf{r})$:

$$V_\xi(\mathbf{r})=eE_\perp\xi(\mathbf{r}), \tag{3}$$

where $\xi(\mathbf{r})$ is the deviation of the helium surface from the planar state and $\mathbf{r}$ is the two-dimensional radius vector along the surface.

A depression in the helium surface is induced by the pressure of the electrons on the surface:

$$P_e=eE_\perp n(\mathbf{r}). \tag{4}$$

Here $n(\mathbf{r})$ is the distribution of electron density over the helium surface. It is natural to assume that a deformation of the surface does not have time to adjust to the rapid vibrations of the electrons in the lattice and, consequently, the quantity $\xi(\mathbf{r})$ is determined by the average value $\langle n(\mathbf{r})\rangle$ of the electron density. The conditions under which such an approach is correct will be indicated below.

Following our previous paper,[7] we represent $\langle n(\mathbf{r})\rangle$ in the following form:

$$\langle n(\mathbf{r})\rangle=\frac{1}{\pi\langle\mathbf{u}^2\rangle}\sum_{\mathbf{l}}\exp\left[-\frac{(\mathbf{r}-\mathbf{l})^2}{\langle\mathbf{u}^2\rangle}\right], \tag{4a}$$

where $\mathbf{l}$ is the position vector of a certain lattice site, and $\langle\mathbf{u}^2\rangle$ is the mean-square displacement of an electron from its equilibrium position. For the moment the value of $\langle\mathbf{u}^2\rangle$ remains undetermined.

Deploying the definitions (4) and (4a), we can solve the problem of the deformation of the helium surface under the action of the pressure (4). The size of the dip $\xi(\mathbf{r})$ in the helium surface near a certain lattice site then turns out to be equal to[7]

$$\xi(\mathbf{l}+\mathbf{u})\approx\xi(\mathbf{l})+\frac{eE_\perp}{\pi\alpha\langle\mathbf{u}^2\rangle}\int_0^{|\mathbf{u}|}\frac{ds}{s}\int_0^s\exp\left(-\frac{x^2}{\langle\mathbf{u}^2\rangle}\right)x\,dx, \tag{5}$$

$0\leqslant|\mathbf{u}|\ll n_s^{-1/2}$. Here $\mathbf{u}$ is a two-dimensional radius vector measured from the center of the given lattice site and α is the surface-tension coefficient.

For displacements $\mathbf{u}$ satisfying the stronger inequality $\mathbf{u}^2\ll\langle\mathbf{u}^2\rangle$, the expression (5) takes the form

$$\xi(\mathbf{l}+\mathbf{u})\approx\xi(\mathbf{l})+eE_\perp\mathbf{u}^2/4\pi\alpha\langle\mathbf{u}^2\rangle. \tag{5a}$$

It must be remarked that the vibration amplitude of electrons localized at the lattice sites has the scale $\mathbf{u}^2\sim\langle\mathbf{u}^2\rangle$. This implies that, for the deformational part V_ξ (3) of the potential energy of the electrons, there is not, strictly speaking, a consistent harmonic approximation, since the corresponding expansion of $\xi(\mathbf{r})$ (5a) has a smaller region of applicability ($\mathbf{u}^2\ll\langle\mathbf{u}^2\rangle$) than is necessary from the point of view of a harmonic description of the dynamical properties of the electrons in the lattice. Nevertheless, aiming principally at qualitative results in the present paper, we shall confine ourselves to investigating the solution of the electron problem in the harmonic approximation, being guided in this by the following considerations.

1. It can be shown, using the more general expression (5) for $\xi(\mathbf{r})$, that the results of the harmonic approximation for the excitation spectrum of the two-dimensional electron crystal are qualitatively correct.

2. There exist well-known methods[9] for taking into account the effect of anharmonicities on the vibration spectrum of different periodic structures with mean-square displacements of relatively large amplitude. These methods will be used in the following to refine the details of the spectrum $\omega_p(q)$ of the Wigner electron crystal on a liquid-helium surface.

