[1)]It can be shown that $d\theta/d\xi < 0$ at $dh/d\omega > 0$, but this is impossible in a shock wave.

[1]C. K. Chu and R. Grass, in: Physics of Hot Plasmas, Adv. in Plasma Physics, Vol. 2, Wiley-Interscience, N.Y. 1969.
[2]A. G. Kulikovskiĭ and G. A. Lyubimov, Magnitnaya gidrodinamika (Magnetohydradynamics), Fizmatgiz, 1962.
[3]Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 32, 1126 (1957) [Sov. Phys. JETP 5, 919 (1957)].
[4]V. D. Shafranov, Zh. Eksp. Teor. Fiz. 34, 1475 (1958) [Sov. Phys. JETP 7, 1019 (1958)].
[5]W. Marshall, Proc. R. Soc. Lond. A233, 367 (1955).
[6]R. Jourdan, Plasma Phys. 11, 57 (1969).
[7]H. K. Sen, Phys. Rev. 102, 5 (1956).
[8]S. H. Schneider, Phys. Fluids 15, 805 (1972).
[9]R. T. Taussig, Phys. Fluids 16, 384 (1973).
[10]L. S. Woods, Plasma Phys. 11, 25 (1969).
[11]G. A. Lyubimov, Fiz. Metal. Metalloved. 25, 179 (1961).
[12]T. P. Coffey, Phys. Fluids 13, 1249 (1970).
[13]P. N. Hu, Phys. Fluids 9, 89 (1966).
[14]P. N. Hu and H. Grad, Phys. Fluids 10, 2597 (1967).
[15]A. L. Velikovich and M. A. Liberman, Fiz. Plazmy 2, 334 (1976) [Sov. J. Plasma Phys. 2, 182 (1976)].
[16]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Fizmatgiz, 1954, p. 88 [Pergamon, 1958].
[17]V. S. Imshennik, Zh. Eksp. Teor. Fiz. 42, 236 (1962) [Sov. Phys. JETP 15, 167 (1962)].
[18]V. A. Belokon', Zh. Eksp. Teor. Fiz. 36, 341 (1959) [Sov. Phys. JETP 9, 235 (1959)].
[19]V. A. Belokon', *ibid.*, 1316 [932].
[20]L. S. Woods, J. Plasma Phys. 3, 435 (1969).
[21]J. W. M. Paul, M. J. Parkinson, J. Sheffield, and L. S. Holmes, Nature (Lond.) 208, 133 (1968).
[22]V. G. Eselevich, A. G. Es'kov, R. Kh. Kurtmullaev, and A. I. Malyutin, Zh. Eksp. Teor. Fiz. 60, 2079 (1971) [Sov. Phys. JETP 33, 1120 (1971)].
[23]S. I. Braginskiĭ, Voprosy teoriya plazmy (Problems of Plasma Theory), Vol. 1, Atomizdat, 1962.
[24]R. S. Sagdeev, *ibid.*, Vol. 4, Atomizdat, 1964.
[25]V. I. Kariman, Nelineĭnye volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersive Media), Nauka, 1973.
[26]A. D. Craig, Plasma Phys. 17, 1111 (1975).
[27]D. H. McNeil, Phys. Fluids 18, 44 (1975).
[28]S. V. Bazdenkov and V. V. Vikhrev, Fiz. Plazmy 1, 451 (1975) [Sov. J. Plasma Phys. 1, 250 (1975)].
[29]V. S. Imshennik, Zh. Eksp. Teor. Fiz. 42, 236 (1962) [Sov. Phys. JETP 15, 167 (1962)].
[30]M. Y. Jaffrin and R. T. Probstein, Phys. Fluids 1, 1658 (1964).
[31]K. Abe, Phys. Fluids 18, 1125 (1975).

