Scale invariance and percolation in a random field

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The problem of percolation in a system of sites is examined by the method of scaling transformation. The position of the percolation level and the character of the variation of the correlation length (the critical exponent ν) are found.

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The problem of percolation in a system of sites in a plane can be formulated in the following way: Let us suppose that the entire plane is marked off into squares of side a. Each square is conducting with probability p or non-conducting with probability $1-p$. It is required to find the critical value $p=p_c$ at which the conducting regions form channels which go to infinity. That is, we are interested in that minimum value $p=p_c$ at which the system begins to conduct as a unit. [1]

The problem formulated above, if $a\to 0$, is equivalent to the problem of percolation in a random field. [2–4] In order to find p_c, let us proceed as follows. Let us change over to squares of side $2a$. A square of side $2a$ will conduct in a direction perpendicular to one of its sides with probability

$$p'=1-(1-p^2)^2. \tag{1}$$

In this case, such a square will be considered conducting. Near the percolation level, linear channels of percolation play a fundamental role. Each square of side $2a$ which strikes such a "direct" channel will again conduct. In other words, if a percolation channel existed in the system of squares of side a, then it will be preserved in the system of squares of side $2a$.

We can further go from squares of side $2a$ to squares of side $4a$. Each square of side $4a$ will be regarded as consisting of four squares of side $2a$. Then the probability p'' that a square of side $4a$ will conduct is again given by the left side of expression (1), but, instead of p on the right side, we must put p'.

If $p'>p$, then also $p''>p'$ and consequently, continuing the described process, we will eventually arrive at a square of side $2^n a$ $(n\to\infty)$, which will conduct with probability 1.

The presence or absence of a percolation channel in the system is preserved at each step. Therefore, at $p'>p$, there was a percolation channel in the system of original squares. Analogously, if $p'<p$, there was no percolation channel. We arrive in this way at a relationship for p_c:

$$p_c=1-(1-p_c^2)^2 \tag{2}$$

In other words, when $p=p_c$, the system does not change under the scaling transformation (1). At the point of formation of a percolation channel the system is scale-invariant.

Relationship (2) is easily generalized for the case of an arbitrary dimensionality d:

$$p_c=1-(1-p_c^2)^{2^{d-1}} \tag{3}$$

The solution of (3) is given in Fig. 1.

The construction described above is analogous to the construction of Kadanoff[7] in the theory of second-order of phase transitions. We note that a relationship analogous to (2) was given earlier[8] for the problem of bonds in a plane. It seems unclear, however, how the relationship of[8] can be generalized for the case $d>2$.

In the derivation of Eq. (2) we actually took into account only rectilinear percolation channels. Let us change now in the manner indicated above from squares of side a to squares of side $2a$, and then to squares of side $4a$. Under such a transition, the square in Fig. 2 will be considered conducting as a unit, while the square in Fig. 3 will be considered nonconducting. In reality, however, the situation is just the opposite. In order to correctly take into account such cases, let us introduce a new quantity $1-\alpha$, the probability that the configuration represented in Fig. 2b conducts in the vertical direction (the probability of contiguity). Let us likewise introduce β, the probability that the system in Fig. 3b is conducting (the probability of conducting along the diagonal).

Under the scaling transformation, the quantities α and β will change along with p. After straightforward but cumbersome calculations we get

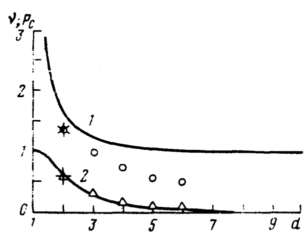

FIG. 1. Curve 1—dependence of the critical exponent ν on the dimensionality of space (calculations on the basis of (3)): *—value of ν, calculated from (5); o—results of the numerical calculations of ν[5] ($d=2$; 3), [6] ($d=4$; 5; 6). Curve 2—dependence of the percolation level on the dimensionality of space (calculations from relationship (3)): +—p_c calculated with the help of (4); Δ—the results of numerical calculations of p_c: ($d=2$; 3), [5] ($d=4$; 5; 6). [6]

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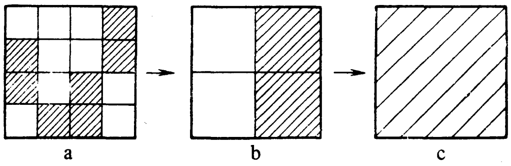

FIG. 2. Transition from an initial system of conducting squares to an effective system under the scaling transformation (3) (the conducting squares are cross-hatched); the square as a whole conducts.

$$p'=2p^2-p^4-2\alpha p^2(1-p^2)+2\beta p^2(1-p)^2+4\alpha\beta p^3(1-p)-\alpha^2 p^4(1-\beta)^2; \tag{4a}$$

$$\alpha'=2(p^2/p')^2\{(1-p)^2-\alpha(1-p)(1-7p+4p^2)+\beta(1-p)^2(1-2p)+2\alpha\beta(1-p)(1-p-3p^2+4p^3)-\alpha^2(1+12p-33p^2+22p^3-3p^4)-\beta^2(1-p)^3(1+p)-\alpha^2\beta(3-28p+45p^2-10p^3-9p^4)+\alpha\beta^2(1-p)^2(3-10p+5p^2)+\alpha^3(1+8p-44p^2+50p^3-13p^4)-\beta^3(1-p)^4\}; \tag{4b}$$

$$\beta'=(p^2/(p'(1-p')))^2\{(1-p)^2(1-p^2)^2(p+p^2-p^3)(2+5p-3p^2-p^3)-2\alpha p(1-p)^3(4+19p-2p^2-74p^3-32p^4+67p^5+27p^6-19p^7-4p^8)+\beta(1-p)^4(3+18p+18p^2-36p^3-59p^4+10p^5+50p^6-2p^7-10p^8)\}. \tag{4c}$$

The change of the system of conducting squares under the scaling transformation (4) is illustrated in Fig. 4. In relationship (4b) only terms up to cubic in α and β are considered, while in (4c) only terms linear in α and β are considered. This is justified by the numerical smallness of the discarded terms.

