Parametric interaction of random waves in nonlinear nondispersive media

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The propagation of intense random waves in nondispersive media with high-frequency absorption is considered. The exact solution of the Burgers equation is used to determine the spectra of random waves at sufficiently large distances. The nonlinear interaction between spectral components is shown to lead to an increase in the energy of the long-wave part of the spectrum. The interaction of sinusoidal and quasimonochromatic waves with broad-band noise is investigated in detail. It is shown that the long-wave part of the resulting spectrum carries information both about the spectrum of the broad-band noise and the parameters of the regular signal, so that measurements on this part of the spectrum can be used to analyze the spectra of the interacting fields.

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INTRODUCTION

There are many problems in hydromechanics, acoustics, radiophysics, and so on, that involve the phenomena occurring during the propagation of high-intensity noise through a nonlinear medium. It is well known that the nonlinear interaction between the spectral components of a random wave leads to the appearance of new components in the spectrum and, if the medium is nondispersive, so that the velocities of all the harmonics are the same, the nonlinearity of the medium may lead to an appreciable modification of the spectrum of the initial wave and to a distortion of its shape.[1–4] The general method for analyzing the spectra of noise waves in nonlinear nondispersive media in which the field propagation is described by the equation of a simple wave was developed in[5–8]. However, the initial profile eventually becomes steeper, and large field gradients appear, so that dissipative effects, which stabilize the shape of the shock front and lead to the attenuation of the wave, begin to play an important role in the wave propagation process. In media with high-frequency absorption, the propagation of finite-amplitude waves is described by the Burgers equations[1–3,9]

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\mu\frac{\partial^2 v}{\partial x^2}, \tag{1}$$

where, for example, in acoustic problems, $v(x,t)$ is the oscillatory velocity and μ represents the dissipation. Depending on the formulation of the problem, the variable t is either the time or the distance between the point of entry into the nonlinear medium and the point of observation, whereas x is, correspondingly, either the distance or the time. The problem of non-Kolmogorov acoustic turbulence was considered on the basis of (1) in[10,11] and the statistical characteristics of quasimonochromatic waves propagating in a nonlinear medium were analyzed in[12,13]. These analyses were confined to the case when $\mu\to 0$ and the initially continuous wave transformed into a sequence of triangular pulses. It is physically clear that when μ is finite, high-frequency absorption will eventually ensure that the wave will follow linear propagation, and the field energy will be concentrated in the long-wave spectrum components. On the other hand, the spectrum and its intensity will then be determined by the initial stage of wave propagation, so that nonlinear effects resulting in a parametric transfer of energy to the low-frequency part of the spectrum will predominate over dissipative effects. In this paper, we investigate the dependence of the asymptotic shape of the wave spectrum for large times on the statistical parameters of the initial field. In particular, we shall consider the interaction of broad-band noise with a regular wave. It will be shown that, owing to the nonlinear interaction between the signal and the noise, the long-wave part of the spectrum carries information both about the noise and about the parameters of the regular signal.

§1. ASYMPTOTIC CHARACTERISTICS OF NONLINEAR RANDOM WAVES

1. We shall use the exact solution of the Burgers equation[14] to determine the statistical characteristics of the random field $v(x,t)$. The required field is then given in terms of the solution of the linear diffusion equation

$$v(x,t)=\frac{\partial}{\partial x}\tilde z(x,t)=-2\mu\frac{\partial}{\partial x}\ln U(x,t), \tag{2}$$

$$\partial U/\partial t=\mu\partial^2 U/\partial x^2. \tag{3}$$

We shall suppose that the field

$$\tilde z_0(x)=\int_{-\infty}^{x} v_0(x')\,dx'$$

is statistically homogeneous in x and has a finite variance D^2. The latter condition is satisfied if the initial field $v_0(x)$ is also statistically homogeneous and its spectrum is proportional to k^α, $\alpha>1$ as $k\to 0$. Transforming to the dimensionless variable $z(x)=\tilde z_0(x)/D$, we can write the initial conditions for the diffusion equation (3) in the form