Translated by J. G. Adashko

Instability and self-refraction of solitons

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(Submitted May 4, 1976)
Zh. Eksp. Teor. Fiz. 71, 1412–1420 (October 1976)

The nonlinear evolution of a two-dimensional soliton with a nonplanar front is investigated in terms of the ray theory. A necessary condition for the stability of an arbitrary soliton in isotropic and anisotropic media is obtained. It turns out that stability is enhanced by anisotropy and that cylindric convergence of the front leads to instability in some cases and to asymptotic stability of the soliton in other cases. The nonlinear stage of self-refraction of the converging and diverging parts of the front is considered. Because of nonlinear defocusing, the field in the focus of a converging soliton remains finite and a cylindric front becomes plane. This is followed by the appearance of a sharp break on the front and a "shock-soliton" type singularity, leading to the destruction of the soliton. General results are applied to the analysis of the behavior of solitons in media with different degrees of nonlinearity.

PACS numbers: 42.65.–k

1. THE GEOMETRIC OPTICS OF SOLITONS

Most problems concerned with nonlinear solitary waves, i.e., solitons, have so far been solved in the one-dimensional formulation. At the same time, the essential point for many physical situations is that the soliton is a "wave layer" moving in space, which may not be strictly plane, and the soliton parameters will, in general, vary along the layer. Kadomtsev and Petviashvili[1] have discussed small deformations of the plane front of a soliton, and have shown that it may become unstable within the framework of the two-dimensional generalization of the Korteweg and de Vries equation. Some nonlinear solutions of this equation were subsequently obtained in[2].

A very effective approach to the solution of two- and three-dimensional problems involving solitons can be developed within the framework of nonlinear geometric optics. This involves the consideration of the variation of amplitude and velocity of a soliton along ray tubes defined by the normals to the soliton front, the local velocity of which depends on the amplitude. This method has already been used to consider the propagation of cylindrically and spherically symmetric solitons and the refraction of solitons in an inhomogeneous medium.[3,4] This analysis was, however, performed in the linear ray-optics approximation when nonlinearity did not affect the distribution of rays even though it was important for the evolution of the wave along the ray tubes. To investigate nonlinear self-refraction effects (which are fundamentally related to the possibility of instability), it is necessary to write down the coupled equations for the ray paths and for the variation of the soliton ampli-

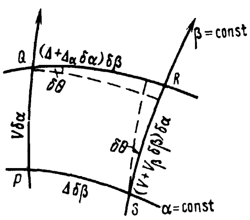

FIG. 1.

tude along the ray. We shall do this in terms of a formulation close to that put forward by Whitham as far back as 1957[5,6] for the description of shock waves.

For the sake of simplicity, we shall confine our attention to the two-dimensional case and will use the orthogonal set of coordinates α, β, defined by the successive positions of the soliton front (α = const) and the normals to it, i.e., the rays (β = const). We shall assume that α is proportional to the time t, i.e., the element of the path traversed by the front is $V(\alpha, \beta)d\alpha$, where V is the local wave velocity. If we consider the curvilinear quadrilateral PQRS in Fig. 1, we can readily determine the change in the ray angle θ in space when α and β are allowed to vary. This yields (see[6]) the following two kinematic equations:

$$V\,\partial\theta/\partial\beta=\partial\Delta/\partial\alpha, \tag{1a}$$

$$\Delta\,\partial\theta/\partial\alpha=-\partial V/\partial\beta, \tag{1b}$$

where Δ is the width of unit (in β) ray tube. To close this system, we must specify the relationship between V and Δ. For a soliton propagating in an undisturbed medium, this can be found from the law of conservation of energy

$$W\Delta=W_0\Delta_0=C, \tag{2}$$

where W is the total energy per unit length of the soliton front. The subscript 0 refers to the initial (for $\alpha=\alpha_0$) position of the front. Since the scale is arbitrary, we shall choose it so that all the ray tubes carry the same amount of energy. The constant C in (2) then independent of β.

