Stochastic breaking of bound states in a system of atoms interacting with a radiation field

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We consider in the framework of a semi-classical theory the interaction of a system of atoms with a radiation field. We show that taking the non-resonance term into account leads to a new effect: the appearance of stochastic trajectories while there are no random parameters in the system. The size of the stochastic region (the stochastic layer) is determined by the dimensionless interaction constant Λ. When $\Lambda \ll 1$ the size of the stochastic layer is of the order of $32\pi \exp(-\pi\sqrt{2}/\Lambda)$. When $\Lambda \gtrsim 1$ one gets complete stochastization of the motion. From the estimates obtained it follows that when $\Lambda \gtrsim 1$, the time to destroy coherence due to stochastic instability can be several times less than the time for spontaneous emission, and this may thereby restrict the limiting coherence of the system.

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§1. INTRODUCTION

The interaction of two-level atoms with a radiation field is one of the basic models for the study of the atom+field system. Notwithstanding the simplified character of the model many physical processes are described within its framework (transformation of the radiation, resonance fluorescence, superradiation, and so on). A theoretical analysis of the model has up to now not been complete, and in the course of time one has detected in it more and more new physical properties.

From the time of the pioneering paper by Dicke[1] which predicted the superradiant state of the system, the ground state of the atom+field system[2] was found to be unstable when

$$\Lambda = (16\pi\rho\mu^2/\hbar\omega)^{1/2} > 1, \tag{1}$$

where Λ is the dimensionless constant for the interaction between the atom and the field, ρ the density of the atoms, μ the matrix element of the dipole moment of the atom, and ω the frequency of the atomic transition, which is the same as the frequency of the field (resonance condition). Quite recently Hepp and Lieb,[3] in a study of the thermodynamic properties of the system in the framework of exactly the same model, have discovered in it a second-order phase transition when condition (1.1) is satisfied and at temperatures below the critical one. In the region of the ordered phase there appears a coherent state of the atoms[4] characterizing the formation of "order from disorder." A new surprising property of the two-level atoms+field system which was described in this paper was the reverse effect of the formation of "disorder from order." In other words, the system in the coherent (superradiant) state is being stochasticized.

We dwell in detail on this effect. When we describe the interaction of the atoms with the field under resonance conditions we neglect the non-resonance terms (the so-called rotating field approximation). The basic results for the spectrum and the dynamics of the system were obtained just in this approximation[2,5–8] (see also the review article[9]). The energy of the atoms is periodically (and, under well defined conditions, aperiodically) changed into field energy and vice versa, characterizing a bound state of the atoms and the field. The neglect of the non-resonance terms is so weakly based because of the formal difficulties which occur, that the generally accepted use of the truncated Hamiltonian of the system has changed into a sort of symbol of faith. We shall show below that the perturbation due to the non-resonance term leads to a breaking of the bound state and that this breaking is stochastic in character, and we determine the region in which this breaking occurs.

The stochastic instability of the motion occurs in a purely dynamic system (i.e., without there being any random parameters or forces present) as the result of the existence of a local instability of trajectories in phase space[10,11]: the distance between initially arbitrarily close points in phase space increases exponentially with time. Such a behavior of the trajectories of the system leads to a decoupling of the time correlations of the fast variables (phases) and to the formation of an equilibrium distribution of the variables of the system (differences in populations, number of field photons) in the region where the breaking occurs, the so-called stochastic layer.

When $\Lambda \ll 1$ the above-mentioned result can be obtained analytically (§3). The width of the stochastic layer is small and lies in the region of the integrals of motion of the system corresponding to a regime close to where the energy is completely transformed from the atoms to the field and vice versa. When Λ increases the width of the stochastic layer increases. A study of the exact equations of motion for $\Lambda \gtrsim 1$ is made by using an electronic computer and we give the corresponding numerical data in §4. It turns out that a value $\Lambda \sim 1$ is, just as in problems about the spectrum and about the phase transition, the critical one: when $\Lambda \gtrsim 1$ the dimensionless width of the stochastic layer is equal to unity. In the system the angular-momentum integral is broken and the system performs a statistical motion on an energy surface. In the Conclusion (§5) we discuss in con-

nection with the results obtained problems about the occurrence of stochastic instabilities in quantum systems and possible conditions for the realization of the obtained effect.

