

from data concerning the critical magnetic field (see below).  
<sup>3</sup>Below we shall utilize the normalized quantity  $\eta N(0)/N_m(0)$ , where  $N_m(0) = 0.94 \times 10^{35}$  states-erg<sup>-1</sup>-cm<sup>-3</sup> denotes the density of electron states in pure, bulk vanadium, corresponding to a coefficient of electronic heat capacity  $\gamma_m = 9.80$  mJ-mole<sup>-1</sup>-K<sup>-2</sup>.<sup>[8]</sup>  
<sup>4</sup>The accuracy of the determination of  $\eta N(0)/N_m(0)$  amounts to  $\sim 8\%$ .  
<sup>5</sup>From  $\eta = 1.20$  and  $\eta/v_F^2 = 0.336 \times 10^{14}$  cm<sup>-2</sup> we obtain the value  $v_F = 1.89 \times 10^7$  cm/sec, which was utilized by us above in order to determine  $\xi_0$ ; see footnote 2.

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## Hydrodynamics of the A phase of superfluid <sup>3</sup>He

G. E. Volovik and V. P. Mineev

L. D. Landau Theoretical Physics Institute and Solid State Physics Institute, USSR Academy of Sciences  
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Starting from the kinetic equation for the quasiparticle distribution function and from considerations of gauge invariance, a complete system of linear hydrodynamic equations for the A phase of superfluid <sup>3</sup>He is obtained. All the thermodynamic quantities and (in the  $\tau$ -approximation) kinetic coefficients appearing in the equations are found and expressed in terms of the Fermi-liquid parameters of the Landau theory. It is shown that in the range of applicability of hydrodynamics ( $\omega\tau \ll 1$ ) there exist two regions of frequency, in which the equations of motion for the orbital angular-momentum vector have a fundamentally different character. At low frequencies this equation is a diffusion equation, while at higher frequencies it describes the propagation of weakly damped orbital waves with a linear dispersion law.

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### 1. INTRODUCTION

A number of experiments carried out recently (measurements of the fourth-sound velocity and heat flows, and experiments with a vibrating wire in the liquid; cf.<sup>[1]</sup>) prove incontrovertibly that the A and B phases of liquid <sup>3</sup>He are superfluid liquids. On the other hand, magnetic-susceptibility measurements and linear and pulsed NMR make it possible to identify these phases with the so-called Anderson-Brinkman-Morel and Balian-Werthamer states (cf. the review<sup>[2]</sup>) for Cooper pairing of <sup>3</sup>He atoms in a state with angular momentum  $l=1$  and spin  $S=1$ .

The Anderson-Brinkman-Morel state, corresponding to the A phase, is characterized by the following order parameter:

$$\hat{\Delta} = i(\hat{\sigma}\mathbf{V})\hat{\sigma}_y(\mathbf{n}\Delta), \quad (1.1)$$

where  $\hat{\sigma}$  are the Pauli matrices,  $\mathbf{V}$  is a unit vector,  $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$ , and  $\Delta = \Delta' + i\Delta''$  where  $\Delta'$  and  $\Delta''$  are real vectors, equal in magnitude ( $|\Delta'| = |\Delta''| = \Delta(T)$ ) and mutually perpendicular. Their vector product  $\Delta' \times \Delta'' = \Delta^2(T)\mathbf{l}$  defines the direction (common for all pairs) of the orbital angular momentum of a pair in the absence of walls or of external fields. Thus, the coordinate (or orbital) part of the order parameter is specified by the position of the triad of vectors  $\Delta'$ ,  $\Delta''$ ,  $\mathbf{l}$ . We shall assume that in equilibrium the vectors  $\Delta'$ ,  $\Delta''$ ,  $\mathbf{l}$  are directed along the axes  $X$ ,  $Y$ ,  $Z$  respectively.

We shall describe small deviations from the equilibrium position as a rotation (specified by a vector  $\theta$  equal

in magnitude to the angle of rotation) of this triad of vectors. In the hydrodynamics regime the vector  $\theta$  is a slowly varying function of the coordinates and time. By virtue of the small magnitude of the spin-orbit interaction (cf. [2]), the dynamics of the vector  $\theta$  can be treated independently of the dynamics of the spin variable  $V$ . In the present paper we shall not be interested in the spin hydrodynamics, the derivation of which can be found, e.g., in the papers of Combescot [3] and Graham and Pleiner, [4] and so we shall discard the spin part of the order parameter (1.1) and consider only its orbital part

$$\Delta_n = n\Delta = \Delta(T) \sin \theta_n \exp(i\phi_n), \quad (1.2)$$

where  $\theta_n$  and  $\phi_n$  are the polar and azimuthal angles of the vector  $n$  (the polar axis is directed along  $l$ ).

Since the phase  $\phi_n$  and modulus  $\Delta(T) \sin \theta_n = \Delta(T) |n \times l|$  of the order parameter depend on the direction of the unit vector  $l$ , the  $A$  phase of  $^3\text{He}$  should exhibit anisotropic properties analogous to those of a nematic liquid crystal in which the axis of each molecule is parallel to a specified direction, common for all the molecules. In particular, the hydrodynamics of the  $A$  phase of  $^3\text{He}$  should be anisotropic, i.e., should contain tensor quantities as the coefficients. Such a system of hydrodynamic equations, with allowance for dissipative terms, was proposed by Graham, [5] starting from considerations of Galilean invariance and invariance under time reversal and from the axial symmetry of the  $A$  phase; however, he left open the question of the calculation of the phenomenological parameters appearing in this system.