$$U(x,0)=\exp\{-\mathrm{Re}\cdot z(x)\}, \tag{4}$$

where $\mathrm{Re}=d/2\mu$ and can be interpreted as the acoustic

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Reynolds number characterizing the relative influence of nonlinear and dissipative effects on the wave propagation process. In view of the statistical homogeneity of $U(x,0)$, the mean field $\langle U\rangle$ will be independent of time. We now introduce the relative fluctuations $\eta(x,t)$ of the field U through the equation

$$U(x,t)=\langle U\rangle(1+\eta(x,t)), \quad (5)$$

and write the solution of the Burgers equation in the form of a series in powers of η:

$$\begin{aligned} v(x,t)&=-2\mu\frac{\partial}{\partial x}\ln\langle U\rangle(1+\eta(x,t))\\ &=-2\mu\frac{\partial}{\partial x}\left\{\eta(x,t)-\frac{1}{2}\eta^2(x,t)+\frac{1}{3}\eta^3(x,t)-\dots\right\}. \end{aligned} \quad (6)$$

The variance $\sigma_\eta^2(t)$ will decrease in the course of time because diffusion will smooth out the inhomogeneities of the initial profile $\eta_0(x)$. To determine the asymptotic properties of the random field, we can therefore restrict our attention to the first term in (6):

$$v(x,t)=-2\mu\partial\eta(x,t)/\partial x. \quad (7)$$

Since $\eta(x,t)$ satisfies the linear diffusion equation, the formula given by (7) is the solution of the Burgers equation for times for which the wave $v(x,t)$ has reached linear propagation, and nonlinear effects which initially give rise to the distortion of the wave profile and to a modification of its spectral composition, reduce to the single fact that the initial conditions for the Burgers equation and the linear diffusion equation are related through the nonlinear transformation (4). Thus, for times for which (7) is valid, the spectrum of the wave $v(x,t)$ satisfying the Burgers equation is given by

$$S_v(k,t)=4\mu^2k^2S_\eta(k,0)e^{-2\mu k^2t}, \quad (8)$$

where $S_\eta(k,0)$ can be expressed in terms of the two-dimensional characteristic function of the process $z(x)$ [Eqs. (4), (5)]:

$$S_\eta(k,0)=\frac{1}{2\pi\Theta_z^2(i\,\mathrm{Re})}\int_{-\infty}^{+\infty}\{\Theta_z(i\,\mathrm{Re},i\,\mathrm{Re};\rho)-\Theta_z^2(i\,\mathrm{Re})\}e^{ik\rho}d\rho. \quad (9)$$

Consequently, to determine the asymptotic behavior of the spectrum of a high-intensity wave, we must know the two-dimensional characteristic function of the process $z(x)$ or, using the connection between $v_0(x)$ and $z(x)$, the characteristic functional of the initial field. In other words, to determine the asymptotic characteristics of the wave, we must have the complete statistical information about the initial field, and it is only for small Reynolds numbers, when nonlinear effects can be neglected, that the spectrum of the wave is linearly related to the spectrum of the initial field.

2. For Gaussian statistics of the initial field $v_0(x)$, the statistical characteristics $U_0(x)$ are completely determined by specifying the spatial spectrum $S_{v0}(k)$ of the profile $v_0(x)$. In view of (2), the correlation function for the process normalized to the dispersion, $z=\tilde{z}_0/D$, is given by

$$B_z(\rho)=\frac{1}{D^2}\int_{-\infty}^{+\infty}\frac{S_{v_0}(k)}{k^2}e^{ik\rho}dk,\qquad D^2=\int_{-\infty}^{+\infty}\frac{S_{v_0}(k)}{k^2}dk. \quad (10)$$

Bearing in mind the Gaussian properties of the field $z(x)$, and using (9), we find that the spectrum $S_\eta(k,0)$, which determines the asymptotic form of the spectrum of the wave for $t\to\infty$, and the variance $\sigma_\eta^2(t)$ are, respectively, given by

$$\begin{aligned} S_\eta(k,0)&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\{\exp(\mathrm{Re}^2B_z(\rho))-1\}\cos k\rho\,d\rho,\\ \sigma_\eta^2(t)&=\int_{-\infty}^{+\infty}S_\eta(k,0)\exp\{-2\mu k^2t\}dk. \end{aligned} \quad (11)$$