§2. SELF-CONSISTENT EQUATIONS OF MOTION

We consider a system of N two-level atoms which interact with a resonant field E in a volume V and which are in a coherent state. We use a semi-classical model to describe this system[6]: the field is assumed to be a classical object and the atoms quantum objects. The state of the i-th atom is described by a wavefunction $\psi_i = a_i\psi^{(1)} + b_i\psi^{(2)}$, which is a superposition of the two basic states of the atom $\psi^{(1)}$ and $\psi^{(2)}$. We introduce the difference in populations and the dipole moment of the atom:

$$n_i = |b_i|^2 - |a_i|^2, \quad M_i = \mu(a_i b_i^* + a_i^* b_i) \equiv \mu m_i, \tag{2.1}$$

where μ is the matrix element of the dipole moment. We define for a system of N atoms the densities of these quantities:

$$n = \frac{1}{N}\sum_{i=1}^{N} n_i, \quad m = \frac{1}{N}\sum_{i=1}^{N} m_i. \tag{2.2}$$

If the field is nearly uniform ($\lambda \gg V^{1/3}$) one can describe the system of atoms as a whole by means of the quantities n and m. The equations of motion have the form

$$\begin{aligned} &\ddot{E} + \omega^2 E = 4\pi\omega^2\mu\rho m, \\ &\ddot{m} + \omega^2 m = -\frac{2\mu\omega}{\hbar} En, \quad \dot{n} = \frac{2\mu}{\hbar\omega}E\dot{m}, \end{aligned} \tag{2.3}$$

where $\rho = N/V$ is the density of the atoms. The difference between the set (2.3) and the one obtained by Jaynes and Cummings[6] lies in the fact that $N \neq 1$ and that the cooperation number[1] r which in the present case is given by the expression

$$r^2 = n^2 + m^2 + \dot{m}^2/\omega^2, \tag{2.4}$$

is also different from unity. The quantity r is an exact integral of motion of the set (2.3). Moreover, Eqs. (2.3) have the energy integral which we write in dimensionless form:

$$C = \dot{e}^2/\omega^2 + e^2 + n - \Lambda e m, \quad e = (4\pi\hbar\omega\rho)^{-1/2}E. \tag{2.5}$$

Here Λ is the dimensionless coupling constant given by (1.1).

It is convenient to introduce action-angle variables:

$$\begin{aligned} &e = (I_e/\omega)^{1/2}\cos\varphi_e, \quad \dot{e} = -(\omega I_e)^{1/2}\sin\varphi_e, \\ &m = (I_m/\omega)^{1/2}\cos\varphi_m, \quad \dot{m} = -(\omega I_m)^{1/2}\sin\varphi_m. \end{aligned} \tag{2.6}$$

We can then write the set (2.3) in the following canonical form

$$\begin{aligned} H &= \omega^2 C = I_e\omega + \omega^2(r^2 - I_m/\omega)^{1/2} - \frac{1}{2}\Lambda\omega(I_e I_m)^{1/2}(\cos\psi + \cos\varphi), \\ \dot{I}_e &= -\frac{\partial H}{\partial\varphi_e} = -\frac{1}{2}\omega\Lambda(I_e I_m)^{1/2}(\sin\psi + \sin\varphi), \\ \dot{\varphi}_e &= \frac{\partial H}{\partial I_e} = \omega - \frac{1}{4}\omega\Lambda\left(\frac{I_m}{I_e}\right)^{1/2}(\cos\psi + \cos\varphi), \\ \dot{I}_m &= 2n\frac{\partial H}{\partial\varphi_m} = -\omega n\Lambda(I_e I_m)^{1/2}(\sin\psi - \sin\varphi), \\ \dot{\varphi}_m &= -2n\frac{\partial H}{\partial I_m} = \omega + \frac{1}{2}\omega\Lambda n\left(\frac{I_e}{I_m}\right)^{1/2}(\cos\psi + \cos\varphi), \\ n^2 &= r^2 - I_m/\omega, \quad \psi \equiv \varphi_e - \varphi_m, \quad \varphi \equiv \varphi_e + \varphi_m. \end{aligned} \tag{2.7}$$

We denote the number of photons per atom by $y = I_e/\omega$ and rewrite (2.7) in the form

$$\begin{aligned} C &= y + n - \frac{1}{2}\Lambda[y(1-n^2)]^{1/2}(\cos\psi + \varepsilon\cos\varphi), \\ \dot{n} &= \frac{\omega}{2}\Lambda[y(1-n^2)]^{1/2}(\sin\psi - \varepsilon\sin\varphi), \\ \dot{\psi} &= -\frac{\omega}{4}\Lambda\left\{\left(\frac{1-n^2}{y}\right)^{1/2} + 2n\left(\frac{y}{1-n^2}\right)^{1/2}\right\}(\cos\psi + \varepsilon\cos\varphi), \\ \dot{y} &= -\frac{\omega}{2}\Lambda[y(1-n^2)]^{1/2}(\sin\psi + \varepsilon\sin\varphi), \\ \dot{\varphi} &= 2\omega - \frac{\omega}{4}\Lambda\left\{\left(\frac{1-n^2}{y}\right)^{1/2} - 2n\left(\frac{y}{1-n^2}\right)^{1/2}\right\}(\cos\psi + \varepsilon\cos\varphi), \end{aligned} \tag{2.8}$$

where we have artificially introduced a parameter ε which takes on values 0 or 1 and have put $r = 1$.