In the present paper we have obtained a complete system of linear hydrodynamic equations for the  $A$  phase, starting from the kinetic equation for the quasi-particle distribution function and from considerations of gauge invariance, just as was done in the paper by Betbeder-Matibet and Nozières [6] for an isotropic superfluid Fermi liquid (see also [7], in which an anisotropic superfluid Fermi liquid, but with a frozen  $l$  equal to its equilibrium value, was considered). All the thermodynamic quantities and (in the  $\tau$ -approximation) kinetic coefficients appearing in the equations are found. They are expressed in terms of the Fermi-liquid parameters of the Landau theory. We emphasize that in the calculation of all the quantities we have not assumed that the interaction leading to the pairing is weak; we use only the smallness of  $T_c/\epsilon_F$ . The point is that strong-coupling effects (cf. the review [2]) are important only in the order-parameter equation, which we have not used, assuming that (1.2) is its solution with a  $\Delta(T)$  which can be taken from experiment.

It is shown that in the hydrodynamic regime  $\omega\tau \ll 1$  there exist two frequency regions, in which the equations of motion for the vector  $l$  have a fundamentally different character. At low frequencies the hydrodynamics of Graham [5] obtains, and the equation for  $l$  is a diffusion equation, while at higher frequencies, but within the limits of applicability of hydrodynamics ( $\omega\tau \ll 1$ ), the equation of motion for  $l$  describes the propagation of yet

another sound—orbital waves with a linear dispersion law.

## 2. THE SYSTEM OF HYDRODYNAMIC EQUATIONS

The hydrodynamics of an isotropic superfluid Fermi liquid was obtained by Betbeder-Matibet and Nozières. [6] According to them, a nonequilibrium state of the system is specified by three variables: the quasi-particle distribution function  $\nu_k(\mathbf{r}, t)$  and the condensate phase  $\varphi(\mathbf{r}, t)$  and density  $\rho(\mathbf{r}, t)$ . The latter two variables are canonically conjugate for fixed  $\nu_k$ . In the case of the  $A$  phase these quantities are supplemented by the unit vector  $l$  indicating the direction of the orbital angular momentum of a pair. In place of the variables  $\varphi(\mathbf{r}, t)$  and  $l(\mathbf{r}, t)$  it is convenient to introduce the vector of the local angle of rotation  $\theta(\mathbf{r}, t)$ , which is connected with them by the relation  $\delta\varphi = -l \cdot \delta\theta$ ,  $\delta l = \delta\theta \times l$ . The variable canonically conjugate to the rotation-angle vector  $\theta$  is the generalized angular momentum  $L$ .

Thus, the state of the  $A$  phase is specified by the following variables:  $\nu_k(\mathbf{r}, t)$ ,  $\theta(\mathbf{r}, t)$  and  $L(\mathbf{r}, t)$ . The total energy of the  $A$  phase obeys the following thermodynamic identity:

$$\delta U = \sum_k E_k \delta \nu_k - M \delta \theta + \omega \delta L, \quad (2.1)$$

where  $\omega$  and  $M$  are respectively the angular velocity and generalized torque.

The equation of motion for the function  $\nu_k$  is the kinetic equation

$$\frac{\partial \nu_k}{\partial t} + \frac{\partial \nu_k}{\partial \mathbf{r}} \frac{\partial E_k}{\partial \mathbf{k}} - \frac{\partial \nu_k}{\partial \mathbf{k}} \frac{\partial E_k}{\partial \mathbf{r}} = J\{\nu_k\}, \quad (2.2)$$

which holds for [6]  $\omega$ ,  $v_F q \ll \Delta(T)$ , where  $\omega$  and  $q$  are the frequency and wave vector, respectively. The equations of motion for the variables  $\theta$  and  $L$  are Hamilton's equations:

$$L = M, \quad \dot{\theta} = \omega. \quad (2.3)$$

We note that in the case of ordinary superfluidity the equations corresponding to Eqs. (2.3) are

$$\frac{1}{2} \dot{\rho} = \frac{\delta U}{\delta \varphi} = -\frac{1}{2} \nabla j, \quad -\dot{\varphi} = 2 \frac{\delta U}{\delta \rho}.$$

As will be seen below, these equations are the  $z$ -components of Eqs. (2.3). The system of equations (2.2)–(2.3) completely describes the system, both in the low-frequency hydrodynamic regime  $\omega\tau \ll 1$  and in the high-frequency regime  $\omega\tau \gg 1$  (but  $\omega \ll \Delta(T)$ ), if the expressions for  $E_k$ ,  $M$  and  $\omega$  as functions of  $\nu_k$ ,  $\theta$  and  $L$  are known.