When $t\to\infty$, high-frequency absorption ensures that the wave energy is concentrated in the long-wave components so that, to determine the asymptotic form of the spectrum $S_v(k,t)$, we must analyze the behavior of $S_\eta(k,0)$ for $k\to 0$. Expanding (11) into a series in powers of Re^2, we obtain

$$S_\eta(k,0)=\mathrm{Re}^2\left[S_z(k)+\frac{\mathrm{Re}^2}{2}S_z(k)\otimes S_z(k)+\dots+\frac{\mathrm{Re}^{2n-2}}{n!}\underbrace{S_z(k)\otimes\dots\otimes S_z(k)}_{n-1}+\dots\right], \quad (12)$$

where $\otimes$ represents convolution. If the initial spectrum near the origin is proportional to k^α, the function $S_z(k)$ will, according to (10), be proportional to $k^{\alpha-2}$. When $1<\alpha<2$, the function $S_z(k)$ has a singularity as $k\to 0$, and the behavior of $S_\eta(k,0)$ near the origin is determined by the first term in the expansion (12), since the remaining terms of the series have a weaker degree of divergence. From (8) and (12), it follows that as $t\to\infty$, the spectrum of the field $v(x,t)$ coincides with the spectrum of the wave in a linear medium and, in particular, the wave energy decreases in proportion to $t^{-(\alpha+1)/2}$. When $\alpha\geqslant 2$, the spectrum of $v(x,t)$ for $t\to\infty$ has the universal form

$$S_v(k,t)=4\mu^2k^2S_\eta(0,0)e^{-2\mu k^2t}, \quad (13)$$

independently of the shape of the initial spectrum, and the wave energy decreases in accordance with the power law

$$E(t)=(\pi\mu/2)^{1/2}S_\eta(0,0)t^{-3/2}. \quad (14)$$

The factor $S_\eta(0,0)$ defines the intensity of the spectrum and the wave energy for large times, and depends on the field correlation function $B_{v0}(\rho)$ and dissipation coefficient μ. We note that, for $\alpha\geqslant 2$ and for sufficiently large times, the energy and the spectrum of an intense field are determined by not only the local behavior of the initial spectrum for $k\to 0$ but also by its entire shape. This is in contrast to the case of a linear medium.

3. Qualitative conclusions with regard to the behavior of the spectrum $S_v(k,t)$ for $k\to 0$ can be derived from the results reported in[5,8] for the spectrum of a random Gaussian field $v(x,t)$ satisfying the equation of a simple wave ($\mu=0$):

$$S_v(k,t)=\frac{\exp(-\sigma^2k^2t^2)}{2\pi k^2t^2}\int_{-\infty}^{+\infty}\{\exp(k^2t^2B_v(\rho))-1\}e^{ik\rho}\,d\rho. \quad (15)$$

When $k\to 0$, equation (15) yields

$$S_v(k,t)=S_v(k)+k^2t^2\{S_v(k)\otimes S_v(k)/2-\sigma^2S_v(k)\}, \quad (16)$$

i.e., if the spectrum $S_v(k)$ is proportional to k^α near the origin, then, for $\alpha<2$, the nonlinear interaction does not lead to a change in its behavior for $k\to 0$, and for $\alpha\geqslant 2$ the spectrum of the wave is proportional to k^2t^2 independently of α as $k\to 0$, and becomes steeper with increasing time. The restriction on the growth of the spectrum for $k\to 0$ is connected with the presence of dissipation in the medium. In a medium with low-frequency dissipation, absorption leads to a steady-state wave profile for sufficiently high attenuation coefficients, and the shape of the wave spectrum is given by (15) and (16) with the time replaced by the characteristic time for linear attenuation.[5,6] In a medium with high-frequency absorption, it is clear from (8) and (12) that the analogous replacement is possible only for small Reynolds numbers but, in general, analysis of the asymptotic spectrum must be based directly on the exact solution of the Burgers equation.