The meaning of these operations consists in the following. It is clear from Eqs. (2.3) and (2.4) that we can introduce the following quantities, which are normalized by r,

$$\begin{aligned} &\bar{y} = y/r, \ \bar{n} = n/r, \ \bar{m} = m/r, \ \bar{\dot{m}} = \dot{m}/r, \\ &\bar{\rho} = \rho r, \ \bar{\Lambda} = \Lambda\sqrt{r}. \end{aligned}$$

Therefore, for the normalized variables, allowance for $r \neq 1$ reduces solely to replacing Λ by $\bar{\Lambda}$, and the form of the equations is retained. We shall omit the bar in the set (2.8) and in what follows everywhere. When $\varepsilon = 1$ the set (2.8) is exactly equivalent to the set (2.7). The resonance approximation for these equations is connected with neglecting terms containing $\cos\varphi$ which is equivalent to $\varepsilon = 0$. In the latter case one gets a closed set of three equations for n, ψ, and y which has an additional integral of motion

$$P = -1/2\Lambda[y(1-n^2)]^{1/2}\cos\psi = \text{const}(\varepsilon = 0). \tag{2.9}$$

The usual argument on which the dropping of the nonresonance term in (2.7) and (2.8) is based is connected with the fact that either $\Lambda \ll 1$ or else the term $\cos\varphi$ is a high-frequency one. In actual fact this is not always true. When $\varepsilon = 0$ the motion of the system is along a closed trajectory which lies on the constant energy surface $C = \text{const}$. in phase space (see Fig. 1, where $\theta = \arccos n$). Each trajectory is determined by the two integrals of motion C and P and corresponds to the nutation of the energy spin of the atoms. Among the family of all trajectories there is one special trajectory (separatrix) for which $C = 1$, $P = 0$. It passes through the points $n = \pm 1$, $\dot{n} = 0$ and describes the aperiodic regime for emission or absorption of the field by the atoms. It is well known that any arbitrarily small perturbations

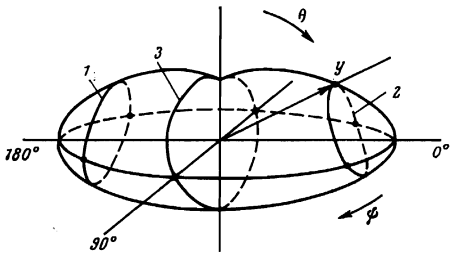

FIG. 1. Constant energy surface $C=1$ in the y, θ, ψ—phase space: 1) $P>0$, 2) $P<0$, 3) $P=0$ separatrix.

lead to very strong distortions of the trajectories which lie in the vicinity of the separatrix. This distortion is stochastic in character, i.e., regular motions of the system under the influence of the perturbation change in the vicinity of the separatrix into random motions[10-12] (see also the review article[13]). The appearance of stochasticity in the set (2.7) or (2.8) leads, as we shall show in what follows, to important physical consequences.

§3. STOCHASTIC DISTORTION OF THE TRAJECTORIES FOR THE CASE OF WEAK COUPLING ($\Lambda \ll 1$)

When $\Lambda \ll 1$ we can study analytically the effect of the non-resonance term on the nature of the motion of the system. We introduce for the unperturbed motion, using the integrals P of (2.9) and C of (2.8), the quantity

$$\mathscr{E}=C-P=n+y, \tag{3.1}$$

which is also an integral of motion. One can directly check that the following equations are valid:

$$\dot{n}=\omega\frac{\partial}{\partial\psi}P(n,\psi;\mathscr{E}), \quad \dot{\psi}=-\omega\frac{\partial}{\partial n}P(n,\psi;\mathscr{E}), \tag{3.2}$$

where we have used (3.1) to make in P the substitution:

$$P(n,\psi;\mathscr{E})=P(n,\psi;\ y=\mathscr{E}-n)=-{}^1/_2\Lambda[(\mathscr{E}-n)(1-n^2)]^{1/4}\cos\psi. \tag{3.3}$$

Therefore, $P(n,\psi,\mathscr{E})$ is the Hamiltonian in the space of the variables n and ψ.

We use (3.2) and (2.8) to write the perturbed equations in the form

$$\begin{aligned}&\dot{n}=\omega\frac{\partial P}{\partial\psi}+\Delta\dot{n}, \quad \dot{\psi}=-\omega\frac{\partial P}{\partial n}+\Delta\dot{\psi},\\ &\Delta\dot{n}=-{}^1/_2\omega\Lambda[(\mathscr{E}-n)(1-n^2)]^{1/4}\sin\varphi,\\ &\Delta\dot{\psi}=-\frac{1}{4}\omega\Lambda\left\{\left[\frac{1-n^2}{\mathscr{E}-n}\right]^{1/4}+2n\left[\frac{\mathscr{E}-n}{1-n^2}\right]^{1/4}\right\}\cos\varphi.\end{aligned} \tag{3.4}$$

The perturbing terms in (3.4) can lead either to a change in the form of the integral of motion P or to its complete breaking.