In the hydrodynamic regime  $\omega\tau \ll 1$ , in place of the quasi-particle distribution function  $\nu_k(\mathbf{r}, t)$  we can introduce hydrodynamic variables: the excitation-flux density  $P$  and entropy density  $S$ , defined by the equalities

$$P = \sum_k k \delta \nu_k, \quad T \delta S = \sum_k (E_k - kv^2) \delta \nu_k,$$

where the velocity  $v^n$  of the normal component and the temperature  $T$  are the variables thermodynamically conjugate to the quantities  $P$  and  $S$ . In this case, (2.1) is rewritten in the form

$$\delta U = T\delta S + v^n \delta P - M\delta\theta + \omega\delta L, \quad (2.4)$$

and the distribution function  $\nu_{\mathbf{k}}(\mathbf{r}, t)$  is close to its local-equilibrium value, which depends on the local energy  $E_{\mathbf{k}}$  and the temperature and velocity of the normal component:

$$\delta\nu_{\mathbf{k}} = \frac{\delta\nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \left( \delta E_{\mathbf{k}} - \mathbf{k}v^n - E_{\mathbf{k}} \frac{\delta T}{T} \right) + g_{\mathbf{k}}, \quad (2.5)$$

$$\nu_{\mathbf{k}}^0 = \frac{1}{2} \left( 1 - \text{th} \frac{E_{\mathbf{k}}}{2T} \right).$$

Here  $g_{\mathbf{k}}$  is the nonequilibrium correction to the distribution function and must be found by solving the kinetic equation.

The equations for  $P$  and  $S$  can be obtained by taking the corresponding moments of the kinetic equation:

$$\dot{P}_i + S_{ij} \nabla_j T = -\nabla_j \sum_{\mathbf{k}} \frac{\partial E_{\mathbf{k}}}{\partial k_j} k_i g_{\mathbf{k}}, \quad (2.6)$$

$$S + \nabla_i (S_{ij} v_j^n) = -\frac{1}{T} \nabla_i \sum_{\mathbf{k}} \frac{\partial E_{\mathbf{k}}}{\partial k_i} E_{\mathbf{k}} g_{\mathbf{k}},$$

where  $S_{ij}$  is the tensor corresponding to the entropy density carried by the normal component (cf. [7]):

$$S_{ij} = -\frac{1}{T} \sum_{\mathbf{k}} k_i \frac{\partial E_{\mathbf{k}}}{\partial k_j} E_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}}, \quad (2.7)$$

and is connected with the entropy density  $S$  by the relation  $S_{ij} = S\delta_{ij}$ .

The system of equations (2.3), (2.6) completely describes the hydrodynamics of the A phase of  $^3\text{He}$ .

Separating out the dissipative terms in the right-hand sides of Eqs. (2.3), (2.6), we rewrite this system in the following form:

$$\begin{aligned} \dot{L}_i - M_i^R &= -\gamma_{L_i L_j} \omega_j + \gamma_{L_i \theta_j} M_j - \gamma_{L_i P_j} v_j^n - \gamma_{L_i S} \delta T, \\ \dot{\theta}_i - \omega_i^R &= -\gamma_{\theta_i L_j} \omega_j + \gamma_{\theta_i \theta_j} M_j - \gamma_{\theta_i P_j} v_j^n - \gamma_{\theta_i S} \delta T, \\ \dot{P}_i + S \nabla_i T &= -\gamma_{P_i L_j} \omega_j + \gamma_{P_i \theta_j} M_j - \gamma_{P_i P_j} v_j^n - \gamma_{P_i S} \delta T, \\ \dot{S} + \nabla_i (S v_i^n) &= -\gamma_{S L_i} \omega_i + \gamma_{S \theta_i} M_i - \gamma_{S P_i} v_i^n - \gamma_{S S} \delta T. \end{aligned} \quad (2.8)$$

Here  $M^R$  and  $\omega^R$  are the equilibrium parts of the generalized torque  $M$  and angular velocity  $\omega$ ; the  $\gamma$  are the different generalized kinetic coefficients, which, in the  $\tau$ -approximation, can be written in the form (see Appendix A)

$$\begin{aligned} \gamma_{L_i L_j} &= -\tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{\partial E_{\mathbf{k}}}{\partial \theta_i} \frac{\partial E_{\mathbf{k}}}{\partial \theta_j}, & \gamma_{\theta_i \theta_j} &= -\tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{\partial E_{\mathbf{k}}}{\partial L_i} \frac{\partial E_{\mathbf{k}}}{\partial L_j}, \\ \gamma_{P_i P_j} &= \tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} k_i k_j \left( \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \nabla \right)^2, & \gamma_{S S} &= \tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{E_{\mathbf{k}}^2}{T^2} \left( \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \nabla \right)^2, \\ \gamma_{L_i \theta_j} &= -\gamma_{\theta_i L_j} = \tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{\partial E_{\mathbf{k}}}{\partial \theta_i} \frac{\partial E_{\mathbf{k}}}{\partial L_j}, \\ \gamma_{L_i P_j} &= \gamma_{P_i L_j} = \tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{\partial E_{\mathbf{k}}}{\partial \theta_i} k_j \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \nabla, \end{aligned}$$

$$\begin{aligned} \gamma_{\theta_i P_j} &= -\gamma_{P_i \theta_j} = -\tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{\partial E_{\mathbf{k}}}{\partial L_i} k_j \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \nabla, \\ \gamma_{S L_i} &= -\gamma_{L_i S} = -\tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{\partial E_{\mathbf{k}}}{\partial \theta_i} \frac{E_{\mathbf{k}}}{T} \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \nabla, \\ \gamma_{\theta_i S} &= \gamma_{S \theta_i} = \tau \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \frac{\partial E_{\mathbf{k}}}{\partial L_i} \frac{E_{\mathbf{k}}}{T} \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \nabla, \\ \gamma_{S P_i} &= \gamma_{P_i S} = 0. \end{aligned} \quad (2.9)$$