4. For small Reynolds numbers, dissipative effects predominate over nonlinear effects and we can confine our attention to a single nonlinear interaction, so that corrections to the spectrum of the linear wave are proportional to Re^2 (12). For broad-band noise, the spectral components due to the presence of the nonlinear interaction are located in the same interval as the main spectrum and, consequently, do not affect its shape, with the exception of a small broadening and a modification of the behavior of the wave spectrum for $k\to 0$. For high-frequency noise, when the initial energy of the wave is concentrated near some frequency k_0, the nonlinear interaction leads to the appearance of new components in the spectrum at the difference and sum frequencies. Thus, for a quasimonochromatic wave with spectrum width Δk, the higher harmonics are cut off and the low-frequency part of the spectrum is formed. The latter is proportional to the convolution of the envelope of the initial spectrum and the factor k^2/k_0^2 [(8) and (12)], and practically does not change for time $t<[\mu(\Delta k)^2]^{-1}$. Its effective width is equal to twice the width of the spectrum of the quasimonochromatic wave, and the relative fraction of energy transferred to the long-wave part of the spectrum is proportional to the square of the Reynolds number and the relative width of the spectrum of the incident wave.

For large Reynolds numbers, we can no longer confine our attention to the inclusion of a single interaction of a spectral component because weak attenuation in the nonlinear medium without dispersion ensures that a large number of harmonics will interact. It can be shown that the inclusion of N-fold nonlinear interaction corresponds to the truncation of the series (12) at the $(N+1)$-th term. The asymptotic behavior of the spectrum will be considered below for certain special cases.

§2. INTERACTION OF BROAD-BAND NOISE WITH REGULAR SIGNAL

1. The problem of the interaction between noise and a regular signal arises in the study of real noise spectra consisting of discrete lines and a continuum, for example, the cavitation spectra. With increasing distance from the source, the spectrum of such waves becomes distorted by the nonlinear interaction and high-frequency absorption connected with the thermal conductivity and viscosity in the medium. It is interesting, for example, to consider in terms of (13) the relative contributions of the continuous and discrete components to the asymptotic shape of the spectrum at large distances from the point of entry. Effects appearing during the nonlinear interaction of noise with a regular signal can also be used to investigate noise fields with the aid of a high-intensity external wave. By varying the parameters of the latter, and confining measurements to a restricted spectral interval, it is possible to determine the spectral composition of the noise.

2. Suppose that, for $t=0$, the field $v_0(x)$ is a superposition of a regular sinusoidal wave of amplitude a_0 and wave number k_0, and broad-band Gaussian noise $\xi_0(x)$ with energy σ^2, characteristic spectrum width γ, and correlation function $B_{\xi_0}(\rho)$ such that $S_{\xi_0}(0)=0$. We shall introduce Reynolds numbers for the broad-band noise through the formula

$$\mathrm{Re}_n^2=\langle \tilde z_0^2(x)\rangle/4\mu^2,$$

where

$$\tilde z_0(x)=\int_{-\infty}^{x}\xi_0(x')\,dx'$$

($\mathrm{Re}_n^2\sim\sigma^2/\mu^2\gamma^2$), and for the monochromatic wave

$$\mathrm{Re}_s=a_0/2\mu k_0,$$

where the initial conditions for the linear diffusion equation (3) can be written in the form

$$U(x,0)=\exp\{-\mathrm{Re}_n z(x)+\mathrm{Re}_s\cos k_0x\}=\exp\{-\mathrm{Re}_n z(x)\}I_0(\mathrm{Re}_s)+2\sum_{n=1}^{\infty}I_n(\mathrm{Re}_s)\exp\{-\mathrm{Re}_n z(x)\}\cos nk_0x, \quad (17)$$

where $I_n(x)$ are the modified Bessel functions. Using the Gaussian properties of $\xi_0(x)$ and equations (5) and (17), we obtain the following expression for the auxiliary field $S_\eta(k,0)$ averaged over one period of the sinusoid:

$$S_\eta(k,0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\exp\{\mathrm{Re}_n^2B_z(\rho)\}\frac{I_0(2\,\mathrm{Re}_s\cos(k_0\rho/2))}{I_0^2(\mathrm{Re}_s)}-1\right\}\cos k\rho\,d\rho$$
$$=S_{\eta n}(k,0)+\sum_{n=1}^{\infty}f_n^2(\mathrm{Re}_s)[\delta(k-nk_0)+\delta(k+nk_0)]$$
$$+\sum_{n=1}^{\infty}f_n^2(\mathrm{Re}_s)[S_{\eta n}(k-nk_0,0)+S_{\eta n}(k+nk_0,0)], \quad (18)$$

where $f_n(x)=I_n(x)/I_0(x)$ and

$$S_{\eta n}(k,0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[\exp\{\mathrm{Re}_n^2B_z(\rho)\}-1]\cos k\rho\,d\rho. \quad (19)$$

The first term in (18) is the spectrum of the process $\eta_0(x)$, (4), (5) for broad-band noise in the absence of the signal, and the second is the spectrum of the sinusoidal signal and corresponds to the well-known Mendousse solution.[3] The last term describes the nonlinear interaction between the signal and the noise. We note that, since the relationship between $v(x,t)$ and $\eta(x,t)$, given by (6), is nonlinear, the last term describes both the distortion of the noise part of the spectrum and the change in the amplitudes of the harmonics of the regular wave.