It is further convenient to change to the variables n and $\dot{n}$. To do this we consider the expression

$$\mathscr{P}\equiv{}^1/_2\omega^2P^2. \tag{3.5}$$

In the unperturbed case it follows from (2.8) that

$$\sin^2\psi=4\dot{n}^2/\omega^2\Lambda^2(\mathscr{E}-n)(1-n^2). \tag{3.6}$$

Substituting (2.9) and (3.6) into (3.5) we get

$$\mathscr{P}=-\frac{\dot{n}^2}{2}+\frac{1}{8}\omega^2\Lambda^2(\mathscr{E}-n)(1-n^2). \tag{3.7}$$

We can easily directly verify that the equations of motion can be written in the following Hamiltonian form:

$$\frac{d}{dt}n=-\frac{\partial\mathscr{P}}{\partial\dot{n}}, \quad \frac{d}{dt}\dot{n}=\frac{\partial\mathscr{P}}{\partial n}. \tag{3.8}$$

Using the integral of motion $\mathscr{P}$ we can easily integrate the set (3.8) and the solution for $n(t)$ has the form[6]

$$\begin{aligned}&n=n_1+(n_2-n_1)\,\mathrm{sn}^2[{}^1/_4\omega\Lambda\sqrt{n_3-n_1}(t+t_0);\ q],\\ &q=[(n_2-n_1)/(n_3-n_1)]^{1/2},\end{aligned} \tag{3.9}$$

where q is the parameter of the Jacobian elliptic function, and $n_1<n_2<n_3$ the roots of the equation

$$(1-n^2)(\mathscr{E}-n)-8\mathscr{P}/\omega^2\Lambda^2=0. \tag{3.10}$$

We find thus in the unperturbed case from (3.6) and (3.9) the functions $n(t)$, $\dot{n}(t)$, and $\psi(t)$ for different values of the integrals of motion $\mathscr{P}$ and $\mathscr{E}$.

We turn to a determination of the equation which describes the change in the integral P under the influence of a perturbation. We have from (3.3), using the canonical nature of the variables n and ψ, the exact relation

$$\dot{P}=\frac{\partial P}{\partial n}\dot{n}+\frac{\partial P}{\partial\psi}\dot{\psi}+\frac{\partial P}{\partial\mathscr{E}}\dot{\mathscr{E}}=\frac{\partial P}{\partial n}\Delta\dot{n}+\frac{\partial P}{\partial\psi}\Delta\dot{\psi}+\frac{\partial P}{\partial\mathscr{E}}\dot{\mathscr{E}}.$$

After substituting the derivatives of P and the definition $\mathscr{E}=C-P$ ($\dot{C}\equiv 0$) we get from this the required equation:

$$\dot{P}={}^1/_8\omega\Lambda^2(1+2n\mathscr{E}-3n^2)\sin(\psi+\varphi)/[1-P/2(\mathscr{E}-n)]. \tag{3.11}$$

As the right-hand side is considered to be a perturbation we can use in it for n, ψ, and φ the unperturbed expressions. In particular, $\varphi\approx 2\omega t$. Of most importance for us will be the region of trajectories close to the separatrix ($P=0$, $\mathscr{E}=C=1$). According to (3.9) the unperturbed motion in the vicinity of the separatrix has the following form: $n(t)$ changes almost from -1 to $+1$ and it stays a very long time near the turning point $n=+1$ which is of a hyperbolic type, since the period of the oscillations tends to ∞ as one approaches the separatrix. Furthermore, we have from (3.10) for the characteristic roots as $\mathscr{P}\to 0$

$$\begin{aligned}&n_1\approx-1+2\mathscr{P}/\omega^2\Lambda^2=-1+P^2/\Lambda^2,\\ &n_{2,3}\approx 1+(\mathscr{E}-1)/2\mp[(\mathscr{E}-1)^2/4+2P^2/\Lambda^2]^{1/4}.\end{aligned} \tag{3.12}$$

Approaching the separatrix means also $\mathscr{E}-1\to 0$, and for sufficiently small values of $\mathscr{E}-1$ we have $n_{2,3}\approx 1\mp 2^{1/2}P/\Lambda$. Hence it follows that the denominator in (3.11) is ~ 1 as the maximum deviation of it from 1 is not more than $\sim\Lambda\ll 1$. We find from (3.12) that the expression $1+2n\mathscr{E}-3n^2\approx -4$ during a small time interval of the order of the period of the small oscillations $2\pi/\omega_0$, when the system is sufficiently far from the singular point $n=+1$. On the other hand, in the vicinity of the

singular point the quantity $1+2n\mathcal{E}-3n^2$ is very small ($\sim P/\Lambda$) during a very long time interval $2\pi/\omega(P)$ where $\omega(P)$ is the non-linear frequency of the oscillations of the system.