In the direction of the normal to the front, the solution is a stationary soliton and, under the usual conditions, when this is a single-parameter solution, the velocity V in an isotropic medium is uniquely related to W (or the amplitude of the soliton A). The set of equations given by (1)–(2) then assumes the form

$$CW_\alpha+W^2V(W)\theta_\beta=0, \tag{3a}$$

$$C\theta_\alpha+WV_W(W)W_\beta=0. \tag{3b}$$

We note that, by virtue of (2), this approach is simpler and more satisfactory for solitons than for shock waves for which the relationship between V and W is determined by the character of the flow behind the shock wave and, to determine it, one must, in general, introduce some additional simplifying assumptions.[6]

In anisotropic media, the normals to the wave front are, as usual, only the phase trajectories, and the energy propagates not only along the normal but also along the front so that, instead of (2), we have $(W\Delta)_\alpha + \Delta^{-1}v(W\Delta)_\beta=0$, where v is the tangential rate of energy transport. The group velocity of a nonlinear wave is, in general, difficult to determine but, in the present case, it is not required. In fact, if, as above, all the tubes have the same energy at the initial time, then, according to the last equation, this will remain so for all time and Eq. (2) will be identically satisfied, as before. Anisotropy, on the other hand, will manifest itself in the fact that the phase velocity V of a soliton depends both on W and on the direction θ, i.e., the additional term $WV_\theta\theta_\beta$ appears on the left-hand side of (3b). Of course, for a linear dispersion-free medium, this leads to the usual ray equations of geometric optics for the motion of the wave front. In particular, for the isotropic case (V = const), equations (1) and (3) yield $\partial\theta/\partial\alpha=0$ (straight rays).

2. STABILITY OF A PLANAR SOLITON

Let us begin by considering the isotropic case. The quasilinear system (3) has two families of characteristics on the α, β plane:

$$d\beta/d\alpha=\pm\Delta^{-1}(VWV_W)^{1/2}. \tag{4}$$

The type of system is thus determined by the sign of the derivative V_W, and the condition for hyperbolic characteristics (necessary condition for stability) is

$$d(V^2)/dW>0. \tag{4'}$$

Small local perturbations will not, in this case, propagate strictly along the rays but will spread over the soliton front.

In the opposite case, the system given by (3) will be elliptic, and this means absolute instability of a plane wave, i.e., small perturbations will grow exponentially. The significance of the condition given by (4) is that, in a stable wave, the front will tend to spread (V increases) in the region of focusing (increasing W) so as to compensate the original perturbation; in the opposite case, the perturbation will grow cumulatively. We note that (4′) is not necessarily equivalent to the condition $dV/dA>0$ (well known in nonlinear optics) because the soliton energy depends not only on the amplitude A but also on its length λ_s.

As an example, let us consider solitons described by the well-known one-dimensional equation

$$\frac{\partial u}{\partial\tau}+au^p\frac{\partial u}{\partial x}+b^2\frac{\partial^3 u}{\partial x^3}=0, \tag{5}$$

where, to be specific, we are assuming that $a>0$, $b^2>0$, $p>0$. If, as usual, (5) describes a traveling wave in a weakly nonlinear system (i.e., x is the "running" coordinate $x-ct$, and $\tau=t$ and $cu_\tau\ll u_x$), the soliton energy (in terms of the variables x, t) is proportional to $A^2\lambda_s$. Using the well-known solution of (5) in the form of a soliton,[7] it can be shown that $\lambda_s\sim A^{-p/2}$, $W\sim A^{2-p/2}$, and $V_W\approx q(p)(2-p/2)^{-1}W^{(3p-4)/(4-p)}$, where $q(p)>0$. Solitons are thus stable for $p<4$ and, in particular, when $p=1$ (Korteweg-de Vries equation) which is, of course, in agreement with the result reported in[1]. However, for

$p > 4$, the soliton is absolutely unstable and its energy falls with increasing amplitude due to the sharp reduction in λ_s. When $p = 4$, the soliton energy is, in general, independent of amplitude in the first order in the nonlinearity parameter, and, to investigate the stability problem, we must perform a more detailed analysis of the dependence of W on A (we note that the formula $W \sim A^2\lambda_s$ will then be invalid in general). When $b^2 < 0$, the stable and unstable cases are interchanged.