Thus, the set of requirements on the deformation potential $V_\xi(\mathbf{r})$ (3) consists of the condition that the deformation $\xi(\mathbf{r})$ be periodic, the possibility of using the adiabatic approximation (4a) for the calculation of $\xi(\mathbf{r})$, and the existence of the harmonic approximation, in which $V_\xi(\mathbf{r})$ has the form (3) with $\xi(\mathbf{r})$ from (5a).

B. Proceeding to the solution of the dynamical electron problem in the harmonic approximation, we write the total potential energy V of the mutual Coulomb interaction of the electrons, each of which sits in the well $V_\xi(\mathbf{r})$ (3), (5a), in the following form:

$$V=\frac{1}{2}\sum_{\mathbf{ll}'ij}A_{ij}(\mathbf{l}-\mathbf{l}')u_i(\mathbf{l})u_j(\mathbf{l}'),$$

$$A_{ij}(\mathbf{l}-\mathbf{l}')=A_{ij}^{0}(\mathbf{l}-\mathbf{l}')-(eE_\perp)^2\delta_{ij}\delta_{\mathbf{ll}'}/2\pi\alpha\langle\mathbf{u}^2\rangle. \tag{6}$$

The tensor $A_{ij}^{(0)}(\mathbf{l}-\mathbf{l}')$ is determined by the Coulomb interaction of the electrons. The explicit form of this tensor is given in the paper[4] by Crandall. It is worth noting that the tensor $A_{ij}^{(0)}(\mathbf{l}-\mathbf{l}')$ possesses the property

$$\sum_{\mathbf{ll}'ij}A_{ij}^{(0)}(\mathbf{l}-\mathbf{l}')=0.$$

At the same time,

$$\sum_{\mathbf{ll}'ij}A_{ij}(\mathbf{l}-\mathbf{l}')\neq 0.$$

This difference arises from the fact that in the formation of the crystal on a planar helium surface a displacement of all electrons by a constant vector $\mathbf{u}\ll\mathbf{a}$ leads to a displacement of the crystal as a whole with respect to the surface, without changing the total potential energy. But in the case of the crystal on a self-consistently deformed helium surface the analogous procedure of displacing all electrons by a small vector $\mathbf{u}\ll\mathbf{a}$ is accompanied by an increase of the deformational potential energy.

According to (6), the deformational part of $A_{ij}(\mathbf{l}-\mathbf{l}')$ arising from $V_\xi(\mathbf{r})$ turns out to be diagonal. This means that the problem of the vibrations of the lattice electrons above a deformed surface is reduced to the already-solved problem of the vibrations above a planar surface by making, in the equations of motion, the replacement

$$\omega^2\to\omega^2-\omega_0^2,\quad \omega_0^2=(eE_\perp)^2/2\pi\alpha m\langle\mathbf{u}^2\rangle. \tag{7}$$

Consequently, the required dispersion law has the following appearance:

$$\omega_p^2(q)=\omega_0^2+\bar{\omega}_p^2(q). \tag{8}$$

Here $\bar{\omega}_p(q)$ is the spectrum of the electron vibrations over a planar helium surface, as determined by formulas (1)–(1a); the index p denotes the polarization (l or t) of the spectrum. In other words, the deformation of the surface leads to the appearance of a threshold frequency ω_0 in both the longitudinal and the transverse branch of the spectrum.

The expression (8) contains the undetermined quantity $\langle\mathbf{u}^2\rangle$, appearing in the denominator of ω_0^2. To eliminate this indeterminacy we shall make use of a well-known relation connecting $\langle\mathbf{u}^2\rangle$ with the spectrum of a lattice of the given type:

$$\langle\mathbf{u}^2\rangle=\frac{\hbar}{mN}\sum_{pq}\frac{1}{2\omega_p(q)}\operatorname{cth}\frac{\hbar\omega_p(q)}{2T}. \tag{9}$$

Here N is the number of electrons and the index p denotes summation over the possible polarizations of the spectrum. When the spectrum (8) is substituted into the right-hand side of (9) an equation for $\langle\mathbf{u}^2\rangle$ arises, the solution of which completes the investigation of the question of the spectrum of the Wigner electron crystal on a helium surface in the harmonic approximation.