Using (2.8) we can thus write instead of (3.11)

$$\dot{P}\approx{}^1/_2\omega\Lambda^2 A(t)\sin\theta, \quad \dot{\theta}\approx 2\omega+O(\Lambda), \tag{3.13}$$

where $A(t)$ is a function in the form of pulses that follow periodically with a period $2\pi/\omega(P)$ and with a height ~ 1 and a width $\sim 2\pi/\omega_0$. The quantity ω_0 determines the frequency of the small oscillations of the atoms + field system and when $\mathcal{E}\sim 1$ according to (3.7) $\omega_0=\omega\Lambda/\sqrt{2}$. Using this feature of $A(t)$ we change from the set (3.13) to equations for discrete transformations

$$P_{m+1}\approx P_m+\Delta P,$$
$$\theta_{m+1}\approx\theta_m+\frac{4\pi\omega}{\omega(P_{m+1})}\approx\theta_m+4\pi\frac{\omega}{\omega(P_m)}-4\pi\frac{\omega}{\omega^2(P_m)}\frac{d\omega(P_m)}{dP_m}\Delta P,$$
$$\Delta P=\frac{\omega\Lambda^2}{2}\int A(t)\sin\theta\,dt, \tag{3.14}$$

where P_m and θ_m are the values of $P(t)$ and $\theta(t)$ immediately preceding the action of the m-th pulse of the function $A(t)$.

The set (3.14) has been well studied (see, e.g.,[13]) and the nature of its solution is determined by the magnitude of the parameter

$$K=\frac{4\pi\omega}{\omega^2(P)}\left|\frac{d\omega(P)}{dP}\right|\Delta P. \tag{3.15}$$

When $K\ll 1$ which means that the change in phase in (3.14) with time due to the perturbation is small, the system performs conditionally periodic oscillations around the unperturbed trajectory—the so-called phase oscillations. On the other hand, when $K\gg 1$ the motion of the system is stochastic and is characterized by the time for the decoupling of the phase correlations t_c:

$$R(t)=\langle\exp i[\theta(t_1)-\theta(t_1+t)]\rangle\propto\exp(-t/t_c). \tag{3.16}$$

It has been shown earlier[12] that when $K\gg 1$ the time t_c is given by the relation

$$t_c=1/\omega(P)\ln K. \tag{3.17}$$

The region where the transition from a regular dynamic motion to a stochastic one takes place can be found from the condition

$$K(P,C)\sim 1. \tag{3.18}$$

Equation (3.18) defines a curve $P_0=P_0(C_0)$ such that for P lying in the region

$$0<|P|<|P_0(C_0)|, \tag{3.19}$$

there occurs a stochastic instability of the trajectories. We call the region (3.19) the stochastic layer. It follows from (3.13) that the stochasticity of the phases θ leads also to stochastic changes in the quantity P. This means that in the region (3.19) the integral of motion is stochastically broken. The time t_D for such a breaking is defined as the time for the diffusion of P.

Using the fact that the change ΔP is small we can write down the Fokker–Planck equation for the distribution function $f(P,t)$:

$$\frac{\partial f}{\partial\tau}=-\frac{\partial}{\partial P}(\langle\Delta P\rangle f)+\frac{1}{2}\frac{\partial^2}{\partial P^2}(\langle(\Delta P)^2\rangle f) \quad (\tau=\omega t), \tag{3.20}$$

where the averaging of ΔP is over θ in the time interval $t_c\ll t\ll t_D$. We have in order of magnitude

$$\frac{\partial f}{\partial t}\approx\frac{1}{t_D}\frac{\partial^2 f}{\partial P^2}, \quad t_D\sim(\max\Delta P)^{-2}. \tag{3.21}$$

Since there is no percolation at the boundaries of the region (3.19) the equilibrium $f(P)$ is constant and this also determines the distribution in P in the stochastic layer.

We evaluate the parameter K on whose magnitude the nature of energy transfer in the atoms + field system depends. According to (3.13) and (3.14) we consider the change in the integral P during one step of the transformation:

$$\Delta P\approx\frac{1}{2}\omega\Lambda^2\int A(t)\sin(2\omega t+\theta_0)\,dt. \tag{3.22}$$

We have already noted that the characteristic frequency for the change in $A(t)$ is equal to $\omega_0=\omega\Lambda/\sqrt{2}\ll\omega$. The integral (3.22) is therefore exponentially small:

$$\max\Delta P\sim 4\sqrt{2}\,\pi\Lambda\exp\left(-\frac{\pi\sqrt{2}\,\alpha}{\Lambda}\right), \tag{3.23}$$

where the constant $\alpha\sim 1$ and where we have used the fact that $A(t)\sim 1$ over a time interval $\sim\pi/\omega_0$.