In the expressions (2.9) the derivatives of  $E_{\mathbf{k}}$  with respect to  $\theta$  and  $L$  are taken at constant  $v^n$  and  $T$ ;  $\tau$  is the temperature-dependent relaxation time. Near  $T_c$  the quantity  $\tau$  coincides with the relaxation time in the normal Fermi liquid. In order that the hydrodynamic equations be closed, it is necessary to relate the quantities  $E_{\mathbf{k}}$ ,  $M^R$ ,  $\omega^R$  and  $P$  to the quantities  $\theta$ ,  $L$ ,  $v^n$  and  $T$ .

### 3. CALCULATION OF THE QUANTITIES APPEARING IN THE HYDRODYNAMIC EQUATIONS

#### Calculation of $E_{\mathbf{k}}$ and $P$

To find  $E_{\mathbf{k}}$  we shall assume, as in [6], that the quasi-particles of the superfluid Fermi liquid are formed from the quasi-particles of the normal Fermi liquid, with a spectrum  $\epsilon_{\mathbf{k}}$  which, by virtue of the smallness of  $T_c/\epsilon_F$ , is not affected by the phase transition to the superfluid state. Therefore, the quasi-particle energy is expressed in terms of  $\epsilon_{\mathbf{k}}$  by

$$\begin{aligned} E_{\mathbf{k}} &= (\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2)^{1/2} + 1/2 (\epsilon_{\mathbf{k}} - \epsilon_{-\mathbf{k}}), \\ \xi_{\mathbf{k}} &= 1/2 (\epsilon_{\mathbf{k}} + \epsilon_{-\mathbf{k}}). \end{aligned} \quad (3.1)$$

The energy of the normal quasi-particles depends on the gradients and time derivative of the phase of the order parameter, which play the role of an external field. This dependence can be obtained by generalizing the expression for  $\epsilon_{\mathbf{k}}$  in the case of an isotropic superfluid Fermi liquid, when the phase  $\phi$  does not depend on  $\mathbf{k}$  (cf. [6-8]):

$$\begin{aligned} \epsilon_{\mathbf{k}} &= H \left( \mathbf{k} + \frac{1}{2} \nabla \phi_{\mathbf{k}}, \quad \mathbf{r} - \frac{1}{2} \frac{\partial \phi_{\mathbf{k}}}{\partial \mathbf{k}} \right), \\ H &= \epsilon_{\mathbf{k}}^0 + \frac{1}{2} \phi_{\mathbf{k}} + \sum_{\mathbf{k}'} f_{\mathbf{k}, \mathbf{k}'} (n_{\mathbf{k}'} - n_{\mathbf{k}}^0), \end{aligned} \quad (3.2)$$

where  $n_{\mathbf{k}}$  is the distribution function of the normal quasi-particles and  $f_{\mathbf{k}, \mathbf{k}'}$  is the Landau Fermi-liquid function. The distribution function  $n_{\mathbf{k}}$  is expressed in terms of the distribution function  $\nu_{\mathbf{k}}(\mathbf{r}, t)$  for the quasi-particles of the superfluid liquid by means of a Bogolyubov transformation, in which, as in (3.2), it is necessary to shift the momentum and coordinate by  $\frac{1}{2} \nabla \phi_{\mathbf{k}}$  and  $-\frac{1}{2} \frac{\partial \phi_{\mathbf{k}}}{\partial \mathbf{k}}$  respectively:

$$\begin{aligned} n_{\mathbf{k}} &= \tilde{n} \left( \mathbf{k} - \frac{1}{2} \nabla \phi_{\mathbf{k}}, \quad \mathbf{r} + \frac{1}{2} \frac{\partial \phi_{\mathbf{k}}}{\partial \mathbf{k}} \right), \\ \tilde{n} &= \frac{1}{2} \left[ \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} (v_{\mathbf{k}} + v_{-\mathbf{k}} - 1) + (v_{\mathbf{k}} - v_{-\mathbf{k}} + 1) \right]. \end{aligned} \quad (3.3)$$

The dependence of the phase  $\phi_{\mathbf{k}}$  on the coordinates and time can be found with the aid of formula (1.2), which shows that the phase of the order parameter coincides with the azimuthal angle  $\phi_n$  of the vector  $\mathbf{n}$ ; under a ro-

tation through an angle  $\delta\theta(\mathbf{r}, t)$ , this angle changes by an amount

$$\delta\phi_{\mathbf{k}} = -2l_n \delta\theta(\mathbf{r}, t), \quad l_n = \frac{1}{2} \frac{[\mathbf{n} \times [\mathbf{l} \times \mathbf{n}]]}{|\mathbf{l} \times \mathbf{n}|^2}.$$

We note that the expressions (3.2) and (3.3) can be derived with the aid of a gauge transformation eliminating the dependence of the phase  $\phi_{\mathbf{k}}$  on  $t$ ,  $\mathbf{r}$  and  $\mathbf{k}$ . Under such a transformation, the term

$$\int d^3r \sum_{\mathbf{k}} \left[ \frac{1}{2} \dot{\phi}_{\mathbf{k}}(\mathbf{r}, t) + \frac{\mathbf{k}\nabla}{2m} \phi_{\mathbf{k}}(\mathbf{r}, t) \right] n_{\mathbf{k}}(\mathbf{r}, t) \quad (3.5)$$

is added to the Hamiltonian of the system.