We note that the sign of the derivative dW/dA may depend on the wave amplitude. Thus, using the results reported in [8], it can be shown that, for a soliton on shallow water, with amplitude close to the critical value (for which it breaks up), dW/dA becomes negative and, according to (4′), the soliton loses stability before it reaches the critical amplitude. Another example is provided by electromagnetic waves in a ferrite, where highly nonlinear relationships are readily produced and correspond, in particular, to large values of the exponent p in (5).

Let us now consider the stability of solitons in anisotropic media. If we add the term $V_\theta W\theta_\beta$ to (3b), then, as noted at the end of the last section, we obtain the following expression for the characteristics:

$$2\Delta\, d\beta/d\alpha = V_\theta \pm (V_\theta^2 + 4VWV_W)^{1/2}. \tag{6}$$

Instability will now arise only for $4VWV_W < -V_\theta^2$. Anisotropy is thus seen to enhance the stability of solitons because the difference between the phase and group velocities results in the deformation of perturbations which spread out over the front and do not suceed in accumulating at a particular point. When the degree of nonlinearity is small and the anisotropy is strong, so that V depends on θ even in the linear approximation, the instability is possible only near the extremal directions for which $V_\theta = 0$.

A special case arises when only the nonlinear correction to the soliton velocity is anisotropic. The term V_θ^2 under the square root in (6) is then of the same order as the nonlinear term. This situation is possible, for example, for magnetoacoustic solitons in plasma in a strong magnetic field H_0. It can readily be shown that anisotropy in the nonlinearity is important near the critical value of the angle between the direction of propagation of the soliton and the magnetic field ($\cot^2\theta_{crit} = m_e/m_i$, where m_e and m_i are, respectively, the masses of the electron and the ion) which, as is well known, defines a narrow region of stability near $\theta = \pi/2$. Anisotropy has an important effect on the true stability region, making it dependent on W. Moreover, allowance for the finite plasma temperature ensures that solitons with very small amplitudes [$A/H_0 \lesssim (c/c_a)^4$, where c is the velocity of sound and c_a is the Alfvén velocity] will be stable for practically all θ, with the exception of the narrow region near θ_{crit}.

Henceforth, we shall confine our attention to isotropic media.

3. STABILITY OF CYLINDRIC SOLITONS

It is clear that, in the isotropic case, Equation (3) has, in addition to a planar solution, a cylindrically symmetric solution of the form

$$\theta = \beta \quad (0 \leqslant \beta \leqslant 2\pi), \quad \Delta(\alpha) = \int_0^\alpha V\, d\alpha', \quad W = C\left(\int_0^\alpha V\, d\alpha'\right)^{-1} \tag{7}$$

where we are assuming that $\alpha = \alpha_0 \pm (t - t_0)$, in which the upper and lower signs refer to the diverging and converging waves, respectively.

Since V depends on W, the expression given by (7) is an implicit equation for W. As in the case of Equation (5) above, we shall now confine our attention to the case of small nonlinearity when V in (7) may be regarded as a constant and we have approximately $W \sim \Delta^{-1} \sim \alpha^{-1}$; the nonlinearity is now represented by the term involving V_W in (3b).