The simplest way to obtain the expression for $\omega(P)$ is from the solution (3.9) the frequency of which for $q'^2=1-q^2\ll 1$ and $C\approx 1$ equals

$$\omega(P)\approx\frac{\pi}{\sqrt{2}}\omega\Lambda\Big/\ln\frac{8\sqrt{2}\,\Lambda}{P}. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.15) we get

$$K=\frac{32\pi}{P}\exp\left(-\frac{\pi\sqrt{2}\,\alpha}{\Lambda}\right). \tag{3.25}$$

Hence, putting $K(P_0, C_0\approx 1)=1$ we get the boundary of the stochastic layer for $C_0=1$:

$$P_0=32\pi\exp(-\pi\sqrt{2}\,\alpha/\Lambda). \tag{3.26}$$

Finally, we can use (3.17) to write down the rate of decay of the nutational motion in the stochastic layer:

$$\gamma_c=1/t_c=\omega(P)\ln K\approx\frac{\pi}{\sqrt{2}}\omega\Lambda\ln K\Big/\ln\frac{8\sqrt{2}\,\Lambda}{P} \quad (P<P_0). \tag{3.27}$$

For small values of Λ the width of the stochastic layer

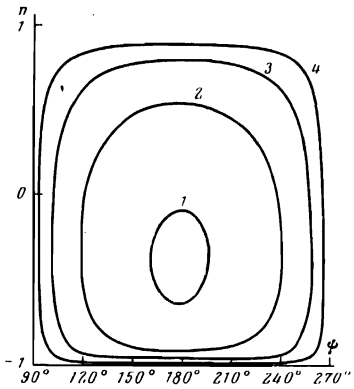

FIG. 2. Unperturbed trajectories for different values of Λ ($C=1$, $P=-0.1$): 1) $\Lambda=0.2$; 2) $\Lambda=0.4$; 3) $\Lambda=1.0$; 4) $\Lambda=3.0$.

is $\sim P_0$ and exponentially small. Thus, for $\Lambda=0.4$ the quantity $P_0 \sim 10^{-3}$ to 10^{-4}. It follows from Eq. (3.7) that such values of P can be reached only in the case when the difference in populations takes the value $n=+1$ with an accuracy of 10^{-3} to 10^{-4}. As this means that there is practically a complete transfer of the energy from the atoms to the field and vice versa one can reach the conclusion that just such a regime is liable to stochastic disruption. It is accompanied by the broadening of all dynamic lines of the spectrum of the vibrations of the system by a magnitude γ_c. To use Eq. (3.27) to estimate the line widths is not at all simple as it is necessary to average in it over the various values of K. However, due to the presence of logarithms it is clear that the quantity γ_c/ω is not too small. We find in the next section from a numerical analysis from $\Lambda=0.4$ that $\gamma_c/\omega \sim 10^{-2}$. From this it follows that it is just the stochastic instability which in a well-defined region of parameters restricts the limiting degree of coherence of the system.

One should also note that there may be also a broadening of the spectrum outside the stochastic layer which is not of a stochastic, but of a dynamic type (this problem will be discussed in detail in §4). When Λ increases the width of the stochastic layer increases and one may expect that its relative width will for $\Lambda \sim 1$ also be ~ 1. A proof of this statement can, however, only be given numerically (§4).

§4. NUMERICAL ANALYSIS OF THE EQUATIONS OF MOTION

We performed a numerical computed analysis of the set (2.8). For convenience we introduced a dimensionless time $\tau=\omega t$. In the general case the set (2.8) has a single integral the energy C and the region of motion in phase space is a three-dimensional volume bounded by a closed surface. When $\varepsilon=0$ there is an additional integral P of (1.9) and the motion is over a two-dimensional closed surface $C=\text{const}$. The quantities C and P determine a two-parameter family of trajectories which are closed curves on the surface (see Fig. 1). These curves correspond to the nutation of the energy spin of the atoms. We give in Fig. 2 the projections onto the (n,ψ) plane of trajectories with different values of the interaction constant Λ. When Λ increases the angles of the trajectories become sharper.

When $\Lambda \ll 1$ and $\varphi=2\tau$ the perturbation from the nonresonance term $\cos\varphi$ reduces to the action of a periodic force on a non-linear system. Far from the separatrix the motion retains under the perturbation its conditionally periodic character[12] and is determined by a new integral $\bar{P}$. This motion can be represented by trajectories wound on a toroidal tube, the axis of which is the unperturbed trajectory. The diameter of the tube is determined by the width of the non-linear resonance between the system and the external force. The projection of such a trajectory on the (n,ψ) plane is a continuously filled layer. This corresponds to an everywhere dense covering of the toroidal tube by trajectories. An example of such a layer is given in Fig. 3. It has a rather well-defined width Δn which corresponds to the appearance of subharmonics near the frequency of the coherent nutation $\omega(P)$. The magnitude of $\omega(P)$ is equal to the frequency for the exchange of energy between the atoms and the field. Due to the perturbation, this process is modulated by a periodic "ripple" with a modulation amplitude Δn.