The formulas (3.1)–(3.4) determine the dependence of  $E_{\mathbf{k}}$  on the variables  $\theta$ ,  $\dot{\theta}$  and  $\nu_{\mathbf{k}}$ :

$$\delta E_{\mathbf{k}} = \sum_{\mathbf{k}'} g_{\mathbf{k}, \mathbf{k}'} \delta \nu_{\mathbf{k}'} + \frac{\partial E_{\mathbf{k}}}{\partial \theta} \Big|_{\nu, \dot{\theta}} \delta \theta + \frac{\partial E_{\mathbf{k}}}{\partial \dot{\theta}} \Big|_{\nu, \theta} \dot{\theta}. \quad (3.6)$$

The variational derivative  $g_{\mathbf{k}, \mathbf{k}'}$  of the quasi-particle energy with respect to the quasi-particle distribution function has the form

$$g_{\mathbf{k}, \mathbf{k}'} = f_{\mathbf{k}, \mathbf{k}'}^u + \frac{\xi_{\mathbf{k}} \xi_{\mathbf{k}'}}{E_{\mathbf{k}} E_{\mathbf{k}'}} \lambda_{\mathbf{k}, \mathbf{k}'}, \quad (3.7)$$

where  $f_{\mathbf{k}, \mathbf{k}'}^u$  is the odd part of the Fermi-liquid function  $f_{\mathbf{k}, \mathbf{k}'}$ , and  $\lambda_{\mathbf{k}, \mathbf{k}'}$  is expressed in terms of the even part  $f_{\mathbf{k}, \mathbf{k}'}^e$  by means of the equation

$$\lambda_{\mathbf{k}, \mathbf{k}'} = f_{\mathbf{k}, \mathbf{k}'}^e - N_F \int \frac{d\Omega''}{4\pi} f_{\mathbf{k}, \mathbf{k}'}^e \chi(\mathbf{n}'') \lambda_{\mathbf{k}, \mathbf{k}'}, \quad (3.8)$$

$$\chi(\mathbf{n}) = - \int_{-\infty}^{\infty} d\xi \frac{|\Delta_{\mathbf{k}}|^2}{E_{\mathbf{k}}^2} \left( \nu_{\mathbf{k}} - \frac{1}{2} \right).$$

The derivatives of  $E_{\mathbf{k}}$  with respect to  $\theta$  and  $\dot{\theta}$  at fixed  $\nu_{\mathbf{k}}$  are equal to

$$\begin{aligned} \frac{\partial E_{\mathbf{k}}}{\partial \theta} \Big|_{\nu, \dot{\theta}} &= \frac{\Delta^2(T)}{E_{\mathbf{k}}} (\mathbf{n} \times [\mathbf{l} \times \mathbf{n}]) \left( 1 + \frac{\xi_{\mathbf{k}}}{N_F} \frac{\partial N_F}{\partial \varepsilon_F} \frac{F_2}{F_2 + 5} \ln \frac{\varepsilon_F}{T_c} \right) \\ &\quad - \nu_F \int \frac{d\Omega'}{4\pi} (\delta_{\mathbf{n}, \mathbf{n}'} + F_{\mathbf{n}, \mathbf{n}'}) l_{\mathbf{n}'} (\mathbf{n}' \nabla), \\ \frac{\partial E_{\mathbf{k}}}{\partial \dot{\theta}} \Big|_{\nu, \theta} &= - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \left[ l_{\mathbf{n}} - N_F \int \frac{d\Omega'}{4\pi} \lambda_{\mathbf{n}, \mathbf{n}'} \chi(\mathbf{n}') l_{\mathbf{n}'} \right], \end{aligned} \quad (3.9)$$

where  $N_F$  is the density of states at the Fermi surface and  $F_{\mathbf{n}, \mathbf{n}'} = N_F f_{\mathbf{n}, \mathbf{n}'}$ .