C. We shall consider the system of equations of motion of the surface electrons in the more general case of the deformation interaction $V_\xi(\mathbf{r})$ (3) with $\xi(\mathbf{r})$ from (5):

$$m\ddot{u}_i(\mathbf{l})=-\sum_{\mathbf{l}'j}A_{ij}^{0}(\mathbf{l}-\mathbf{l}')u_j(\mathbf{l}')-eE_\perp\frac{\partial}{\partial u_i}\xi(\mathbf{l}+\mathbf{u}), \tag{10}$$

where $A_{ij}^{0}(\mathbf{l}-\mathbf{l}')$ is given by (6). The question is: how does the dispersion law of the given system of equations of motion behave in the region of small wave numbers?

To answer this question we shall make use of the circumstance that, in the absence of the deformational interaction, the solution of the corresponding dynamical problem leads to the following long-wavelength asymptotic forms of the vibration spectrum of the electron crystal: $\bar{\omega}_l|_{q\to0}\propto q^{1/2}$, $\bar{\omega}_t|_{q\to0}\propto q$.[4] Thus, in the limit $q\to0$ in the system (10) we can neglect the Coulomb part of the problem, both for the longitudinal and for the transverse vibrations. As a result

$$m\ddot{u}_i\approx-\frac{(eE_\perp)^2}{\pi\alpha\langle\mathbf{u}^2\rangle}\frac{\partial}{\partial u_i}\int_0^{|\mathbf{u}|}\frac{ds}{s}\int_0^{s}\exp\left(-\frac{x^2}{\langle\mathbf{u}^2\rangle}\right)x\,dx,$$

$$\mathbf{u}^2=u_x^2+u_y^2. \tag{10a}$$

With the aid of the replacements

$$u_i^2\to\langle\mathbf{u}^2\rangle\tilde{u}_i^2,\ u^2\to\langle\mathbf{u}^2\rangle\tilde{u}^2,\ t\to\omega_0^{-1}\tau,$$

$$\omega_0^2=(eE_\perp)^2/\pi\alpha m\langle\mathbf{u}^2\rangle \tag{10b}$$

Eq. (10a) reduces to the dimensionless form

$$\frac{\partial^2\tilde{u}_i}{\partial\tau^2}=-\frac{\partial}{\partial\tilde{u}_i}\int_0^{\tilde{u}}\frac{ds}{s}\int_0^{s}\exp(-x^2)x\,dx.$$

This means that the scale of variation of u and ω_0 is determined, to within numbers of order unity, by the expressions (10b). It is clear that the structure of ω_0 from (10b) is analogous to that of ω_0 from (7). This agreement serves as an argument that the harmonic approximation is reasonable, at least for the long-wavelength limit or for vibrations with small dispersion.

Investigation of limiting cases

It is possible to obtain an analytic solution of Eq. (9) for $\langle\mathbf{u}^2\rangle$ only in certain limiting situations.

A. We shall consider first the case of low electron concentration and sufficiently low temperatures: $T\ll\hbar\omega_0$. Neglecting the dispersion of $\omega_p(q)$ in this limit and taking into account the asymptotic behavior $\coth(\hbar\omega_0/2T)\to1$, it is not difficult to obtain from (9) the relation

$$\langle\mathbf{u}^2\rangle\approx\hbar/m\omega_0. \tag{9a}$$

Substituting (9a) into the definition (7) of ω_0, we find an explicit expression for ω_0:

$$\omega_0(0)\approx(eE_\perp)^2/2\pi\hbar\alpha. \tag{11}$$

The result (11) corresponds to the characteristic frequency ω_0 of a single electron localized at the helium surface by the action of the clamping field $E_\perp$. The structure that arises (localized electron + surface deformation) has been dubbed a surface anion.[10] The characteristics of this structure (the localization energy $\delta\varepsilon$ and effective mass M) are finite provided that the capillary constant $\varkappa$ of liquid helium is finite:

$$M=\rho(eE_\perp)^2/16\alpha^2\varkappa,\quad \varkappa^2=\rho g/\alpha, \tag{11a}$$

where g is the acceleration due to gravity and ρ is the density of the helium.