3. Let us now consider, to begin with, the transformation of the spectrum of broad-band noise in a nonlinear medium. During the initial stage, when nonlinear effects predominate over dissipative effects, the spectrum of the wave broadens in both directions away from the "central" frequency $k \approx \gamma$, and its energy center shifts toward higher frequencies.[5] Dissipation leads to a reduction in the wave energy and limits the nonlinear interaction process. Consequently, for sufficiently large times, the more effective attenuation of high-frequency components ensures that the energy width of the spectrum is reduced. The maximum width of the spectrum of the noise propagating in the nonlinear medium is determined by the Reynolds number. For $\mathrm{Re}_n \ll 1$, we can expand the exponential in (19) into a series in Re_n, so that

$$S_{\eta n}(k,0) = \mathrm{Re}_n^2 S_z(k) \tag{20}$$

and the spectrum of the broad-band noise becomes practically identical with the spectrum in a linear medium.

When $\mathrm{Re}_n \gg 1$, we can retain the first two terms of the expansion $B_z(\rho) = 1 - \gamma^2\rho^2 + \ldots$, in (19), so that

$$S_{\eta n}(k,0) = \frac{\exp(\mathrm{Re}_n^2)}{(4\pi)^{1/2}\,\mathrm{Re}_n\,\gamma} \exp\left\{-\frac{k^2}{4\,\mathrm{Re}_n^2\,\gamma^2}\right\}. \tag{21}$$

Hence, it is clear that, for $\mathrm{Re}_n \gg 1$, the effective width Γ_n of the spectrum is proportional to $\gamma\,\mathrm{Re}_n$. For sufficiently large distances, the wave spectrum has the universal form given by (13), and the factor $S_\eta(0,0)$ is then equal to $\exp(\mathrm{Re}_n^2)/(4)^{1/2}\,\mathrm{Re}_n$. Hence, it follows that the slope of the spectrum $S_v(k,t)$ of an intense wave for $k \to 0$, and its energy, are much greater than in a linear medium. This is due to the parametric transfer of energy to the long-wave part of the spectrum.

4. The nonlinear interaction between broad-band noise and a regular signal leads both to the appearance of noise modulation of the signal harmonics and to the modification of the spectral composition of the noise itself. In particular, it produces an additional contribution to the low-frequency part of the spectrum. It is clear from (18) that the influence of the determined signal on the transfer of energy to the long-wave region will be important only when the noise spectrum broadened in the nonlinear medium intersects the harmonics of the monochromatic wave, i.e., when $\Gamma_n \gtrsim k_0$. When $\mathrm{Re}_n \ll 1$, the width of the noise spectrum in the nonlinear medium is practically constant, and the regular wave leads to an increase in the amplitudes of the low-frequency components only when the discrete line $k = k_0$ lies on a background of broad-band noise ($k_0 \lesssim \Gamma_n \approx \gamma$). When $\mathrm{Re}_n \gg 1$, the influence of the monochromatic wave must be taken into account if $k_0 \lesssim \Gamma_n \approx \gamma\,\mathrm{Re}_n$.

For a weak signal ($\mathrm{Re}_s \ll 1$), when the spectrum of the regular wave consists of a single discrete line $k = k_0$, we have only the single interaction between the signal and the noise, and

$$S_\eta(0,0) = S_{\eta n}(0,0) + S_{\eta n}(k_0,0)\frac{\mathrm{Re}_0^2}{2} = S_{\eta n}(0,0) + S_{\eta n}(k_0,0)\frac{a_0^2}{8\mu^2 k_0^2}, \tag{22}$$

i.e., the additional contribution to the low-frequency part of the spectrum is proportional to the spectrum of the auxiliary field $\eta_0(x)$ at the frequency $k = k_0$. Thus, by varying the signal frequency and measuring the low-frequency part of the spectrum, it is possible to determine the spectrum $S_{\eta n}(k,0)$ and hence the initial shape of the noise spectrum $S_{\xi_0}(k)$.