In local equilibrium, using the expression for  $\delta \nu_{\mathbf{k}}$  from (2.5), keeping only the first three harmonics of the function  $F_{\mathbf{n}, \mathbf{n}'}$ , and neglecting (by virtue of the smallness of  $T_c / \varepsilon_F$ ) the temperature dependence of  $E_{\mathbf{k}}$ , we obtain for  $\delta E_{\mathbf{k}}$  the following dependence of  $E_{\mathbf{k}}$  on  $\theta$  and  $\dot{\theta}$ :

$$\begin{aligned} \delta E_{\mathbf{k}} &= - \frac{l_n (\mathbf{k}\nabla) \theta}{m} + \frac{m}{3m^*} F_1 k_j \\ &+ \frac{\Delta^2(T)}{E_{\mathbf{k}}} (\mathbf{n} \times [\delta\theta \mathbf{n} \times \mathbf{l}]) \left( 1 + \frac{\xi_{\mathbf{k}}}{N_F} \frac{\partial N_F}{\partial \varepsilon_F} \frac{F_2}{F_2 + 5} \ln \frac{\varepsilon_F}{T_c} \right) \\ &+ \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \left[ \frac{\dot{\theta}}{2(1+F_0)} - l_n \dot{\theta}_{\perp} \left( 1 - \frac{1}{2} \frac{F_2}{F_2 + 5} [\mathbf{n} \times \mathbf{l}]^2 \right) \right]. \end{aligned} \quad (3.10)$$

Here  $\delta\theta_{\perp} = \delta\theta - \mathbf{l}(\mathbf{l} \cdot \delta\theta) = \mathbf{l} \times \delta\mathbf{l}$ , and  $\mathbf{j}$  is the liquid flux density

$$\mathbf{j} = \sum_{\mathbf{k}} \mathbf{k} n_{\mathbf{k}} = \mathbf{P} + \mathbf{j}_0, \quad (3.11)$$

where  $\mathbf{P}$  is the flux of the excitations and  $\mathbf{j}_0$  is the superfluid flux, which, by virtue of (3.3), has the form

$$\mathbf{j}_0 = \frac{1}{2} \sum_{\mathbf{k}} \mathbf{k} \left( \frac{\partial \phi_{\mathbf{k}}}{\partial \mathbf{k}} \nabla \bar{n}_{\mathbf{k}} - \frac{\partial \bar{n}_{\mathbf{k}}}{\partial \mathbf{k}} \nabla \phi_{\mathbf{k}} \right). \quad (3.12)$$

Calculation of  $\mathbf{j}_0$  with neglect of terms  $\sim \Delta(T)/\varepsilon_F$  gives

$$\mathbf{j}_0 = \frac{\rho}{2m} \nabla \varphi + \frac{\rho}{4m} ((1\nabla)\theta_{\perp} + l(\nabla\theta_{\perp})). \quad (3.13)$$

Substituting the variation  $\delta \nu_{\mathbf{k}}$  of the distribution function in local equilibrium into the expression for  $\mathbf{P}$  and using (3.10), we obtain for  $\mathbf{j}$

$$\mathbf{j}_i = \frac{1}{2m} \rho_{ij}^s \nabla_j \varphi + \rho_{ij}^n \nu_j^n + \frac{\rho}{4m} (1 - \Phi_{\parallel}) \left( \frac{1\nabla\theta_{\perp i}}{1 + 1/3 F_1 \Phi_{\perp}} + \frac{l_i \nabla\theta_{\perp}}{1 + 1/3 F_1 \Phi_{\parallel}} \right), \quad (3.14)$$

where  $\rho_{ij}^s$  and  $\rho_{ij}^n$  are respectively the tensors of the superfluid and normal components, satisfying the relation

$$\begin{aligned} \rho_{ij}^s + \rho_{ij}^n &= \rho \delta_{ij}, \quad \rho_{ij}^n = \rho \frac{m^*}{m} \Phi_{ij} \left( \hat{l} + \frac{1}{3} \hat{\Phi} \right)_{ij}^{-1}, \\ \Phi_{ij} &= - \frac{3}{N_F} \sum_{\mathbf{k}} \frac{\partial \nu_{\mathbf{k}}}{\partial E_{\mathbf{k}}} n_{\mathbf{k}} l_j = \Phi_{\parallel} l_i l_j + \Phi_{\perp} (\delta_{ij} - l_i l_j). \end{aligned} \quad (3.15)$$

The expression (3.14) coincides with the expression for the flux obtained by Cross<sup>[9]</sup> from a microscopic theory. The formulas (3.11), (3.12), (3.14) determine the momentum  $\mathbf{P}$  of the quasi-particles:

$$\begin{aligned} \mathbf{P} &= \sum_{\mathbf{k}} \mathbf{k} \delta \nu_{\mathbf{k}} = \hat{\rho}^n \left( \mathbf{v}^n - \frac{1}{2m} \nabla \varphi \right) \\ &- \frac{\rho}{4m} \left[ \frac{\Phi_{\parallel} + 1/3 F_1 \Phi_{\perp}}{1 + 1/3 F_1 \Phi_{\perp}} (1\nabla)\theta_{\perp} + \frac{\Phi_{\parallel} (1 + 1/3 F_1)}{1 + 1/3 F_1 \Phi_{\parallel}} l(\nabla\theta_{\perp}) \right]. \end{aligned}$$

Calculation of the generalized angular momentum  $\mathbf{L}$

For this, as previously,<sup>[10]</sup> we shall make use of the gauge transformation, under which the expression (3.5) is added to the Hamiltonian. The first part of this extra term is equal to

$$- \int d^3r \dot{\theta}(\mathbf{r}, t) \sum_{\mathbf{k}} l_n n_{\mathbf{k}}(\mathbf{r}, t).$$