The derivation of the relation (11) from the general equation (9) gives an indication of the correct asymptotic behavior of ω_0 as T and n_s tend to zero. Written out to the first nonvanishing terms in T and n_s, the expressions for ω_0^2 and $\langle u^2\rangle$ have the following appearance:

$$\omega_0(T,\,n_s)\approx\omega_0(0)\,(1-\delta),\quad \langle\mathbf{u}^2\rangle\approx\hbar\,(1+\delta)/m\omega_0(0),$$
$$\delta=2\exp\left(-\frac{\hbar\omega_0}{T}\right)-\frac{1}{6}\,\frac{\omega_l^2(q_m)}{\omega_0^2}-\frac{1}{8}\,\frac{\omega_t^2(q_m)}{\omega_0^2}\ll 1, \tag{11b}$$

where $\omega_0(0)$ is given by formula (11) and ω_p^2 by formula (8), and q_m is the maximum value of q: $q_m^2=\gamma^2 n_s$, $\gamma\approx 1$. The formulas (11b) generalize our results in[10] for ω_0 to the case of finite (low) temperatures and n_s.

B. It is interesting to note that the vibrations of a Wigner crystal composed of the electrons on a helium surface become dispersionless not only at small but also at large values of the density n_s. In fact, to produce a finite density of electrons on a helium surface it is necessary to use a clamping field $E_\perp$ of intensity

$$E_\perp\geqslant 2\pi e n_s. \tag{12}$$

Otherwise, the electrons, under the influence of their own field, will move away from the surface into the volume of the gas phase. It is of importance for the following that compensation of the external field $E_\perp$ by the intrinsic field of the electrons occurs over a distance of the order of the lattice constant, i.e., of the order of $n_s^{-1/2}$, from the helium surface. Consequently, if the localization length λ of the electron wavefunction in the direction of the gas phase satisfies the inequality $\lambda\ll n_s^{-1/2}$ (which is always the case), the surface electrons themselves are situated in an uncompensated electric field $E_\perp$.

Taking into account what has been said, we write down the condition for small dispersion for the longitudinal and transverse vibrations of the electron crystal:

$$\omega_0^2\gg\bar\omega_l^2(q_m),\quad \omega_0^2\gg\bar\omega_t^2(q_m). \tag{13}$$

Using the definition $q_m^2=\gamma^2 n_s$ ($\gamma\approx 1$), ω_0 from (11), and an intensity $E_\perp$ equal to its minimum value from (12), namely, $E_\perp=2\pi e n_s$, one can convince oneself without difficulty that, e.g., the inequality (13) for the longitudinal vibrations takes the form

$$n_s>n_s^*,\quad (n_s^*)^{5/2}=2\gamma\hbar^2\alpha/\pi m e^6. \tag{13a}$$

In other words, in the region $n_s>n_s^*$ the dispersion of the longitudinal vibrations can be neglected in comparison with ω_0. Using the numerical value $\alpha=0.36$ erg/cm^2, we find from (13a) the value of the limiting density above which the dispersion of the longitudinal vibrations becomes small: $n_s^*\approx 10^{11}$ cm^{-2}.

For the transverse vibrations the corresponding inequality (13) has, to within a constant of order unity, a form analogous to (13a). Thus, in the region $n>n_s^*$, as for $n_s\to 0$ also, the dispersion of the vibrations of the electron crystal on a helium surface disappears. The only difference is that in order to produce a finite value of ω_0 in the limit $n_s\to 0$ it is necessary to use a field $E_\perp\gg 2\pi e n_s$. But if $n\gtrsim n_s^*$ the dispersion already vanishes automatically for the minimum field $E_\perp=2\pi e n_s$ ensuring stability of the electron system on the helium surface.