When $\mathrm{Re}_n \gg 1$, the nonlinear interaction between the noise and the signal harmonics becomes important. Hence, the components of the spectrum $S_{\eta n}(k,0)$ provide a contribution to $S_\eta(0,0)$ at a series of frequencies $k = nk_0$, $n = 1, 2, \ldots, N$, where $N^2 \approx \mathrm{Re}_s$ is the number of harmonics of the intense regular wave generated in the linear medium. We then have the following expression for $S_\eta(0,0)$:

$$S_\eta(0,0) = \sum_{n=-\infty}^{+\infty} f_n^2(\mathrm{Re}_s) S_{\eta n}(nk_0,0)$$
$$= \begin{cases} S_{\eta n}(0,0) + 2S_{\eta n}(k_0,0), & \Gamma_n \ll k_0 \\ B_{\eta n}(0)/k_0 = [\exp(\mathrm{Re}_n^2) - 1]/k_0, & k_0 \ll \Gamma_n \ll Nk_0. \\ S_{\eta n}(0,0)\sqrt{\pi\,\mathrm{Re}_n}, & \Gamma_n \gg Nk_0. \end{cases} \tag{23}$$

It is clear from (23) that, as the amplitude of the regular wave increases, an additional contribution to the low-frequency part of the spectrum is provided by the components $S_{\eta n}(k,0)$ at increasingly higher frequencies. Consequently, the dependence of the slope of the field $S_v(k,t)$ for $k \to 0$, or of its energy at sufficiently large times, on the regular-wave amplitude can be used to determine the spectrum of the auxiliary field $S_{\eta n}(k,0)$ and, consequently, the initial spectrum of the noise at frequencies $k = nk_0$. In particular, when k_0 is much smaller than the characteristic noise frequency, measurements in the low-frequency part of the spectrum can be used to determine the shape of the initial spectrum with sufficient detail.

To estimate the spectrum width of the noise wave, we can use the fact that, for a sufficiently intense signal, when $Nk_0 \gg \Gamma_n$, the effectiveness of the transformation of energy is independent of the regular-wave amplitude and is determined by its frequency k_0. For broad-band noise, we find, by comparing (20) and (21) with (23), that, for $\mathrm{Re}_n \ll 1$ (regular wave on a weak noise background: $\gamma \gg k_0$), the slope of the spectrum $S_v(k,t)$ for $k \to 0$ increases due to the interaction between the signal and the noise by a factor of γ/k_0, and for $\mathrm{Re}_n \gg 1$ by a factor of $\mathrm{Re}_n\,\gamma/k_0$.

The finite spectral-line width of the quasimonochromatic wave ensures that its energy is transferred as a result of the nonlinear interaction not only to the shortwave but also the long-wave parts of the spectrum. The appearance of low-frequency components in the quasimonochromatic-wave spectrum ensures that the interaction between broad-band noise and the quasimonochromatic wave is more effective than the interaction with the regular signal. The additional transfer of energy to the long-wave part of the spectrum may turn out to be important even for $\Gamma_n \ll k_0$ when the regular wave has practically no influence on the low-frequency part of the spectrum.

5. To determine the spectral composition of the noise we can also use the components near $k = nk_0$ due to the interaction with the regular signal. When $\Gamma_n \ll k_0$, it follows from (19) that the above spectral bands do not overlap and one can speak of noise pedestals near the monochromatic lines or, in other words, the random modulation of the regular wave and its harmonics. We shall consider the situation in which $\mu k_0^2 t \gg 1$, i.e., we need not take into account the higher harmonics in the wave spectrum. From (17), (2), and (3), we have the following expression for the high-frequency component of the wave at the frequency $k \approx k_0$:

$$v(x,t) = -4\mu f_1(\mathrm{Re}_s) \frac{\partial}{\partial x} \frac{\int \exp\{-\mathrm{Re}_n z(x')\} \cos k_0 x' G(x-x',t)\,dx'}{\int \exp\{-\mathrm{Re}_n z(x')\} G(x-x',t)\,dx'},$$

$$G(\rho, t) = (4\pi\mu t)^{-1/2} \exp\{-\rho^2/4\mu t\}. \quad (24)$$

It follows from this expression that the intensity of a randomly modulated signal is proportional to Re_s^2 for small signal Reynolds numbers and is independent of the amplitude a_0 for $\mathrm{Re}_s \gg 1$ when the transfer of the noise-signal energy from the region $k \approx k_0$ toward higher frequencies becomes important.