The system (4) describes the change of p, α, and β on going from squares of side $2^{n-1}a$ to squares of side $2^n a$. Going from squares of side a to squares of side $2a$, it is necessary to set $\alpha=\beta=0$. With subsequent iterations either $p\to 0$ or $p\to 1$. The presence or absence of a percolation channel in the system is preserved at each step of the scaling transformation (4). Consequently, there was no percolation channel in the first case and there was a channel in the second. The stationary point of the transformation (4)

$$p'(p^*, \alpha^*, \beta^*)=p^*, \quad \alpha'(p^*, \alpha^*, \beta^*)=\alpha^*, \quad \beta'(p^*, \alpha^*, \beta^*)=\beta^*,$$

$p^*=0.5182$; $\alpha^*=0.1274$; $\beta^*=0.6513$ determines the critical probability p_c.

Near the percolation level ($p<p_c$), the correlation length ξ (the mean dimension of the conducting region) behaves like $\xi\sim(p_c-p)^{-\nu}$. It is well known[9] that the critical exponent ν is determined by the largest eigenvalue λ_{max} of the matrix of the linearized scaling transformation (4)

$$\left.\frac{\partial(p', \alpha', \beta')}{\partial(p, \alpha, \beta)}\right|_{p=p^*;\ \alpha=\alpha^*;\ \beta=\beta^*}. \tag{5}$$

As a result, we have $\nu=\ln 2/\ln\lambda_{max}=1.3267$. If the eigenvector corresponding to this eigenvalue is known, then, following Niemeyer and van Leeuwen,[10] it is possible to calculate $p_c=0.5872$.

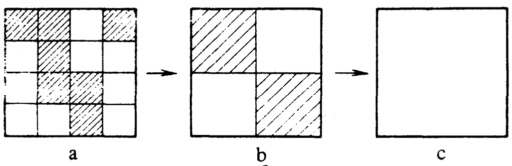

FIG. 3. Transition from an initial system of conducting squares to an effective system of conducting squares under the scaling transformation (3) (the conducting squares are crosshatched); the square as a whole does not conduct.

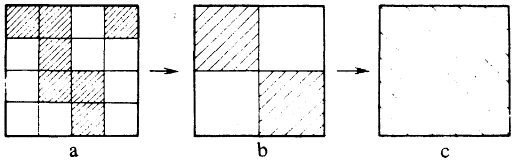

FIG. 4. Transition to an effective system of conducting squares under the scaling transformation (4) (the conducting squares are crosshatched).

In the transformation (4) the possibilities of percolation "around a square," which arise in connection with succeeding iterations, are not taken into account. Such an approximation is analogous to the well-known approximation of local interaction for an Ising system.[9]

In Fig. 1 is also shown the critical exponent ν for $d>2$, calculated on the basis of relationship (3). As $d\to\infty$, the exponent ν found in this way approaches 1, which contradicts the results of numerical calculations.[6] This, apparently, is connected with the greater role assumed by non-rectilinear percolation channels with increasing dimensionality of space. To find relations analogous to (4) and taking properly into account such channels for $d>2$ calls for cumbersome calculations. With the method developed it is possible to calculate also other critical exponents in the problem of percolation in a random field.

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[1]B. I. Shklovskiĭ and A. L. Éfros, Usp. Fiz. Nauk. **117**, 401 (1975) [Sov. Phys. Usp. **18**, 845 (1976)].
[2]A. M. Dykhne, Zh. Eksp. Teor. Fiz. **59**, 111, (1970) [Sov. Phys. JETP **32**, 63 (1971)].
[3]A. S. Skal, B. I. Shklovskiĭ, and A. L. Efros, Pis'ma Zh. Eksp. Teor. Fiz. **17**, 522 (1973) [JETP Lett. **17**, 377 (1973)].
[4]A. N. Lagar'kov and A. K. Sarychev, Zh. Eksp. Teor. Fiz. **68**, 641 (1975) [Sov. Phys. JETP **41**, 317 (1975)].
[5]M. E. Levinshteĭn, B. I. Shklovskiĭ, M. S. Shur, and A. L. Éfros, Zh. Eksp. Teor. Fiz. **69**, 386 (1975) [Sov. Phys. JETP **42**, 197 (1976)].
[6]S. Kirkpatrick, Phys. Rev. Lett. **36**, 69 (1976).
[7]L. P. Kadanoff, Physics **2**, 263 (1966).
[8]A. P. Young and R. B. Stinchcombe, J. Phys. C **8**, L535 (1975).
[9]K. G. Wilson and J. Kogut, Phys. Rep. **2C**, 75 (1974).
[10]Th. Niemeyer and J. M. J. van Leeuwen, Physica **71**, 17 (1974).

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