We now consider the behavior of small perturbations against a background of diverging or converging waves (7). Substituting $\theta = \beta + \theta'(\alpha, \beta)$, $W = (W_0\alpha_0/\alpha)[1 + s(\alpha, \beta)]$ in (3), where θ' and s are small, and assuming, to be specific, that $V_W = qW^n$, we can readily show that the energy modulation coefficient s is given by the equation

$$\xi_{\alpha\alpha} = B\alpha^{-(n+3)}\xi_{\beta\beta} = 0, \tag{8}$$

where $\xi = \alpha s$ and $B = (q/V)(C/V)^{n+1}$. Substituting $\xi \sim \exp(im\beta)$, where m is an integer, we can readily express the solution of (8) for each angular mode in terms of cylinder functions:

$$s \sim \alpha^{-1/2} Z_{-1/(n+1)}\left(-\frac{2mB^{1/2}}{n}\alpha^{-(n+1)/2}\right). \tag{9}$$

This solution is completely determined by specifying the initial (for $\alpha = \alpha_0$) distribution of s and θ' with β (and, in general, contains functions of the first and second kind). Hence, we can readily establish the asymptotic behavior of the modulation coefficient for large (diverging soliton) and small (converging soliton) values of α. Let us now return to (5) when $n = (3p - 4)/(4 - p)$.

In the hyperbolic case ($p < 4$, $B > 0$), the asymptotic behavior of the function s in the diverging wave ($\alpha \to \infty$) is qualitatively the same for all p. In particular, the modulation depth s remains constant just as in the case of a plane wave with the difference, however, that, for oscillations with β, the quantity s is an aperiodic function of α. In a converging wave, the quantity s oscillates with α as $\alpha \to 0$ with the increasing frequency $\omega \sim \alpha^{-(n+3)/2}$, whereas the amplitude of s varies as $\alpha^{(n-1)/4}$. Therefore, in the Korteweg-de Vries equation ($p = 1$, $n = -1/3$), s increases as $\alpha^{-1/3}$, and the converging wave turns out to be unstable even in the hyperbolic case (an analogous instability is known for strong shock waves [6]). However, when $p = 2$ ($n = 1$), the amplitude of s remains constant, and when $p = 3$ ($n = 5$), the modulation depth decreases in proportion to α, i.e., a converging wave is absolutely stable even in the first approximation.

In the elliptic case ($p > 4$, $B < 0$), the cylindric wave, like the plane wave, is always unstable. It is interesting, however, that, in the converging wave, $s \sim \alpha^{-1}$ for any p, i.e., the instability is much weaker than in the plane wave. Conversely, in the converging wave, s increases very rapidly (and the increase is different for different p), for example, $s \sim \alpha^{-7/2} e^{\alpha^5}$ for $p=5$. At first sight, this would appear to be a paradoxical situation, but it is explained by the fact that, in this case, the nonlinearity parameter increases with decreasing W for the perturbations (WV_W). Cylindric divergence of a soliton is thus seen to have a radical effect on instability character and conditions.

It is important to note that delta-type perturbations localized on a cylindric front will behave differently from the angular modes with finite m (considered above) because they correspond to the asymptotic behavior as $m \to \infty$. It follows from (9) and directly from (8) that, in the hyperbolic case, such perturbations propagate along characteristics of the form

$$\beta-\beta_0=\pm\frac{2B^{1/2}}{n+1}\left(\alpha_0^{-(n+1)/2}-\alpha^{-(n+1)/2}\right). \tag{10}$$

In a stable diverging wave, therefore, and in contrast to the plane wave, local perturbations spread out as $\alpha \to \infty$ only within a finite angle [we are again concerned with (5)]. The amplitude of s in this type of perturbation varies as $\alpha^{(n-1)/4}$, i.e., the modulation depth increases as $\alpha^{-1/3}$ for $p=1$ in the converging wave (as above), and we have the converse situation in the diverging wave for $p=3$ (this differs from the corresponding result for finite m).