C. To conclude the discussion of the low-temperature limit, we shall give the solution of Eq. (9) in the Debye approximation for arbitrary n_s and $T=0$. In this case the quantity $\langle u^2\rangle$ (9), represented in the form

$$\langle\mathbf{u}^2\rangle=\langle\mathbf{u}_l^2\rangle+\langle\mathbf{u}_t^2\rangle,$$

turns out, after replacement of the corresponding sums by integrals and use of the long-wavelength asymptotic forms of $\omega_p(q)$ (8), to be equal to

$$\langle\mathbf{u}^2\rangle=\frac{\hbar}{2\pi m n_s}\left\{s^{-2}[\omega_t(q_m)-\omega_0]+\frac{1}{3\sigma^2}\,(\sigma q_m-2\omega_0^2)\,\omega_l(q_m)+\frac{2}{3}\,\frac{\omega_0^2}{\sigma^2}\right\},$$
$$q_m^2=\gamma^2 n_s,\quad \gamma^2=24\pi/5,\quad \sigma=2\pi e^2 n_s/m,\quad s^2=e^2 n_s^{1/2}/m. \tag{14}$$

The value of q_m in the relation (14) has been chosen using the condition that Eq. (14) goes over into (9a) as $n_s\to 0$.

Combining (14) with the definition (7) of ω_0, we find an equation for ω_0. Written in dimensionless form, this equation has the following appearance:

$$\frac{4\pi}{5.4}\,x=\Omega^2\left\{\left(\Omega^2+\frac{\gamma^2}{5.4}\right)^{1/2}-\Omega+\frac{\gamma}{3\pi}\left(\Omega^2+\frac{2\pi\gamma}{5.4}\right)^{1/2}\right.$$
$$\left.-\frac{5.4}{3\pi^2}\,\Omega^2\left[\left(\Omega^2+\frac{2\pi\gamma}{5.4}\right)^{1/2}-\Omega\right]\right\},$$
$$\Omega=\omega_0/\omega_c,\quad x=\omega_0(0)/\omega_c,\quad \omega_c^2=5.4e^2 n_s^{2/3}/m; \tag{15}$$

its solution is

$$\Omega\cong\begin{cases} x, & x\gg 1\\ \Omega_{min}=1.88, & x\ll 1\end{cases}. \tag{15a}$$

The value of $\Omega_{\min}$ is determined by equating the expression in curly brackets in (15) to zero.

As the characteristic frequencies $\omega_0(0)$ and ω_c Eq. (15) contains the frequency $\omega_0(0)$ from (11) and also the frequency ω_c that arises in the problem of the vibrations of a given electron in a field of stationary neighbors. This specific choice of the characteristic frequencies $\omega_0(0)$ and ω_c has been made with the purpose of comparing the results for ω_0 that follow from the relation (15) with the definition of ω_0 used by us previously.[7] There, the calculation of the effect of the Coulomb interaction on the value of ω_0 was carried out, following,[3] in the static model, in which all electrons in the lattice, apart from the given one, are stationary. The corresponding expression for Ω from[7] has the form

$$\Omega=\frac{1}{2}x+\left(\frac{1}{4}x^2+1\right)^{1/2},\quad \Omega\approx\begin{cases}x, & x\gg1\\ 1, & x\ll1\end{cases}. \tag{16}$$

Thus, the determinations (15a) and (16) of Ω are qualitatively the same over the whole range of x.