Noting that the addition to the Hamiltonian should have the form  $- \int d^3\theta \cdot \mathbf{L}$ , we obtain the following expression for the variation of the generalized angular momentum:

$$\delta \mathbf{L} = \sum_{\mathbf{k}} l_n \delta n_{\mathbf{k}}. \quad (3.16)$$

Substituting  $\delta n_{\mathbf{k}}$  from (3.3) for local equilibrium, we have

$$\delta \mathbf{L} = \hat{\chi} \dot{\theta} + 1/2 [\delta\theta \times \mathbf{L}^{sp}], \quad (3.17)$$

where the generalized susceptibility  $\hat{\chi}$  and the density  $\mathbf{L}^{sp}$  of the spontaneous angular momentum of the  $A$  phase are equal to

$$\begin{aligned} \chi_{ij} &= \chi_{\parallel} l_i l_j + \chi_{\perp} (\delta_{ij} - l_i l_j), \\ \chi_{\parallel} &= \frac{N_F}{4(1+F_0)}, \quad \chi_{\perp} = \frac{N_F}{8} \left( \ln \left( \frac{\Delta(T)}{\omega} \right) + \frac{\pi i}{2} - \frac{5}{6} \frac{F_2}{F_2 + 5} \right), \\ \mathbf{L}^{sp} &= \frac{1}{6} \frac{5}{F_2 + 5} \hbar \Delta^2(T) \frac{\partial N_F}{\partial \varepsilon_F} \ln \frac{\varepsilon_F}{T_c}. \end{aligned} \quad (3.18)$$

In calculating  $\chi_{ij}$  and  $L^{sp}$  we have confined ourselves to just the first three harmonics of the function  $F_{n,r}$ .

We emphasize that the generalized angular momentum  $L$  differs from the total angular-momentum density  $\tilde{L}$ , the variation of which in a departure from equilibrium has the form

$$\delta\tilde{L} = \chi\delta\theta + [\delta\theta \times L^{sp}]. \quad (3.19)$$

The difference in the coefficients of  $L^{sp}$  in formulas (3.17) and (3.19) arises from the fact that the Lagrangian of the system contains, beside the term quadratic in  $\hat{\theta}$ , a term proportional to the product of  $\hat{\theta}$  and  $\theta$ , viz.,  $-\frac{1}{2}\hat{\theta}[\delta\theta \times L]$ .

The value of the angular-momentum density  $\tilde{L}$  can be obtained directly from the expression for the superfluid-flux density  $j_0$  (cf. (3.12)) by integration by parts:

$$\begin{aligned} \int d^3r \delta\tilde{L} &= \int d^3r [r j_0] = -\frac{1}{2} \int d^3r \sum_k \left( [k \times \frac{\partial \tilde{n}_k}{\partial k}] \delta\phi_k - [k \times \frac{\partial \phi_k}{\partial k}] \delta \tilde{n}_k \right) \\ &= \int d^3r \left( \sum_k l_k \delta n_k + \frac{1}{2} [\delta\theta \times L^{sp}] \right) = \int d^3r (\chi \hat{\theta} + [\delta\theta \times L^{sp}]). \end{aligned} \quad (3.20)$$

Thus, the angular momentum  $\tilde{L}$  is the total angular momentum of the super-fluid flow. For more detail about this, see Appendix B.

In formula (3.17), the generalized angular momentum is expressed in terms of the variables  $\theta$  and  $\hat{\theta}$ . But if we make use of the variables  $\theta$  and  $L$ , this formula gives the dependence of the angular velocity  $\omega$  on  $\theta$  and  $L$ :

$$\omega = \hat{\chi}^{-1}(\delta L - 1/2[\delta\theta \times L^{sp}]). \quad (3.21)$$

### Calculation of $M^R$

We shall consider the longitudinal and transverse components of the vector  $M$  separately. Inasmuch as  $\delta\theta_z = -\delta\varphi$ , it follows from the gauge transformation that

$$M_z = \frac{\delta E}{\delta\varphi} = -\frac{1}{2m} \nabla j. \quad (3.22)$$

We note that, by virtue of (3.16) and (3.4),

$$\delta L_z = \frac{1}{2} \sum_k \delta n_k = \frac{1}{2m} \delta\rho$$

and, consequently, the  $z$ -component of the equation  $\dot{L} = M$  is none other than the continuity equation

$$\dot{\rho} + \nabla j = 0.$$

We shall represent the perpendicular component  $M_{\perp}^R$  in the form

$$\delta M_{\perp}^R = \frac{\partial M_{\perp}^R}{\partial \theta_{\perp i}} \delta \theta_{\perp i} + \frac{\partial M_{\perp}^R}{\partial \varphi} \delta \varphi + \frac{\partial M_{\perp}^R}{\partial L_j} \delta L_j + \frac{\partial M_{\perp}^R}{\partial v_j^n} \delta v_j^n + \frac{\partial M_{\perp}^R}{\partial T} \delta T. \quad (3.23)$$