4. NONLINEAR SELF-REFRACTION AND DEFOCUSING OF SOLITONS

We must now consider the essentially nonlinear solutions of (3) in the hyperbolic case. These solutions can, as usual, be investigated with the aid of the characteristics (4) which correspond to the Riemann invariants

$$J_\pm=\theta\mp\int\left(\frac{V_W}{WV}\right)^{1/2}dW \tag{11}$$

[in particular, for (5), we have $J_\pm=\theta\pm\text{const}\times W^{p/(4-p)}$] and two families of simple waves traveling over the soliton front with velocity that increases with increasing local values of W. We note at once that, in general, the propagation of such waves leads to a "turnover" effect, i.e., to a sharp change in the direction of the rays and the amplitude of the solitons. An analogous situation in the case of shock waves was described by Whitham[6] by the phrase "shock-shock." In the present case, we may be dealing with a shock-soliton combination, i.e., a stationary change in W and θ (front break) travling along the soliton front. Integrating (3) with respect to β in the neighborhood of this break, we obtain the following formal boundary conditions relating the changes in θ and W:

$$\theta_2-\theta_1=\Gamma\int\limits_{W_1}^{W_2}\frac{\Delta dW}{WV(W)},\quad \theta_2-\theta_1=\Gamma^{-1}\int\limits_{W_1}^{W_2}(WV_W/\Delta)dW, \tag{12}$$

where Γ is the rate of displacement of the break over the front. However (as in the case of shock waves), the actual existence of such "jumps" in an open question. Various quantities undergo a sharp change in the region of a jump, and this gives rise to diffraction which produces the emission of energy in a "nonsoliton" form and may destroy the soliton (see below).

Singularities of nonlinear refraction of solitons are qualitatively determined by the relationship between the subsequent effects, each of which manifests itself within its own particular interval in α: a) convergence and divergence of the wave front or of its individual segments, which occur even in the linear approximation, b) spreading of perturbations over the soliton front, which is determined by the characteristics (5), and c) the turning over of the spreading perturbations, leading to the formation of "jumps" and the destruction of the soliton.

Let us elucidate these effects by considering the following example which has an exact solution. Suppose that, at the initial time, $\alpha_0=R_0/V_0$ the soliton front is cylindric with radius of curvature R_0 for $2\theta_0<\pi$ and plane elsewhere (Fig. 2a), i.e.,

$$\theta(\alpha=\alpha_0)=\beta,\ |\beta|<\theta_0;\ -\theta_0,\ \beta<-\beta_0;\ \theta_0,\ \beta>\beta_0. \tag{13}$$

The initial distribution of the energy W_0 over the front is assumed to be uniform. Let us begin by considering a converging wave (as before, we shall consider the case $\alpha<\alpha_0$, where $\alpha=0$ corresponds to the time of focusing of the front in the linear case). The characteristics of the system given by (3) for this wave are shown in Fig. 2b. As is usual for hyperbolic systems, one can readily distinguish between different regions, i.e., regions of rest I, where the front remains plane, regions of simple waves II, where one of the Riemann invariants in (11) is a constant, and regions of interaction III, where the front is cylindric and (7) is valid. The characteristics corresponding to the last region have the form given by (10).

Figure 2b provides a clear illustration of the overall features of the process. As α decreases from α_0, plane and cylindric regions are separated by expanding simple wave regions, where the front is less curved than in the central region; the analytic solution for this region is given by (4) and (11). Subsequently, for a certain $\alpha_p>0$,

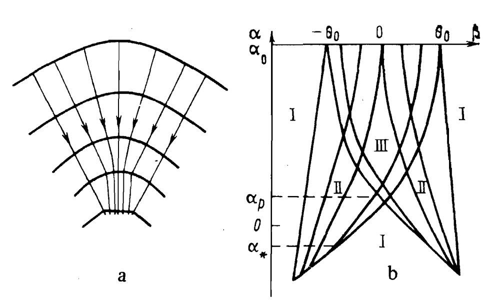

FIG. 2. Focusing of a cylindric soliton: a) rays and fronts, b) characteristics on the (α, β) plane.