The shape of the wave spectrum near $k \approx k_0$ depends on the parameter $\mu k_0 \gamma t$, which characterizes the nonuniformity of high-frequency absorption in the frequency band of the modulated signal. When $\mu k_0 \gamma t \ll 1$, all the harmonics appearing as a result of the interaction between the signal and the noise at frequencies $k \approx k_0$ are attenuated practically in the same way. Neglecting the change in the low-frequency noise, we can readily show, using (24), that for such times

$$v(x,t) \approx -4\mu f_1(\mathrm{Re}_s) \frac{\partial}{\partial x} \cos k_0 (x - \xi_0(x) t).$$

When the initial field $\xi_0(x)$ has Gaussian statistics, the signal spectrum has the form

$$S_v(\Omega, t) = 8\mu^2 f_1^2(\mathrm{Re}_s)(k_0+\Omega)^2 \exp(-2\mu k_0^2 t) \Big\{ \exp(-\sigma^2 k_0^2 t^2)\,\delta(\Omega)$$

$$+ \exp(-\sigma^2 k_0^2 t^2) \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\exp\{B_{\xi_0}(\rho) k_0^2 t^2\} - 1] \cos \Omega\rho \, d\rho \Big\}, \quad (25)$$

where $\Omega = k - k_0$. We note that the shape of the pedestal spectrum and the additional attenuation of the mean field are independent of the dissipation parameter μ and, for $\gamma \ll k_0$, agree with the results in[6]. The attenuation of the regular component may be regarded as the additional attenuation of sound due to the interaction with extraneous noise. We note, however, that there is an important difference between high- and low-frequency noise. Thus, whilst the wave energy is scattered into a broad spectral interval in the case of propagation of sound in a medium with high-frequency noise, the situation in a medium with low-frequency noise is different in that the scattered energy is concentrated in a relatively narrow spectral interval near the carrier frequency of the signal. In particular, it is readily verified with the aid of (25) that the low-frequency noise has practically no effect on the resultant energy of the regular signal and its pedestal.

When $\sigma k_0 t \ll 1$, so that only a single interaction between noise and the regular wave is important, a pedestal is formed near $k \approx k_0$ which approximately repeats the shape of the low-frequency noise spectrum.[6] When $\sigma k_0 t \gg 1$, the interaction of the low-frequency noise with the pedestal becomes important, and this results in a broadening of the spectrum of the wave. It can be shown from (25) that the spectrum of the pedestal is then Gaussian in shape with a width $\gamma\sigma k_0 t$, whatever the shape of the noise spectrum $\xi_0(x)$, i.e., the width of the spectrum of the low-frequency wave becomes much greater than the bandwidth of the low-frequency noise.

When $\mu k_0 \gamma t \gg 1$, the nonuniformity of high-frequency absorption within the bandwidth of the modulated signal begins to have an effect. Neglecting the change in the low-frequency noise, and using Gaussian statistics for $\xi_0(x)$, we obtain from (24) the following expression for the mean field of the modulated signal:

$$\langle v(x,t) \rangle = 4\mu k_0 f_1(\mathrm{Re}_s) \exp(\mathrm{Re}_n^2)$$

$$\cdot \int_{-\infty}^{+\infty} \exp\{-\mathrm{Re}_n^2 B_z(x')\} \sin k_0 (x - x') G(x', t)\, dx'. \quad (26)$$

Consider a noise spectrum with correlation function $B_{\xi_0}(\rho) = \sigma^2(1 - \gamma^2\rho^2)\exp(-\gamma^2\rho^2/2)$, for which $\mathrm{Re}_n = \sigma/2\mu\gamma$. Expanding the integrand in (26) into a series in powers of Re_n^2, and recalling that the low-frequency component is still undistorted, we obtain the following expression for the mean field after integration and summation of the series:

$$\langle v(x,t) \rangle = 4\mu k_0 f_1(\mathrm{Re}_c) \sin k_0 x e^{-\mu k_0^2 t} \cdot \exp\{-\mathrm{Re}_n^2 (e^{2(\mu k_0 \gamma t)^2} - 1)\}. \quad (27)$$