The real parameters determining the variation of the frequency ω_0 are the intensity $E_\perp$ and surface electron density n_s. Here it is necessary to remember that in the region of small $n_s \ll n_s^*$ the magnitude of $E_\perp$ can appreciably exceed $2\pi e n_s$. But in the region $n_s \sim n_s^*$ it is technically difficult to violate the equality $E_\perp \approx 2\pi e n_s$. Therefore, to obtain a complete picture of the dependence of ω_0 on $E_\perp$ and n_s it is natural to take for the characteristic clamping field $E_\perp$ the expression $E_\perp = E_\perp^0 + 2\pi e n_s$, where $E_\perp^0$ is the clamping field for $n_s \to 0$, and supplement the expressions for Ω in terms of x by the dependence of x on n_s and $E_\perp^0$:

$$x=\frac{\omega_0(0)}{\omega_c}=\frac{em^{1/2}(E_\perp{}^0+2\pi e n_s)^2}{2\pi\cdot 5.4^{1/2}\hbar\alpha n_s^{3/4}}. \tag{17}$$

According to (17), the quantity x in the region of small n_s and finite values of $E_\perp^0 \gg 2\pi e n_s$ increases without limit: $x \propto n_s^{-3/4}$. With increase of n_s the value of x decreases, passes through a minimum and, in the region of large values of n_s, again begins to grow, like $x \propto n_s^{5/4}$. When the asymptotic form $\Omega(x) \to x$ as $x \to \infty$ is taken into account, this behavior of x with monotonic increase of n_s is evidence of a relative increase in the role of the threshold frequency in the dispersion law of the collective vibrations of the electron crystal on a helium surface at small and large values of n_s. This conclusion correlates with the phenomenon, noted in the preceding section, of the disappearance of dispersion in $\omega_p(q)$ at small and large values of n_s.

D. In the region of finite temperatures it is possible to obtain an explicit expression for $\langle \mathbf{u}^2\rangle$ only in the limit $\omega_0 \ll \bar{\omega}_p(q_m)$ $(p=l, t)$, $\omega_0 \ll T$. The first two inequalities permit us in the calculation of the integrals from (9) in the Debye approximation to neglect the frequency ω_0 everywhere except in the contribution to $\langle \mathbf{u}^2\rangle$ from the transverse vibrations, which diverges logarithmically at the lower limit of the frequencies. As a result,

$$\langle \mathbf{u}^2\rangle=\langle \mathbf{u}_l{}^2\rangle+\langle \mathbf{u}_t{}^2\rangle, \tag{18}$$

$$\langle \mathbf{u}_l{}^2\rangle=\frac{b^{1/2}}{6\pi(2\pi)^{1/2}}n_s^{-3/4}+\frac{\zeta(3)}{2\pi^3 b n^{3/2}}\left(\frac{T}{V_c}\right)^3,\quad V_c=e^2n_s^{1/2}, \tag{18a}$$

$$\langle \mathbf{u}_t{}^2\rangle=\frac{\hbar\omega_t(q_m)}{4\pi m n_s s^2}+\frac{T}{2\pi m n_s s^2}\ln\frac{1-\exp[-\hbar\omega_t(q_m)/T]}{1-\exp(-\hbar\omega_0/T)}, \tag{18b}$$

where b is the Bohr radius $b=\hbar^2/me^2$, $\zeta(x)$ is the Rieman ζ-function and s is the sound velocity from (1a).

Using now the assumption $\omega_0 \ll T$ and also the relation (7) between ω_0 and $\langle \mathbf{u}^2\rangle$, we bring Eq. (18) to the form

$$\begin{gathered} z=\nu+\ln z,\\ z=\langle \mathbf{u}^2\rangle/u_T{}^2,\quad u_T{}^2=T/4\pi m s^2 n_s,\quad \omega_T{}^2=\frac{(eE_\perp)^2}{2\pi\alpha m u_T{}^2},\\ \nu=\frac{\langle \mathbf{u}_l{}^2\rangle}{u_T{}^2}+\frac{\hbar s q_m}{T}+2\ln\left[\frac{T}{\hbar\omega_T}\left(1-\exp\left(-\frac{\hbar s q_m}{T}\right)\right)\right]. \end{gathered} \tag{19}$$

This equation has two roots ($z_1>1$ and $z_2<1$), but only one of them ($z>1$) satisfies the inequality $\hbar\omega_0 \ll T$ used above. In this limit the number ν is large and, therefore, to solve Eq. (19) we can make use of the method of successive approximations.