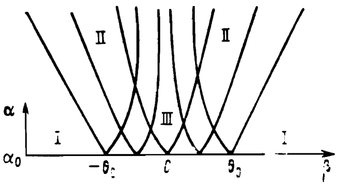

FIG. 3. Characteristics for a cylindrically diverging soliton.

the cylindric region vanishes altogether and, instead, a flat segment appears in the central region of the front. Subsequently, for $\alpha = \alpha_*$, the characteristics begin to intersect in region II, and this corresponds to a break on the front.

To obtain quantitative results, we again start with (5). Since region III is bounded by the characteristics given by (10), emerging from the points $\pm\theta_0$, we can readily find the coordinate α_p for which the cylindric part of the front disappears:

$$\alpha_p = \alpha_0(1+\theta_0/M_0^{1/2})^{-2/(n+1)}, \quad M_0 = ((n+1)/2)^2\alpha_0^{n+1}B^{-1}, \tag{14}$$

where M_0 is a dimensionless quantity (Mach number) defining the relative nonlinearity in the initial wave. Hence, it is clear that, for small θ_0, the perturbation will rapidly travel outward along the characteristics, leaving a plane region with a somewhat larger value of W than in the initial wave. If, on the other hand, $\theta_0 \gg M_0^{1/2}$, then $\alpha_p \ll \alpha_0$ and the front converges (in the same way as in the absence of nonlinearity) right down to a small neighborhood of the focus (linear ray approximation). However, the intensity sharply increases near the focus, and the perturbations again spread out over the front. It follows that the intensity remains finite (defocusing) everywhere, including the focus ($\alpha = 0$), even without taking into account diffraction effects, and the front becomes plane in the central region. The maximum value of W is $W_0(\alpha_0/\alpha_p)$.

To determine the time at which the break appears on the front, we must consider the characteristic (4) in region II with the initial conditions on the curve (10) bounding region III. Well-known methods can be used to show that the "turnover" effect occurs first on the bounding characteristics separating region II from the central undisturbed region I, and the corresponding value of α is

$$\alpha_* = \alpha_p\left(\frac{n-1}{n+3}\right) = \alpha_p\left(\frac{p}{2}-1\right), \tag{15}$$

where α_p is given by (14). In particular, when $p=1$ (Korteweg-de Vries equation), $\alpha_* = -\alpha_p/2$, i.e., α_* lies behind the focal points (in the linear approximation); when $p=2$, we have $\alpha_* = 0$ (α_* at the level of the focus) and, finally, when $p=3$, we have $\alpha_* = \alpha_p/2$, i.e., the front break is formed closer to the focus but always after the transformation of the cylindric converging part of the front into the plane front. The nonlinear focusing problem can therefore be solved completely, at least for all $\alpha > \alpha_*$ ($t < \alpha_0 - \alpha_*$).

Let us now consider a diverging wave with the same initial conditions (13). The behavior of the characteristics (which now lie in the region $\alpha > 0$) is shown in Fig. 3. Here, we have the following important differences from the case considered above. Firstly, the cylindrically diverging part of the front becomes plane only for sufficiently small values of θ_0. In fact, instead of (14), we now have

$$\alpha_p = \alpha_0(1-\theta_0/M_0^{1/2})^{-2/(n+1)}, \tag{16}$$

and when $\theta_0 > M_0^{1/2}$ the characteristics bounding the interaction region III do not intersect at all (see the end of the preceding section). The central part of the front remains cylindric. Secondly, in the simple wave regions II, the characteristics diverge (rarefaction waves) and the "turnover" of the wave, i.e., the formation of breaks on the front, does not occur.