As expected, when $\mu k_0 \gamma t \ll 1$, the additional attenuation of the mean field is determined only by the total energy σ^2 of the noise, and the expression for the mean field becomes identical with (25). On the other hand, when $\mu k_0 \gamma t \gg 1$, the additional attenuation of the mean field begins to depend appreciably on the spectral composition of the noise, and increases rapidly with increasing noise spectrum width.

Nonuniform attenuation will also substantially modify the shape of the pedestal. The expression for the pedestal for $\mathrm{Re}_n \ll 1$ and $\mathrm{Re}_n \exp(\mu k_0 \gamma t) \ll 1$ is

$$S_p(\Omega, t) = 8\mu^2 f_1^2(\mathrm{Re}_s)(k_0+\Omega)^2 \Omega^{-2} S_{\xi_0}(\Omega, 0)[e^{-2\mu k_0 \Omega t} - 1]^2 e^{-2\mu k_0^2 t}. \quad (28)$$

Hence, it is clear that when $\mu k_0 \gamma t \gg 1$ and $\Omega > 0$, the shape of the pedestal is proportional to the spectrum of the noise integral, i.e., $S_p(\Omega, t) \sim S_{\xi_0}(\Omega, 0)/\Omega^2$, and is attenuated in the same way as a regular harmonic. On the other hand, pedestal frequencies $(\Omega < 0)$ are attenuated much more slowly, and this leads to a shift of the energy center of the spectrum toward lower frequencies. When $\mathrm{Re}_n \exp(\mu k_0 \gamma t) \gtrsim 1$, the total power within the pedestal is approximately equal to the power in a regular harmonic, i.e., the random modulation index of the signal becomes high. Interactions between the low-frequency noise and the pedestal harmonics, which lead to the broadening of the spectrum mainly in the direction of lower frequencies, then become important.

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Interaction of charged particles with strong monochromatic radiation in an inhomogeneous medium

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Multiphoton processes of stimulated absorption and emission by a charged particle in an external electromagnetic field incident on the interface between two media are considered. The effect can manifest itself in a broadening of the energy spectrum of the particle beam. It is also shown that the stimulated processes lead to modulation of the beam at the frequency of the external field.

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1. Charged particles that move uniformly in optically inhomogeneous media can radiate electromagnetic waves. Examples are the transition radiation at the interface between two media, radiation in a layered medium, and diffraction radiation by different screens. These phenomena have been sufficiently well studied both theoretically and experimentally.[1–3]

It is of interest to ascertain the variation of the character of this radiation in the presence of an external electromagnetic field, particularly a laser field. In contrast to ordinary spontaneous emission, this process has a stimulated character: the particle can not only radiate but also absorb quanta of the external field. These two phenomena are of definite interest both from the point of view of acceleration of charged particles by an external electromagnetic field and from the point of view of amplifying electromagnetic radiation by charged particles, as in the case of stimulated Čerenkov radiation.[4,5] In addition, as a result of such a stimulated interaction, the beams are modulated at the external-field frequency or its multiples, which is also of definite interest.

The present paper is devoted to a study of the interaction of an external monochromatic electromagnetic radiation with charged particles crossing the interface between two media, i.e., the analog of stimulated transition radiation.

2. We assume as usual that the particles themselves do not interact with the medium. Let the medium be inhomogeneous along z and let a plane linearly polarized electromagnetic wave of frequency ω propagate in the same direction. We seek the solution of the Klein-Gordon equation

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2}=\left[M^2c^4+c^2\hat{p}_x^2+c^2\hat{p}_z^2+c^2\left(\hat{p}_y-\frac{e}{c}A_y\right)^2\right]\psi \tag{1}$$

in the form

$$\psi=\exp[i\hbar^{-1}(\mathbf{pr}-Et)]\varphi, \tag{2}$$

where φ is a function that varies slowly in comparison with the exponential function. This approximation is valid if the following inequalities are satisfied:

