the left of the transition and the conducting network determines that on the right. For $R_d > R_m\sigma_m/\sigma_d$ the dielectric is found to have a determining role in a certain region to the right of the transition as well (curve 4).

It is interesting to note that for $\mu_d < \mu_m$ our results differ not only quantitatively but also qualitatively from the predictions of effective-medium theory.[12] This theory leads to a monotonic decrease of $R(x)$ with increase of x. For example, in the case $R_m = R_d \equiv R$ the effective-medium theory gives $R(x) = R = \text{const}$. But in reality $R(x)$ should have a sharp maximum.

We assumed above that the equality (28) is fulfilled. If this is not so, the two-band model is not entirely adequate for $x < x_c$. This may be connected with the neglect of the Hall current generated in the active dielectric layers of the conducting network. However, we are confident that, even in this case, the formula (27) gives a qualitatively correct description of the behavior of $R(x)$.

I am grateful to A. L. Éfros for very useful discussions about the article.

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Translated by P. J. Shepherd

Anomalous penetration of an electromagnetic field into a metal with diffuse reflection of electrons by the specimen boundary

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(Submitted June 26, 1976)
Zh. Eksp. Teor. Fiz. 72, 296–307 (January 1977)

A theory is constructed for anomalous penetration (AP) of an electromagnetic wave into a metal placed in a magnetic field parallel to its surface. The reflection of electrons from the metal-vacuum interface is assumed to be diffuse. AP of the field occurs along a chain of electron trajectories. It is shown that under anomalous-skin-effect conditions in the radiofrequency range, the field distribution contains four spikes, at distances from the boundary of one, two, three, and four cyclotron diameters. The first three spikes have a distinct spatial structure, whereas the fourth exists against the background of a smooth quasiharmonic distribution. At distances exceeding the region of existence of the last spike, the field has a quasiharmonic character.

PACS numbers: 73.90.+f

1. INTRODUCTION

The effect of anomalous penetration (AP) of an electromagnetic wave into a metal along a chain of electron trajectories, in a magnetic field **H** parallel to the surface of the specimen, is well known in the physics of metals (see Fig. 1). It has been observed experimentally by Gantmakher[1] and investigated theoretically in an article by one of the authors.[2] A large number of papers have now been devoted to this phenomenon (see the review[3] and also the article[4]). There is at present extensive experimental material on the observation of AP of the trajectory type in many metals. Nevertheless there has so far been lacking a systematic theory of the trajectorial transfer of an electromagnetic wave with allowance for the interaction of the electrons with the specimen surface. The reason lies in the mathematical difficulties that arise when one takes account of this interaction and that lead to a complex character of the field distribution in the metal. In order to circumvent these difficulties, qualitative considerations have been introduced. It has been supposed that a good approximation is the distribution of the electric field **E** in an infinite specimen with a current sheet, simulating the skin layer δ (see, for example, [3]). In other words, it has been assumed that the principal role in AP is played by electrons that do not collide with the metal–vacuum interface, and for which it is possible to use the results that are valid in an infinite specimen. Thus for a wave polarized perpendicular to **H**, the spatial distri-

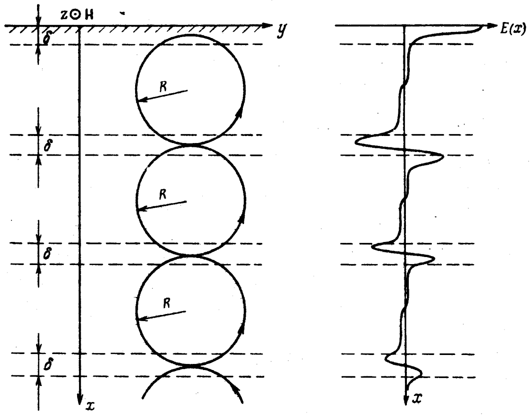

FIG. 1. Picture of anomalous penetration of an electromagnetic field into a metal with a diffuse boundary. On the left is depicted a chain of trajectories of volume electrons that participate in AP. On the right is shown the schematic form of the field distribution in the specimen.

bution of the field $E(x)$ is determined by the following expression[2,3]:

$$E(x)=-\frac{2E'(0)\delta}{\pi}\int_0^\infty d\xi\frac{\xi\cos(x\xi/\delta)}{\xi^3-i+i(4\delta/\pi R\xi)^{1/2}\sin(2R\xi/\delta-\pi/4)}. \tag{1.1}$$

Here the x axis is directed along the internal normal to the metal surface; R is the maximum Larmor radius of an electron orbit in field $\mathbf{H}$; $2E'(0)$ is the discontinuity of the derivative of the electric field at the interface $x=0$; and δ is the depth of the skin layer.

Analysis of formula (1.1) shows that it describes three spikes of the field $\mathbf{E}$, at distances of one, two, and three cyclotron diameters $2R$ from the surface. At larger depths, the field has a quasiharmonic character. Finiteness of the number of spikes of the electromagnetic field, in a metal placed in a parallel magnetic field $\mathbf{H}$, is a general law of the trajectorial transfer effect for all actual types of closed Fermi surface (the case of a cylindrical Fermi surface is an exception). The physical reason for such behavior of the field of the wave is the following. The existence of a spike is due to the occurrence of a packet of shortwave harmonics in the spectral expansion of $E(x)$, with characteristic values of the wave vector $k\sim\delta^{-1}$. On the other hand, from general considerations it is clear that with increasing distance from the surface, information about the field $\mathbf{E}$ is determined by an ever increasing contribution of the longwave part of the spectrum, with $k\ll\delta^{-1}$. It is the competition of these two factors that leads to the finite number of spikes.

The actual distribution of the field $E(x)$ in a bounded metal differs from formula (1.1) and in general depends on the character of the scattering of the electrons by the specimen boundary. Therefore the number of spikes and their shape must also depend on the law of reflection of electrons by the surface. We have recently solved the problem of AP in a metal with a specular boundary.[5] In that paper it was shown that the number of spikes is two and that their shape differs importantly from the shape determined from (1.1). This is explained physically by the fact that with specular reflection in a parallel magnetic field $\mathbf{H}$, in addition to the electronic states that exist in an infinite metal, there appears a group of electrons that are bound to the surface (grazing electrons). The surface electrons form a fundamental skin layer at the metal–vacuum interface and thus exert an important influence on the trajectorial transfer of the electromagnetic wave.

In the present paper, a theory is developed for AP along a chain of trajectories in a metal with a diffuse boundary. A constant magnetic field $\mathbf{H}$ is applied parallel to the specimen surface. It might seem that in this case one could apply the concept of effective participation in trajectorial transfer only by electrons that do not collide with the boundary. But a detailed analysis presented in the present paper shows that the influence of the metal surface leads to a significant change of the picture of AP, as compared with the results of the theoretical papers.[2,3] Specifically, the range of existence of singularities of the field $E(x)$ is broadened. The number of spikes is found to be four, and their shape differs from that obtained from the expression (1.1).

The difference in the number of spikes and in their shape, in the diffuse and the specular cases, makes it possible to determine the nature of the scattering of electrons by the metal boundary by experiments on AP.

2. STATEMENT OF THE PROBLEM. SOLUTION OF MAXWELL'S EQUATION

We consider a metallic half-space located in a constant and uniform magnetic field $\mathbf{H}$. The vector $\mathbf{H}$ lies in the metal–vacuum interface. We choose the coordinate system as follows: the y and z axes lie in the metal surface (the plane $x=0$), the z axis is parallel to $\mathbf{H}$, and the x axis is directed along the internal normal to the specimen surface, as is shown in the figure.

Let there be incident on the interface $x=0$ a plane monochromatic wave of frequency ω, whose $\mathbf{E}$ vector is polarized perpendicular to $\mathbf{H}$ ($\mathbf{E}\parallel y$). The direction of propagation of the wave coincides with the x axis. The electric field $\mathbf{E}=\{0, E, 0\}$ inside the metal, $x\geqslant 0$, depends only on the coordinate x; that is, $E=E(x)\exp(-i\omega t)$.

We introduce the Fourier transformation

$$E(x)=\frac{1}{\pi}\int_0^\infty dk\mathscr{E}(k)\cos(kx), \quad \mathscr{E}(k)=2\int_0^\infty dx\,E(x)\cos(kx). \tag{2.1}$$

Maxwell's equation for the Fourier component $\mathscr{E}(k)$ of the field has the form

$$k^2\mathscr{E}(k)+2E'(0)=4\pi i\omega c^{-2}j(k), \tag{2.2}$$

where $j(k)$ is the Fourier transform of the current density, and where c is the velocity of light; the prime indicates differentiation with respect to the argument.

Because of the spatial inhomogeneity of the problem, the relation between the Fourier components of the field $\mathscr{E}(k)$ and of the current $j(k)$ is of integral nature:

$$j(k)=\mathscr{K}(k)\mathscr{E}(k)-\frac{1}{\pi}\int_0^\infty dk'\,Q(k,k')\mathscr{E}(k'). \tag{2.3}$$

Here $\mathcal{K}(k)$ is the Fourier transform of the conductivity of an infinite metal; the kernel $Q(k, k')$ takes account of the influence of the boundary $x=0$ on the electrical conductivity of the specimen. Exact expressions for the kernels $\mathcal{K}(k)$ and $Q(k, k')$, for an arbitrary coefficient ρ of reflection from the metal surface ($0 \leq \rho \leq 1$), are given in[6].

We consider the case of a radiofrequency anomalous skin effect,

$$|\gamma| \ll 1 \ll kR \qquad (2.4)$$

with diffuse scattering of the electrons by the interface $x=0$; that is, when

$$|\gamma| \ll 1-\rho. \qquad (2.5)$$

Here $\gamma=(\nu-i\omega)/\Omega$, $R=v/\Omega$ is the maximum Larmor radius of an orbit, $\Omega=eH/mc$ is the cyclotron frequency of revolution of an electron in the magnetic field $\mathbf{H}$, v is the Fermi velocity, e is the absolute value of the charge, m is the effective mass, and ν is the frequency of collision of the electrons with the scatterers.

The condition (2.4) consists of three inequalities, namely $|\gamma| \ll 1$, $kR \gg 1$, and $|\gamma| \ll kR$. The first of these selects the region of low frequencies (strong magnetic fields $\mathbf{H}$). The second inequality indicates anomalousness with respect to the magnetic field, $R \gg \delta \sim k^{-1}$. The third is the condition for an anomalous skin effect, and equivalently the requirement that the thickness δ of the skin layer be small in comparison with the effective free-path length of the electrons, $v/|\nu-i\omega| \gg \delta$.

We shall not present the standard procedure for obtaining the asymptotic behavior of $\mathcal{K}(k)$ and $Q(k, k')$. We shall at once write the result. For simplicity we shall restrict ourselves to the case of a quadratic and isotropic law of dispersion of the electrons (an alkali metal). The formulas given below, however, are also applicable, with insignificant changes, when the section of the Fermi surface by a plane perpendicular to $\mathbf{H}$ is a convex closed curve of arbitrary shape.

The asymptotic behavior of the electrical conductivity $\mathcal{K}(k)$ of an infinite metal, under the condition (2.4), has the form

$$\mathcal{K}(k) \approx \frac{3\sigma}{4kR} - \frac{3\sigma}{\sqrt{2\pi}\, kR} \frac{\sin(2kR-\pi/4)}{(2kR)^{1/2}}, \qquad (2.6)$$

where $\sigma=Ne^2/m(\nu-i\omega)$; N is the concentration of the electrons.

The integral conductivity kernel $Q(k, k')$, when the inequalities (2.4) and (2.5) are satisfied, is asymptotically the sum of four terms:

$$\begin{aligned} Q(k,k') \approx {} & \frac{3\sigma}{8R}\left[\frac{4}{\pi}(kk')^{-1/2}\int_0^{\pi/2} d\theta \sin^2\theta \frac{\sin[2(k-k')R\sin\theta]}{k-k'} + \frac{(kk')^{-1/2}}{k+k'}\right] \\ & + \frac{3\sigma}{2R\sqrt{2\pi}} \frac{(kk')^{-1/2}}{k-k'}\left[\frac{\cos(2kR-\pi/4)}{(2kR)^{1/2}} - \frac{\cos(2k'R-\pi/4)}{(2k'R)^{1/2}}\right] \\ & - \frac{3\sigma}{2R\sqrt{2\pi}} \frac{(kk')^{-1/2}}{k+k'}\left[\frac{\sin(2kR-\pi/4)}{(2kR)^{1/2}} + \frac{\sin(2k'R-\pi/4)}{(2k'R)^{1/2}}\right] \\ & - \frac{3\sigma}{2R\sqrt{2\pi}} \frac{(kk')^{-1/2}}{k+k'} \frac{\cos[2(k+k')R-\pi/4]}{[2(k+k')R]^{1/2}}. \end{aligned} \qquad (2.7)$$

The expression (2.7) for $Q(k, k')$ is due to the contribution of volume electrons, which do not interact with the specimen boundary. The contribution of surface electrons to the conductivity is small, in comparison with (2.7), according to the parameter $|\gamma|/(1-\rho) \ll 1$.

The first terms in formulas (2.6) and (2.7) describe the anomalous skin effect. The remaining terms, which oscillate rapidly according to $2kR$, insure AP on electron trajectories of the central section of the Fermi surface. The AP terms in the conductivity of a metallic half-space contain, as compared with the smooth part, the smallness parameter $(2kR)^{-1/2}$, which is equal to the relative number of electrons that effectively participate in trajectorial transfer.

For convenience in later consideration, we introduce the notation

$$\mathcal{E}(k) = -2E'(0)\delta^2 F(k\delta), \quad \delta = (2c^2R/3\pi\omega\sigma)^{1/3}. \qquad (2.8)$$

The quantity δ is the depth of the skin layer in an anomalous skin effect in a strong magnetic field $\mathbf{H}$.

We substitute the asymptotic expressions (2.6) and (2.7) in the right side of (2.2), then in Maxwell's equation go over from the Fourier component $\mathcal{E}(k)$ to the function $F(k\delta)$. We also introduce the dimensionless wave number $\xi=k\delta$. As a result, we obtain the following integral equation for the function $F(\xi)$:

$$\begin{aligned} & (\xi^3-2i)F(\xi) + \frac{4t}{\pi^2}\int_0^{\pi/2} d\theta \sin^2\theta \int_0^\infty \frac{dx}{\sqrt{x}} F(\xi x) \frac{\sin[2R\xi\sin\theta(x-1)/\delta]}{x-1} \\ & + \frac{i}{\pi}\int_0^\infty \frac{dx}{\sqrt{x}} \frac{F(\xi x)}{x+1} = \xi - 2i\left(\frac{4\delta}{\pi R\xi}\right)^{1/2} \sin\left(\frac{2R}{\delta}\xi - \frac{\pi}{4}\right) \\ & \times\left\{F(\xi) - \frac{1}{2\pi}\int_0^\infty \frac{dx}{\sqrt{x}}\frac{F(\xi x)}{x+1} - \frac{1}{2\pi}\int_0^\infty \frac{dx}{x} F(\xi x) \frac{\sin[2R\xi(x-1)/\delta]}{x-1}\right\} \\ & + i\left(\frac{4\delta}{\pi R\xi}\right)^{1/2} \cos\left(\frac{2R}{\delta}\xi - \frac{\pi}{4}\right) \frac{1}{\pi}\int_0^\infty \frac{dx}{\sqrt{x}} F(\xi x) \frac{1-x^{-1/2}\cos[2R\xi(x-1)/\delta]}{x-1} \\ & + i\left(\frac{4\delta}{\pi R\xi}\right)^{1/2} \frac{1}{\pi}\int_0^\infty \frac{dx}{\sqrt{x}}\frac{F(\xi x)}{x+1} \cdot \\ & \times\left\{\frac{\sin\left(\frac{2R}{\delta}\xi x - \frac{\pi}{4}\right)}{\sqrt{x}} + \frac{\cos\left[\frac{2R}{\delta}\xi(x+1) - \frac{\pi}{4}\right]}{\sqrt{x+1}}\right\}. \end{aligned} \qquad (2.9)$$

We shall carry out the solution of (2.9) by perturbation theory, making use of the smallness of the AP terms in comparison with the smooth terms. The answer is expressed in the form of a sum

$$F(\xi) = F_0(\xi) + i(4\delta/\pi R\xi)^{1/2} F_1(\xi). \qquad (2.10)$$

The function $F_0(\xi)$ is the solution of the unperturbed problem and is determined from equation (2.2) with a current density corresponding to the anomalous skin effect; that is,

$$(\xi^3-i)F_0(\xi) + \frac{i}{\pi}\int_0^\infty \frac{dx}{\sqrt{x}}\frac{F_0(\xi x)}{x+1} = \xi. \qquad (2.11)$$

Equation (2.11) was solved by Hartmann and Luttinger[7] by means of the Mellin transformation. The function

$F_0(\xi)$ is expressed as a contour integral in the complex z plane[7]:

$$F_0(\xi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\,\xi^z M(z),\quad -2<c=\mathrm{Re}\,z<0. \tag{2.12}$$

The Mellin transform $M(z)$ is regular in the vertical strip $-7/2<\mathrm{Re}z<2$, with the exception of the three singular points $z=-2$, $z=0$, and $z=1$. At these points $M(z)$ has simple poles with residues 1, $-2\exp(i\pi/3)/3$, and $-i/2$ respectively. It should be mentioned that in the interval $(-7/2, 2)$ the function $M(z)$ vanishes (all zeros are simple) at the points $z=-3$, $z=\frac{1}{2}$, and $z=\frac{3}{2}$. The function $M(z)$ satisfies the difference equation

$$\frac{M(z-3)}{M(z)}=-2i\frac{\sin^2(\pi z/2)}{\cos(\pi z)} \tag{2.13}$$

and is described, within the strip of regularity $(-7/2, 0)$ by the formula

$$M(z)=\frac{\pi}{3}\frac{\exp[\pi i(z+2)/6]}{\sin[\pi(z+2)/3]}\exp\left(-\int_{-2}^{z}dz'\left\{\frac{2\pi}{3}\frac{z'+3/2}{\sin(2\pi z')}+\frac{\pi}{2\sin(\pi z')}+\frac{2\pi}{9\sin[2\pi(z'+1)/3]}-\frac{2\pi}{9\sin[2\pi(z'-1)/3]}\right\}\right). \tag{2.14}$$

We shall later need the values of the function $M(z)$ at the points $z=-1$ and $z=-\frac{1}{2}$. We write them:

$$M(-1)=\frac{\pi}{\sqrt{3}}\exp\left(\frac{\pi i}{6}\right),\quad M(-1/2)=\frac{\pi}{\sqrt{2}}\exp\left(\frac{\pi i}{4}\right),$$
$$M'(-1)=-\frac{\pi^2(\sqrt{3}-11i)}{108}-\frac{\pi e^{\pi i/6}}{3\sqrt{3}}. \tag{2.15}$$

It follows from the expression (2.12) that for $\xi\to\infty$

$$F_0(\xi)\approx\xi^{-2}$$

and for $\xi\to 0$

$$F_0(\xi)\approx\frac{1+i\sqrt{3}}{3}+\frac{i}{2}\xi+\frac{1}{\pi}\xi^2\left[\frac{1-i\sqrt{3}}{\sqrt{3}}\ln\xi+\frac{i\sqrt{3}-1}{3\sqrt{3}}\frac{\pi}{54}(11+i\sqrt{3})\right]. \tag{2.16}$$

We next find $F_1(\xi)$. We note that in Eq. (2.9) the terms containing $F_1(\xi)$ under the integral sign are small in comparison with the corresponding terms outside the integral and may be neglected. Among the terms containing $F_0(\xi)$ on the right side of equation (2.9), it is necessary to keep only the term containing $\sin(2R\xi/\delta-\pi/4)$ and the smooth term in the integral next to $\cos(2R\xi/\delta-\pi/4)$. The integral must then be understood as the principal value. As a result we get

$$F_1(\xi)=\frac{\Phi(\xi)\sin(2R\xi/\delta-\pi/4)+\chi(\xi)\cos(2R\xi/\delta-\pi/4)}{\xi^3-2i+2i(4\delta/\pi R\xi)^{1/2}\sin(2R\xi/\delta-\pi/4)}. \tag{2.17}$$

Here the following notation has been introduced:

$$\Phi(\xi)=\frac{1}{\pi}\int_0^\infty\frac{dx\,F_0(\xi x)}{\sqrt{x}\ \ x+1}-F_0(\xi)=-i\xi+i\xi^3F_0(\xi)$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dz\,M(z)\xi^z\frac{2\sin^2(\pi z/2)}{\cos(\pi z)},\quad -1/2<c=\mathrm{Re}\,z<1. \tag{2.18}$$

The function $\chi(\xi)$ is defined by

$$\chi(\xi)=\frac{1}{\pi}\int_0\frac{dx\,F_0(\xi x)}{\sqrt{x}\ \ x-1}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dz\,\xi^z M(z)\,\mathrm{tg}(\pi z),\quad -1/2<c=\mathrm{Re}\,z<2. \tag{2.19}$$

From formulas (2.18) and (2.19) it is not difficult to obtain the asymptotic forms of the functions $\Phi(\xi)$ and $\chi(\xi)$. For $\xi\to\infty$,

$$\Phi(\xi)\approx\frac{\exp(\pi i/4)}{\sqrt{2\xi}}\approx-\chi(\xi), \tag{2.20}$$

for $\xi\to 0$,

$$\Phi(\xi)\approx\xi^3\left(-\frac{1}{\sqrt{3}}-\frac{1}{2}\xi+\frac{1}{\pi}\xi^2\ln\xi-\frac{\xi^2}{3\pi}-\frac{\sqrt{3}}{54}\xi^2\right)$$
$$-i\xi\left(1-\frac{1}{3}\xi^3-\frac{1}{\pi\sqrt{3}}\xi^4\ln\xi+\frac{\xi^4}{3\pi\sqrt{3}}-\frac{11}{54}\xi^4\right), \tag{2.21}$$
$$\chi(\xi)\approx\xi^2\left(\frac{1}{\sqrt{3}}-\frac{1}{\pi}\xi^2\ln\xi-\frac{4}{9\sqrt{3}}\xi^2+\frac{\xi^2}{3\pi}-\frac{1}{2}\xi^3\right)-i\xi^2\left(1+\frac{1}{6}\xi^2+\frac{1}{2\sqrt{3}}\xi^3\right).$$

It is interesting to compare the solution (2.10) of the integral equation with the corresponding expression for an infinite metal. In an unbounded specimen, the role of $F_0(\xi)$ is played by the function $\xi/(\xi^3-i)$. The denominators of the oscillatory terms are of the same form in the two cases (the value of δ in an infinite metal is smaller than (2.8) by a factor $2^{1/3}$), whereas the numerators are different. In the approximation of an unbounded specimen, the numerator of the oscillatory part is proportional to $\sin(2R\xi/\delta-\pi/4)$, while in formula (2.17) there occurs a term $\chi(\xi)\cos(2R\xi/\delta-\pi/4)$. The identical form of the denominators indicates that the mechanism of AP is, as before, determined by electrons that do not collide with the boundary of the specimen. But the scattering at the surface significantly changes the picture of the field distribution as a whole. In particular, the presence of the second term in the numerator of (2.17) leads to the appearance of a fourth spike of $E(x)$.

3. THE FIELD DISTRIBUTION

The spatial structure of the electromagnetic field in the metal is determined by the solution $F(\xi)$ of the integral equation (2.9) according to the formula

$$E(x)=-\frac{2\delta}{\pi}E'(0)\int_0^\infty d\xi\,F(\xi)\cos\left(\frac{x}{\delta}\xi\right). \tag{3.1}$$

After substitution in (3.1) of the expression for $F(\xi)$ in the form (2.10), (2.17), we obtain for $E(x)$ the following representation:

$$E(x)=E_0(x)+\frac{i\delta}{\pi}E'(0)\left(\frac{4\delta}{\pi R}\right)^{1/2}$$
$$\times\int_0^\infty\frac{d\xi}{\sqrt{\xi}}\frac{\Phi(\xi)\cos[(x-2R)\xi/\delta-\pi/4]+\chi(\xi)\sin[(x-2R)\xi/\delta-\pi/4]}{\xi^3-2i+2i(4\delta/\pi R\xi)^{1/2}\sin(2R\xi/\delta-\pi/4)}. \tag{3.2}$$

Here $E_0(x)$ describes the smooth component of the field, due to $F_0(\xi)$. To obtain it, the function $F_0(\xi)$ must be substituted for $F(\xi)$ in formula (3.1). The second term contains the spikes of $E(x)$. In the expression (3.2) we have omitted a smooth term that is smaller than $E_0(x)$ by a factor $(\pi R/4\delta)^{1/2}\gg 1$. It coincides in form with the second term in (3.2) if, in it, x is replaced by $-x$.

Formula (3.2) represents the solution of the problem of the field distribution in the metal. Numerical integration of this expression can give a graphical description of the change of $E(x)$ with distance. Unfortunately we do not possess such technical resources. We shall therefore carry out an analytical investigation of formula (3.2).

We consider the structure of $E_0(x)$. On substituting in (3.1) the integral representation (2.12) of $F_0(\xi)$, we obtain $E_0(x)$ in the form

$$E_0(x) = -\frac{2\delta}{\pi} E'(0) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \left(\frac{x}{\delta}\right)^{-z} M(z-1)\Gamma(z)\cos\left(\frac{\pi z}{2}\right), \quad 0<c=\mathrm{Re}\, z<2. \tag{3.3}$$

At large distances from the surface, $\delta \ll x \ll R$, the behavior of $E_0(x)$ is described by an asymptotic expression that is the sum of the residues of the integrand, with sign reversed, at the simple poles $z=2$ and $z=3$:

$$E_0(x) \approx -\frac{iE(0)}{2M(-1)} \left(\frac{x}{\delta}\right)^{-2} \left[1 + \frac{4M(-1)}{\pi}\left(\frac{x}{\delta}\right)^{-1}\right]. \tag{3.4}$$

From (3.4) it is seen that with diffuse reflection of the electrons by the specimen boundary, the smooth part $E_0(x)$ of the field falls off faster, in the interior of the metal, than in the specular case (compare with formula (4.8) of [5], where $E_0(x) \sim x^{-3/2}$).

To analyze the second term of (3.2), it is necessary to expand it as a series in the small parameter $(4\delta/\pi R)^{1/2} \ll 1$. For this purpose we add to and subtract from the integrand a term with the same structure, but without the term containing $\sin(2R\xi/\delta - \pi/4)$ in the denominator. The integral of the added term describes the first spike. The remaining term (the difference between the original expression and the added term) is small in comparison with the first term of the expansion in the parameter $(4\delta/\pi R)^{1/2} \ll 1$. It contains the remaining spikes. From the difference thus obtained, by the same procedure, the second and third spikes are separated out. Separation of each successive spike adds to the remaining term a multiplier $(4\delta/\pi R\xi)^{1/2}$; therefore the expansion procedure is valid as long as the integrals over ξ remain finite at small ξ. Consequently the term containing $\Phi(\xi)$ in the exact formula (3.2) can be decomposed three times, and the term proportional to $\chi(\xi)$ five times. As a result, the expression for $E(x)$ takes the form

$$E(x) = E_0(x) + \frac{E'(0)\delta}{\pi} \sum_{n=1}^{5} \left(\frac{4\delta}{\pi R}\right)^{n/2} \Psi_n\left(\frac{x-2nR}{\delta}\right). \tag{3.5}$$

Formula (3.5) is convenient in that each term in it describes the field of the wave in its own region of variation of the coordinate x. Thus $E_0(x)$ is the field of the fundamental skin layer $0 \leqslant x \ll R$. The functions $\Psi_n((x-2nR)/\delta)$ describe the distribution of $E(x)$ in the vicinity of the n-th singularity; that is, when $|x-2nR| \ll R$. In the sum over n only nonmonotonic terms are left, and each of the functions $\Psi_n(t)$ is written to the lowest approximation in the parameter $(4\delta/\pi R)^{1/2} \ll 1$. The small smooth part of the field produced by the second term in (3.2) has been neglected in comparison with $E_0(x)$.

The first three spikes of $E(x)$ are determined by the expressions

TABLE I. Asymptotic forms of the spike terms for $|t| \gg 1$.

n	$\mathrm{Re}\,\Psi_n(t)$	$\mathrm{Im}\,\Psi_n(t)$
1	$\frac{315\sqrt{\pi}}{64}\lvert t\rvert^{-9/2}\Theta(t) - \frac{\sqrt{3\pi}}{8}\lvert t\rvert^{-5/2}\Theta(-t)$	$\frac{945\sqrt{3\pi}}{128}\lvert t\rvert^{-11/2}\Theta(t) - \frac{\sqrt{\pi}}{4}\lvert t\rvert^{-3/2}\Theta(-t)$
2	$\frac{1}{4\sqrt{3}}t^{-2}$	$-\frac{1}{4}t^{-1}$
3	$-\frac{75\sqrt{\pi}}{128}\lvert t\rvert^{-7/2}\Theta(t) - \frac{1}{16}\sqrt{\frac{\pi}{3}}\lvert t\rvert^{-3/2}\Theta(-t)$	$-\frac{175\sqrt{3\pi}}{256}\lvert t\rvert^{-9/2}\Theta(t) - \frac{\sqrt{\pi}}{8}\lvert t\rvert^{-1/2}\Theta(-t)$
4	$-\frac{1}{16\sqrt{3}}t^{-1}$	$\frac{1}{16}t^{-1}$

$$\Psi_1(t) = i\int_0^\infty \frac{d\xi}{\sqrt{\xi}} \frac{\Phi(\xi)\cos(\xi t-\pi/4) + \chi(\xi)\sin(\xi t-\pi/4)}{\xi^3-2i}, \tag{3.6}$$

$$\Psi_2(t) = \int_0^\infty \frac{d\xi}{\xi} \frac{\chi(\xi)\cos(\xi t) - \Phi(\xi)\sin(\xi t)}{(\xi^3-2i)^2}, \tag{3.7}$$

$$\Psi_3(t) = i\int_0^\infty \frac{d\xi}{\xi^{3/2}} \frac{\chi(\xi)\cos(\xi t-\pi/4) - \Phi(\xi)\sin(\xi t-\pi/4)}{(\xi^3-2i)^3}. \tag{3.8}$$

These spikes are characterized by oscillations of $E(x)$ that are localized at distances within some skin layer δ from the centers $x=2nR$ $(n=1, 2, 3)$, and that decrease rapidly at the edges of the line, for $|t| \gg 1$. The value of the field on the wings of the spikes decreases as a power of the parameter $\delta/|x-2nR| \ll 1$. The asymptotic forms of the functions (3.6)–(3.8) at large values of the arguments $|t|$ are given in Table I. From it, it is evident that the first and third spikes are sharply asymmetric: the field on the left wing $(2nR - x \gg \delta)$ attenuates much more slowly than on the right $(x-2nR \gg \delta)$. Such a behavior of the field is apparently observed experimentally.[3] The amplitude of the second spike falls off symmetrically with respect to the center $x=4R$. Near the centers of the singularities, the field varies linearly with distance.

The structure of the first three spikes indicates that their fields within the boundaries of the line are determined by the values of the wave number k from the fundamental skin layer $(k \sim \delta^{-1})$. Small values $k \sim R^{-1} \ll \delta^{-1}$ can have an effect only on the asymptotic behavior of the field far from the center, where $\delta \ll |x-2nR| \lesssim R$. The steeper the law of decay of $E(x)$ in the wings, the smaller is the contribution from $k \sim R^{-1}$ in the spectrum of the field of the spike. With increase of distance from the boundary $x=0$, the relative weight of a longwave component in the spectral resolution of $E(x)$ increases. Correspondingly, the decrease of the functions $\Psi_n(t)$ with increase of the argument also becomes smoother with increase of the number n (see table).

In the vicinity of $x=8R$ and $x=10R$ the field distribution is determined by the formulas

$$\Psi_4(t) = -\int_0^\infty \frac{d\xi}{\xi^2} \frac{\chi(\xi)\sin(\xi t)}{(\xi^3-2i)^4} - \int_0^\infty \frac{d\xi}{\xi^2} \frac{\Phi(\xi)\cos(\xi t)}{(\xi^3-2i)^2} \left[(\xi^3-2i)^2 + \frac{16\delta}{\pi R\xi}\sin^2(2R\xi/\delta-\pi/4)\right]^{-1}, \tag{3.9}$$

$$\Psi_5(t) = -i\int_0^\infty \frac{d\xi}{\xi^{5/2}} \frac{\chi(\xi)\sin(\xi t-\pi/4)}{(\xi^3-2i)^5} - i\int_0^\infty \frac{d\xi}{\xi^{5/2}} \frac{\Phi(\xi)\cos(\xi t-\pi/4)}{(\xi^3-2i)^3} \left[(\xi^3-2i)^2 + \frac{16\delta}{\pi R\xi}\sin^2(2R\xi/\delta-\pi/4)\right]^{-1}. \tag{3.10}$$

The first terms in the expressions (3.9) and (3.10) have a purely "spike" character. Their structure is similar to the structure of the integrals (3.6)–(3.8). The asymptotic form of the spike term of (3.9) for large $|t| \gg 1$ is given in the table.

The functions $\Psi_4(t)$ and $\Psi_5(t)$ are interesting in that they contain, along with spike terms, terms of another type—the second terms in (3.9) and (3.10). It is they that contain the qualitative difference of structure of $\Psi_4(t)$ and $\Psi_5(t)$ from the usual spike structure. Mathematically, these additional terms represent the remainder term in the expansion of the integral in (3.2), containing the function $\Phi(\xi)$, as a series in the small parameter $(4\delta/\pi R)^{1/2}$. We shall pause to analyze them.

We consider the second term in formula (3.9). The rapidly oscillating function $\sin^2(2R\xi/\delta - \pi/4)$ may be replaced, in the approximation under consideration, by its mean value $\frac{1}{2}$. Since the function $\Phi(\xi)/\xi^2 \sim \xi^{-1}$ at small ξ (see (2.21)), in the denominator of the integrand it is not permissible to neglect the small quantity $8\delta/\pi R\xi$. Otherwise the integral diverges logarithmically at the lower limit. This indicates that in this term there is an appreciable contribution from $\xi \sim \delta/R$ $(k \sim R^{-1})$. From the physical point of view, allowance for the term $8\delta/\pi R\xi$ means that in the formation of the center of the spike, the role of small $k \sim R^{-1}$ is comparable with the contribution of values $k \sim \delta^{-1}$ from the fundamental skin layer. The contribution of small $\xi \sim \delta/R \ll 1$ to the term under consideration, in the vicinity of the center of the line $|t| \gtrsim 1$ $(|x - 8R| \gtrsim \delta)$ is asymptotically

$$16\int_0^\infty \frac{d\xi}{\xi^2}\frac{\Phi(\xi)\cos(t\xi)}{(\xi^3-2i)^2}\left[(\xi^3-2i)^2+\frac{8\delta}{\pi R\xi}\right]^{-1}$$
$$\approx -\pi\exp\left(-i\frac{2\delta}{\pi R}|t|\right)+i\left[\operatorname{ci}\left(\frac{2\delta}{\pi R}|t|\right)\cos\left(\frac{2\delta}{\pi R}t\right)\right.$$
$$\left.+\operatorname{si}\left(\frac{2\delta}{\pi R}|t|\right)\sin\left(\frac{2\delta}{\pi R}|t|\right)\right]. \qquad (3.11)$$

Here $\operatorname{ci}(z)$ and $\operatorname{si}(z)$ are the cosine and sine integrals, respectively.[8] The expression (3.11) describes a quasiharmonic component of the wave field. A characteristic scale for its variation is R, whereas the spike terms are localized at distances of order δ. We note that formula (3.11) describes the behavior of $\Psi_4(t)$ at the edges of the line, $\delta \ll |x - 8R| \lesssim R$.

The quasiharmonic field, which first appears in $\Psi_4(t)$, penetrates into the depth of the metal, remaining practically unchanged in amplitude. It is easy to demonstrate this by calculating the asymptotic form of the quasiharmonic term contained in $\Psi_5(t)$. In the range $|t| \gtrsim 1$ $(|x - 10R| \gtrsim \delta)$ we have

$$32\int_0^\infty \frac{d\xi}{\xi^{5/2}}\frac{\Phi(\xi)\cos(\xi t-\pi/4)}{(\xi^3-2i)^3}\left[(\xi^3-2i)^2+\frac{8\delta}{\pi R\xi}\right]^{-1}$$
$$\approx \pi\left(\frac{\pi R}{2\delta}\right)^{1/2}\left[\sqrt{2}\,S\left(\sqrt{\frac{2\delta}{\pi R}|t|}\right)\sin\left(\frac{\pi}{2}\Theta(t)+\frac{2\delta}{\pi R}t\right)\right.$$
$$\left.-\sqrt{2}\,C\left(\sqrt{\frac{2\delta}{\pi R}|t|}\right)\cos\left(\frac{\pi}{2}\Theta(t)-\frac{2\delta}{\pi R}t\right)-i\cos\left(\frac{\pi}{4}-\frac{2\delta}{\pi R}t\right)\right]. \qquad (3.12)$$

Here $S(z)$ and $C(z)$ are the Fresnel integrals,[8] and $\Theta(x)$ is the unit step function ($\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$).

The appearance of quasiharmonic terms in $\Psi_4(t)$ and $\Psi_5(t)$ indicates that in the analysis of the behavior of $E(x)$, the anomalous-skin-effect approximation $kR \gg 1$ is no longer completely adequate in the vicinities of $x = 8R$ and $x = 10R$. Nevertheless formulas (3.9) and (3.10) give a qualitatively correct description of the actual situation.

The functions $\Psi_4(t)$ and $\Psi_5(t)$ illustrate graphically the transition from spikes to a quasiharmonic variation of $E(x)$ with increase of the distance x from the boundary. In $\Psi_4(t)$ the magnitude of the spike in the vicinity of the center of the line is of the same order as the quasiharmonic term. In $\Psi_5(t)$ the amplitude of the quasiharmonic field is larger than the field of the spike by a factor $(\pi R/2\delta)^{1/2} \gg 1$, and it makes no sense to speak of a spike in $\Psi_5(t)$.

The field distribution $E(x)$ at distances $x \gtrsim 12R$ is described by the integral in (3.2) that contains the function $\chi(\xi)$. In this range, the integral in question has a quasiharmonic structure. In view of the qualitative nature of the approximation being used, we shall not give the explicit form of the asymptotic approximation to the field in the regions $x \sim 12R$ and $x \sim 14R$. We remark only that in the region $|x - 12R| \gtrsim \delta$ the distribution $E(x)$ is described by an expression similar to (3.11), and for $|x - 14R| \gtrsim \delta$ by a formula of the type (3.12).

Summarizing what has been said above, we can state that the field distribution $E(x)$ contains four spikes. The first three spikes are clearly expressed, whereas the fourth exists against the background of a quasiharmonic distribution.

The analysis of the formulas obtained in this section for the $E(x)$ distribution relates to the case of low frequencies, $\omega \ll \nu$. Under this condition the depth δ of the skin layer may be considered a real quantity, and the real and imaginary parts of the ratio $E(x)/E'(0)$ are determined by the values of $\operatorname{Re}\Psi_n(t)$ and $\operatorname{Im}\Psi_n(t)$, respectively. The results of the present paper are actually valid over a much broader range of frequencies $\omega, \nu \ll \Omega$, when the value of δ is in general complex (see (2.8)). The general situation is easily analyzed by a method similar to that set forth above. The occurrence of a phase in δ leads to the result that the real and imaginary parts of $E(x)/E'(0)$ will be determined by combinations of $\operatorname{Re}\Psi_n(t)$ and $\operatorname{Im}\Psi_n(t)$ in the region of each spike, and the field-distribution picture will become somewhat more complicated than in the low-frequency case.

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Translated by W. F. Brown, Jr.

Dynamics of fluctuations at multicritical points

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(Submitted June 28, 1976)
Zh. Eksp. Teor. Fiz. 72, 308–316 (January 1977)

The dynamics of phase transitions at multicritical points is investigated. Systems whose free energy can be represented in a Ginzburg-Landau form with one order parameter are considered. The dependence of the critical dynamics on conservations laws in the system is considered. Different types of universal dynamical behavior are distinguished. The effect of many-particle excitations on the critical-damping frequency is established. The logarithmic corrections to the theory of dynamic scale invariance that arise from the interaction of fluctuations at a tricritical point are found. The possibility that these effects are manifested in nondegenerate metamagnets is discussed.

PACS numbers: 64.60.Kw

One of the most attractive features of critical phenomena is their universality. This consists in the fact that the critical exponents and a number of other quantities depend only on such characteristics of physical systems as the dimensionality, the number of components of the order parameter, and the symmetry.[1] In the dynamics of critical fluctuations the concept of universality takes on a narrower meaning—the conservation laws for the order parameter and the energy in the system also become important. (Here it is understood that the system is on a critical line in the space of the thermodynamic variables and external fields.)

However, curves of continuous transitions can terminate at certain points, passing into phase-coexistence curves. Such behavior occurs at certain points on the phase diagrams of a ^3He–^4He mixture, metamagnets, and compressible magnets.[2–10] Such points are critical for several phases at once (ordered, disordered and mixed), and the number σ of phases determines the order of the critical points. It has been established that the static properties of phase transitions change substantially at these points.[11–13] The changes that the dynamics of fluctuations can undergo at multicritical points are unknown. The role of the conservation laws at such points has also not been elucidated. The present paper is devoted to examining this group of questions.

1. At an ordinary critical point ($\sigma=2$) two-particle excitations of the energy density can decay into one-particle excitations of the order parameter ψ, changing the damping of the latter.[14] But in the case of higher critical points the fluctuations of ψ may be coupled nonlinearly with excitations consisting of more than two fluctuations. We show below that if such excitations are not damped they determine the dynamics of the critical fluctuations in many respects.

The local field $\rho_k(\mathbf{x})$ of excitations consisting of k particles can be constructed in the form of a product of k fluctuating fields of the n-component order parameter $\psi(\mathbf{x})$. In the general case the quantity ρ_k is a tensor of rank k. Its components (or their linear combinations) should be such that the operations of the symmetry group of the free energy do not change them. This ensures the conservation of the total field ρ, equal to the integral of the local field $\rho_k(\mathbf{x})$ over the volume (henceforth we shall speak simply of the conservation of the field $\rho_k(\mathbf{x})$).

We shall consider a free energy that is isotropic in the space of the components of the order parameter, when the field ρ_k is characterized by a tensor contracted with respect to the maximum possible number of pairs of indices. As a result, for even k the field ρ_k is a scalar field and for odd k it is a vector field with n components. For many-particle excitations to be manifested in the critical dynamics it is important that, in the equations of motion, the nonlinearities due to the coupling of one-particle and many-particle excitations do not fall off as we go to large lengths (small frequencies); this imposes restrictions on the scaling dimensions of the nonlinearities. This requirement is inherently linked with the condition that the inclusion of many-particle excitations in the equations of motion should not change the equilibrium distribution function of the field $\psi(x)$.

We write the free energy of the system in the form of the functional

$$\frac{\mathscr{H}(\psi)}{T}=\int d^dx\left[\frac{1}{2}|\nabla\psi(\mathbf{x})|^2+\sum_{k=1}^{\sigma}u_{2k}(\psi\psi)^k\right], \qquad (1)$$

where d is the dimensionality of the system. We shall describe the dynamics of the model by the equation