When P decreases the trajectories approach the separatrix[1)] and in some neighborhood of the separatrix fall into the stochastic layer. We have already noted that there occurs in such a layer a local instability of trajectories in phase space: the distance between two close trajectories increases exponentially with time. This property of the trajectories can be put as the basis for the observation of the stochastic instability in the numerical analysis.[10] We define the distance between two trajectories by the expression

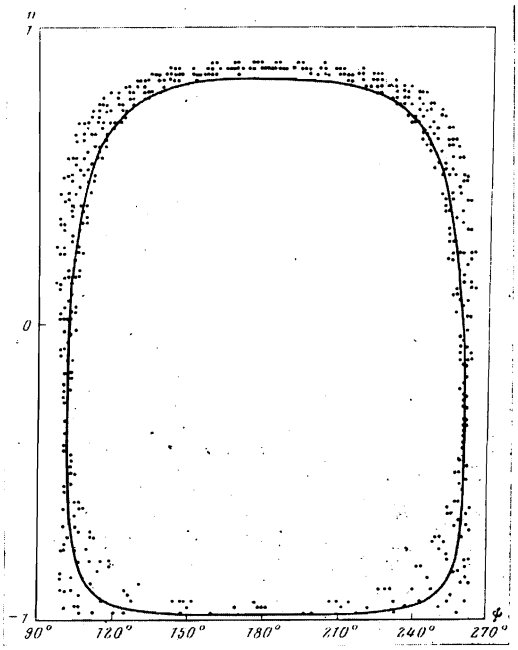

FIG. 3. Points of the perturbed trajectory $\Lambda=0.4$, $C=1$, $P=0.04$. The solid curve is the unperturbed trajectory.

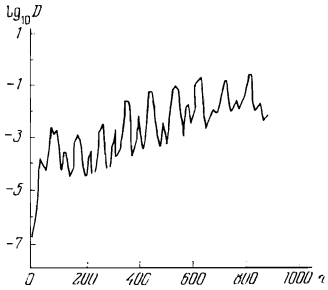

FIG. 4. The time-dependence of the distance between initially close trajectories. $C=1$, $\Lambda=0.4$, $P=10^{-5}$.

$$D=\left[(n_1-n_2)^2+(y_1-y_2)^2+\left\{\frac{\psi_1-\psi_2}{2\pi}\right\}+\left\{\frac{\varphi_1-\varphi_2}{2\pi}\right\}\right]^{1/2}, \qquad (4.1)$$

where the brackets $\{\ldots\}$ indicate the fractional part of the argument. The characteristic time for the mixing of the trajectories in phase space is defined as the quantity which is the reciprocal of the growth rate γ_c of the local instability, i.e.,

$$D\sim D_0\exp(\gamma_c\tau/\omega). \qquad (4.2)$$

The numerical analysis of the set (2.8) confirms the results of §3 that for $\Lambda\ll 1$ and sufficiently small P the motion becomes stochastic. The boundary of the stochastic layer corresponds to the estimate from Eq. (3.26). An example of the exponential increase of D with time is given in Fig. 4. For small values of Λ the quantity log$D$ fluctuates strongly with time and on average increases linearly until D becomes ~1. It follows from Fig. 4 that $\gamma_c/\omega\sim 10^{-2}$.

When Λ increases the stochasticity region of the motion increases steeply and when $\Lambda>1$ it covers the whole phase space (see Figs. 5 and 6). The growth rate γ_c then increases trongly (Fig. 7): for $\Lambda=0.9$, $\gamma_c/\omega\approx 0.1$; for $\Lambda=3$, $\gamma_c/\omega\approx 0.35$.

From the physical point of view the appearance of stochasticity means the following. The perturbation breaks the integral P. The energy spin performs not only nutational oscillations, but also precesses. This leads to random changes in sign of P. As a result of such a motion the coherence of the energy transfer process from the atoms to the field and vice versa is destroyed. The correlation function of the dipole moment

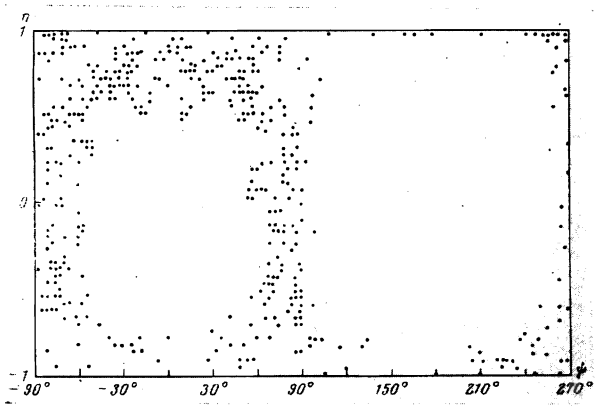

FIG. 5. Stochastic trajectory for $C=1$, $\Lambda=0.9$, $P=10^{-5}$.