For the k-th approximation we obtain the formula

$$\langle \mathbf{u}^2\rangle_k=z^{(k)}u_T{}^2, \tag{20}$$

where $k=0, 1, 2, \ldots$; $z^{(0)}=\nu$; $z^{(k)}=\nu(1+\varepsilon_1)\ldots(1+\varepsilon_k)$;

$$\varepsilon_k=\ln\nu/\nu^k\ll1,\quad \text{if}\quad k\geqslant1.$$

For $\nu \gg 1$ we can confine ourselves to the approximation with $k=1$:

$$\langle \mathbf{u}^2\rangle\approx u_T{}^2(\nu+\ln\nu). \tag{20a}$$

The corresponding expression for ω_0 is obtained by substituting (20) or (20a) into formula (7).

Passing on to the discussion of the results of this section we note first of all the absence of a logarithmic divergence in the expression (20a) for $\langle \mathbf{u}^2\rangle$. The divergence in $\langle \mathbf{u}^2\rangle$ that is characteristic for two-dimensional systems and arises on account of the contribution of the long-wavelength transverse vibrations to $\langle \mathbf{u}^2\rangle$ is cut off in the present case at the finite value ω_0, as a result of which the expression for $\langle \mathbf{u}^2\rangle$ turns out to be finite and independent of the size of the system. It is necessary, however, to bear in mind that the absence of this divergence is in fact a consequence of the approximation chosen in the present paper, in which the shape of the deformation of the helium surface is static and perfectly periodic. In reality, for very low frequencies the assumption that the deformation is static is no longer reasonable. In this region of frequencies an electron and its deformational well begin to move self-consistently as a certain new quasi-particle (a surface anion) with a strongly renormalized (in comparison with the electronic) effective mass M. The scale of this mass can be estimated in analogy with the determination of M (11a), in which the capillary length is replaced by the spacing between the electrons in the lattice:

$$M\approx\rho(eE_\perp)^2/16\alpha^2 n_s^{1/2}. \tag{21}$$

As a result, in addition to the optical vibrations with the spectrum (8) the lattice of electrons on a helium surface should possess a very soft branch of acoustic vibrations, having the dispersion law (1), (1a) with effective mass M from (21). The presence of acoustic vibrations in the system of surface anions leads to the usual logarithmic divergence of the magnitude of the mean-square displacements of the surface anions about their equilibrium positions.

Unfortunately, the concept of the effective mass of a surface anion positioned at a lattice site does not have a well-defined meaning (in comparison with the definition of M from (11a) for a single anion), since in the lattice problem the surface anions touch each other and, consequently, interact strongly with each other. In view of this the expression (21) for M can be used only as an estimate for arriving at various suggestive considerations. In particular, the possibility of subdividing the problem of the calculation of the vibration spectrum of the electron crystal on a helium surface into an optical and an acoustic part is based on using the fact that the parameter $m/M \ll 1$ is small. In reality this ratio is of order $m/M \lesssim 10^{-5}$.

Here it is already pertinent to discuss the second adiabaticity condition, which permits us to define the electron pressure on the helium surface in the form (4), (4a). In order that the helium surface respond only to the mean electron density, it is necessary that the inequality

$$\omega_e \gg \omega_\xi, \tag{22}$$

which requires that the characteristic frequency ω_e of an electron at a site be greater than the characteristic frequency ω_ξ of capillary waves with wavelength $\lambda \sim \langle u^2 \rangle^{1/2}$, be fulfilled. This choice of scale for the capillary wavelength follows from the definition (5) for $\xi(\mathbf{r})$. Taking into account the capillary-wave dispersion law $\omega^2 = \alpha\rho^{-1}q^3$, where q is the wave number of the vibrations, and noting that the bound $\omega_e \gtrsim \hbar/m\langle u^2 \rangle$ is valid for the characteristic electron frequency (the sign > refers to the high-temperature case), we find from (22) the relation

$$\langle u^2 \rangle \ll \hbar^2\rho/m^2\alpha. \tag{22a}$$

This inequality is fulfilled up to $\langle u^2 \rangle^{1/2} \lesssim 10^{-1}$ cm, i.e., in the whole region 10^6 cm$^{-2} \lesssim n_s \lesssim 10^{10}$ cm^{-2} of interest to us.