Finally, let us briefly consider the range of validity of the above results. In the linear theory, geometric optics is valid for $\Lambda^2 \gg R\lambda$, where λ is the wavelength, Λ is the characteristic scale of variation of θ and W along the front, and R is the characteristic radius of curvature of the front (we are considering the "worst" case when $R \gg \Lambda$). For solitons (just as for the shock waves considered in[5,6]) this condition will not, in general, be sufficient. In fact, to ensure that the wave can be regarded as locally close to a soliton, we must ensure that all the non-one-dimensional corrections which distinguish the exact field equations from the local equations such as (5) are small. Formally, this is wholly consistent with the usual geometric-optics approximation with the physical reservation, however, that (5) is usually obtained from the original system only in the approximation of weak nonlinearity and dispersion. This gives rise to an additional condition of the form $\lambda_s \ll RM$ (M, is, as before, the nonlinearity parameter), which is compatible with the linear condition if $M \gg (\lambda_s/\Lambda)^2$. For example, for a soliton on water, this means that $(\Lambda A)^2 \gg h^4$, where h is the depth of the water and A is the height of the solitary wave.

It is clear that diffraction restricts the influence of the above nonlinear effects. Thus, within the framework of the formula given by (4), the characteristic distance L within which the instability of the plane soliton develops is reduced as the scale Λ of the perturbed segments of the front is reduced. It is readily seen that, when diffraction is taken into account, we obtain the optimum scale $\Lambda_{opt} \sim \lambda_s/M^{1/2}$, which corresponds to $L_{min} \sim \lambda_s/M$. Similarly, in the problem of focusing, which was considered above, the maximum intensity in the focus is achieved for $\alpha_p \sim \lambda_s/V\theta_0$ and then $W_{max} \sim W_0\Lambda/\lambda_s$ (we note that, when $\alpha_p \ll \lambda_s/\theta_0^2$, the diffraction of the soliton in the region of the focus can be described by the formulas of the linear theory; cf. the analogous approach in the case of shock waves[9]). We emphasize that, in the present case, diffraction has a more radical effect than in the quasioptics of harmonic nonlinear waves because, when the diffraction terms in the original equation are of the order of nonlinear terms, the existence of the quasistationary soliton is impossible and the soliton is destroyed (by radiation into the "nonsoliton" region).

Finally, the question of the validity of the "linear ray optics" approximation, [3] when self-refraction can be neglected, can readily be resolved, remembering, however, the nonlinear transformation of the soliton along the ray. This is meaningful in the case where the cross section of a ray tube undergoes an appreciable change within intervals of the order of λ_s/M (small in comparison with $\Lambda/M^{1/2}$) for which longitudinal nonlinear wave distortion is significant.

Analysis of more complicated cases of nonlinear self-refraction of a soliton within the framework of (1)–(3) can be performed with the aid of a transformation of the travel-time curve which enables us to obtain a more general solution [for example, for (5) with $p = 1, 2$] for both the hyperbolic and elliptic cases, although the final expression is not explicit and, as a rule, must be investigated on a computer.

[1]B. V. Kadomtsev and V. I. Petviashvili, Dokl. Akad. Nauk SSSR **192**, 753 (1970) [Sov. Phys. Dokl. **15**, 539 (1970)].

[2]V. E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. 22, 364 (1975) [JETP Lett. 22, 172 (1975)].

[3]L. A. Ostrovskiĭ, Izv. Akad. Nauk SSSR Fiz. Atmosfery i Okeana 11, 67 (1975).

[4]K. A. Gorschkov, L. A. Ostrovsky, and E. N. Pelinovsky, Proc. IEEE 62, 1511 (1974).

[5]G. B. Whitham, J. Fluid Mech. 2, 146 (1957).

[6]G. B. Whitham, Linear and Nonlinear Waves, Wiley Interscience, New York, 1974.

[7]N. Zabusky, Nonlinear Partial Differential Equations. Academic Press, New York, 1967, p. 223.

[8]M. S. Longuet-Higgins and J. Fenlon, Proc. R. Soc. Aer. A 340, 471 (1974).

[9]L. A. Ostrovskiĭ and A. M. Sutin, Dokl. Akad. Nauk SSSR 221, 1300 (1975) [Sov. Phys. Dokl. 20, 275 (1975)].

Translated by S. Chomet