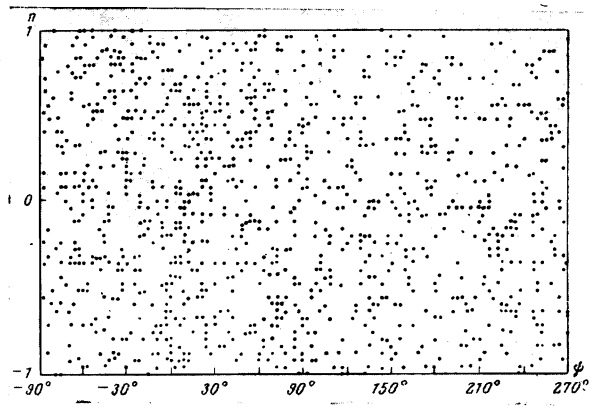

FIG. 6. Stochastic trajectory for $C=1$, $\Lambda=1.8$, $P=10^{-5}$.

which determines the susceptibility has a width γ_c which is equivalent to a broadening of the line of the transition. The number of field photons y also changes stochastically with time, almost uniformly filling the allowed region of phase space (Fig. 8). The set of Eqs. (2.8) thus contains in it a source of stochasticity without any external or other additional factors. Critical for the system is a value of the interaction constant $\Lambda\sim 1$, as for $\Lambda>1$ there is no regular motion of the system (except, perhaps, for small islands of stability which do not show up in the numerical analysis). In other words, a coherent nutation process is principally impossible for $\Lambda>1$.

§5. CONCLUSION

It follows from our results that in the exact model for the interaction of two-level atoms with a resonance radiation field the motion of the system can be of two natures, depending on the parameters: regular periodic or stochastic. The regions of the one or the other kind of motion are determined by the magnitude of the interaction constant Λ. The value $\Lambda\sim 1$ turns out to be the critical one and leads to an anomalously large region of stochasticity. We give characteristic estimates for Λ. For $\lambda=10$ μm, $\mu=10^{-18}$ esu the value of $\Lambda\sim 1$ corresponds to a density $\rho\sim 4\times 10^{21}$ cm^{-3}. For rotational transitions of molecules $\lambda\sim 250$ μm and $\mu=3\times 10^{-18}$ esu (for instance in the HF gas) we have $\Lambda\sim 1$ for $\rho\sim 10^{20}$ cm^{-3}. This means that the critical value of Λ can be reached only at rather high pressures.

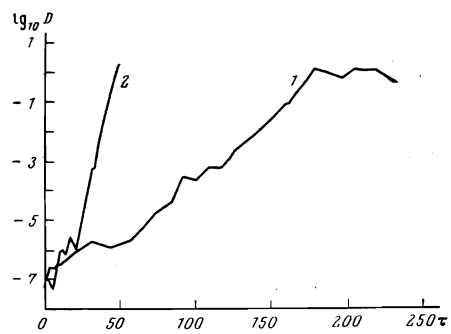

FIG. 7. Time-dependence of the distance between initially close trajectories. $C=1$, $P=10^{-5}$, and 1) $\Lambda=0.9$, 2) $\Lambda=3$.

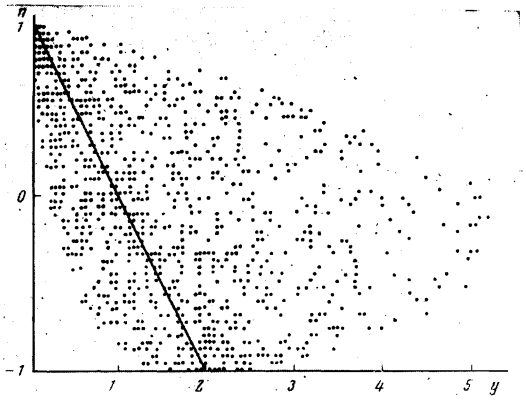

FIG. 8. Stochastic trajectory in the n, y-plane. $C=1$, $\Lambda=1.8$, $P=10^{-5}$. The straight line depicts the unperturbed trajectory.

In the experiments by Skribanowitz *et al.*[14] when observing superradiance in the HF gas stochasticity effects could not manifest themselves in view of the very low pressure. Notwithstanding the fact that the criterion $\Lambda \gtrsim 1$ corresponds to extremal conditions for optical systems, the effect discovered here has a fundamental value as one is dealing with the appearance of stochasticity in quantum systems. Although the problem at the present remains unexplained, the model considered is yet another example (see also[15]) in which the stochasticity destroys the energy spectrum of a quantum system. Indeed, quantization of the truncated Hamiltonian of the system without the non-resonance term leads to a two-parameter energy spectrum (with respect to the number of integrals of motion).[9] The stochastic instability breaks one of them (P) and the energy spectrum becomes not only a one-parameter one, but also, apparently, a quasi-random one, as in[15]. This shows that the analysis of the interaction of radiation with matter under conditions where there is a strong coupling must as a matter of principle take into account the unremovable statistical nature of the motion which in quantum optics so far has not been considered.

[1)]We chose for all perturbed trajectories an initial phase $\varphi_0 = \frac{1}{2}\pi$ and initial values n_0, y_0, ψ_0 pertaining to the unperturbed trajectory which is determined by the quantity P which is indicated in the legends to the figures.

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