Among the results of this section it is worth noting also the expression (18a) for $\langle u_l \rangle^2$. Using this expression we can ascertain in which temperature region for a given density n_s there exist only longitudinal vibrations with the spectrum (1) and determine where we may expect transverse vibrations and a threshold frequency ω_0 to appear as the temperature is lowered. This estimate is of real interest in connection with the experiments of[6], which make it possible to detect longitudinal vibrations in the system of surface electrons.

The boundary temperature separating the regions of purely longitudinal vibrations and the region in which transverse vibrations may possibly appear is determined by the order-of-magnitude relation

$$\langle u_l^2 \rangle \lesssim n_s^{-1}. \tag{23}$$

Neglecting the contribution of the zero-point vibrations in the expression (18a) for $\langle u_l^2 \rangle$ and substituting the simplified expression for $\langle u_l^2 \rangle$ into (23), we find an expression for the required boundary temperature T^*:

$$T^* \approx V_c \left(\frac{2\pi b n_s^{1/2}}{\zeta(3)}\right)^{-1/3}. \tag{23a}$$

Substituting into (23a) the density n_s used in[6], it is not difficult to estimate the corresponding temperature T^*: $n_s \approx 10^8$ cm^{-2}, $T^* \lesssim 0.1$ K. This temperature is appreciably below the temperatures $T \lesssim 0.5$ at which the measurements in[6] were carried out.

CONCLUSION

We shall summarize some of the results. One of the necessary conditions for the existence of a Wigner crystal is the requirement $\langle u^2 \rangle \ll n_s^{-1}$. In the case of the formation of an electron crystal on a liquid-helium surface this same requirement is sufficient for the appearance of deformation effects due to the local pressure on the surface by the electrons located at the lattice sites and clamped to the free surface by a strong electric field.

The onset of deformation of the helium surface leads to a number of observable consequences, the most interesting of which is the appearance of a threshold frequency in the spectrum of both the longitudinal and the transverse vibrations of the crystal. In the present paper a self-consistent system of equations is proposed making it possible to display the effect of the deformation of the helium surface on the spectrum of the electron crystal on the liquid-vapor interface of the liquid helium. Solving this system of equations in the harmonic approximation gives a determination of a number of quantitative characteristics of the threshold frequency ω_0 in the electron spectrum. In particular, we have shown that the dispersion of the vibrations of the Wigner crystal becomes small not only in the region of low surface electron density but also in the region $n_s > n_s^*$, where n_s^* is given by (13a). It has been noted that in conditions of small dispersion the solution in the harmonic approximation of the problem of the spectrum of the Wigner crystal on a self-consistently deformed helium surface has not only qualitative but also quantitative significance.

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1)An electron system has low density if the distance between the electrons in the system is much greater than the electron Bohr radius.

2)It should be noted that in the determination of the spectrum of the transverse vibrations of a two-dimensional Wigner crystal in[2,4] the transverse electric fields that arise on excitation of the transverse vibrations in the electron crystal were not taken into account, and this, in the general case, is inaccurate. As shown in[5] using the example of a three-dimensional Wigner crystal, allowance for these fields appreciably alters the transverse-vibration spectrum of the crystal in the long-wavelength approximation. However, the corresponding calculation carried out by analogy with[5] for a two-dimensional crystal (the actual calculations are omitted) shows that if the surface electron density $n_s \lesssim 10^{10}$ cm^{-2} and the size of the system $L \lesssim 1$ cm, the effect of the transverse electric fields on the structure of the transverse-vibration spectrum can be neglected in the long-wavelength approximation.

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