The values of the last four derivatives are found by equating mixed derivatives of the functional  $U - P \cdot v^n - TS$ :

$$\begin{aligned} \frac{\partial M_{\perp}^R}{\partial \varphi} &= -\frac{\partial M_{\perp}^R}{\partial \theta_{\perp i}} \\ &= \frac{\rho}{2m^2} (1 - \Phi_{\parallel}) \left( \frac{1}{1 + 1/3 F_1 \Phi_{\perp}} + \frac{1}{1 + 1/3 F_1 \Phi_{\parallel}} \right) (1V)(\nabla - 1(1V)), \\ \frac{\partial M_{\perp}^R}{\partial L_j} &= -\frac{\partial \omega_j^R}{\partial \theta_{\perp i}} = -\frac{1}{2\chi_{\perp}} e_{ij} n L_n^{sp}, \\ \frac{\partial M_{\perp}^R}{\partial T} &= \frac{\partial S}{\partial \theta_{\perp i}} \approx 0, \\ \frac{\partial M_{\perp}^R}{\partial v_j^n} &= \frac{\partial P_j}{\partial \theta_{\perp i}} = -\frac{\rho}{4m} \left[ \frac{\Phi_{\parallel} + 1/3 F_1 \Phi_{\perp}}{1 + 1/3 F_1 \Phi_{\perp}} (\delta_{ij} - l_i l_j) 1V \right. \\ &\quad \left. + \frac{\Phi_{\parallel} (1 + 1/3 F_1)}{1 + 1/3 F_1 \Phi_{\parallel}} l_j (\nabla_i - l_i \nabla) \right]. \end{aligned} \quad (3.24)$$

It remains to elucidate the form of the tensor  $\partial M_{\perp}^R / \partial \theta_{\perp i j}$ . Inasmuch as the energy  $U$  depends quadratically on the spatial derivatives of  $\theta_{\perp}$ , this tensor can be expressed in terms of three temperature-dependence coefficients:

$$\frac{\partial M_{\perp}^R}{\partial \theta_{\perp i j}} = [C_1(T) \delta_{ij} \delta_{mn} + 1/2 C_2(T) (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) + C_3(T) \delta_{ij} l_m l_n] \nabla_m \nabla_n. \quad (3.25)$$

To calculate the coefficients  $C_1$ ,  $C_2$  and  $C_3$  it is no longer sufficient to use the technique developed above. We need to take into account the dependence of the phase of the order parameter on the space and time derivatives of the vector  $\theta$  (cf. Appendix B). The results of the calculations coincide with those of the microscopic derivation carried out in the papers by Cross<sup>[9]</sup> and Wölfle.<sup>[11]</sup> Near  $T_c$ ,

$$C_3 = 2C_1 = \rho_0^2 / 4m^2, \quad (3.26)$$

and, compared with these, the coefficient  $C_2$  is small in the parameter  $\Delta(T)/T_c$ .

### Calculation of the kinetic coefficients

The kinetic coefficients defined by the expressions (2.9) are obtained by substituting formula (3.10) for  $E_k$  into these expressions. Taking into account, each time, only the leading terms in  $\omega/\Delta(T)$ ,  $qv_F/\Delta(T)$  and  $\Delta(T)/\epsilon_F$ , we have

$$\begin{aligned} \gamma_{L, L_j} &= \gamma_{LL} (\delta_{ij} - l_i l_j), \quad \gamma_{LL} = \frac{\tau \Delta^2(T) N_F}{6} \left[ \Phi_{\parallel} - 3 \int \frac{d\Omega}{4\pi} (nl)^2 (1 - \chi(n)) \right], \\ \gamma_{L, \theta_j} &= -e_{ij} l_p \frac{\tau \Delta^2(T)}{4\chi_{\perp}} \frac{\partial N_F}{\partial \epsilon_F} \frac{5 + F_2 \ln(\epsilon_F/T_c)}{5 + F_2} \\ &\quad \times \int \frac{d\Omega}{4\pi} (nl)^2 (1 - \chi(n)) \left( 1 - \frac{5}{2} \frac{F_2 [n \times l]^2}{F_2 + 5} \right) \\ \gamma_{L, P_j v_j^n} &= -\frac{1}{2} \gamma_{L, L} (\text{rot } v^n) + e_{ij} l_p l_m (\nabla_m v_j^n + \nabla_j v_m^n) \\ &\quad \times \left[ -\frac{\gamma_{LL}}{2} + \frac{\tau \Delta^2(T) N_F}{2} \left( 3 + \frac{m^2 v_F^2}{N_F} \frac{\partial N_F}{\partial \epsilon_F} \frac{F_2 \ln(\epsilon_F/T_c)}{F_2 + 5} \right) \right. \\ &\quad \left. \times \int \frac{d\Omega}{4\pi} (nl)^2 [n \times l]^2 (1 - \chi(n)) \right], \\ \gamma_{\theta, \theta_j} &= l_i l_j \frac{4\tau}{N_F} \int \frac{d\Omega}{4\pi} (1 - \chi(n)) + (\delta_{ij} - l_i l_j) \frac{\tau N_F}{8\chi_{\perp}^2} \\ &\quad \times \int \frac{d\Omega}{4\pi} \frac{(nl)^2}{[n \times l]^2} (1 - \chi(n)) \left( 1 - \frac{5}{2} \frac{F_2 [n \times l]^2}{F_2 + 5} \right), \\ \gamma_{\theta, P_j} &= -\frac{6\tau\rho}{m N_F} \int \frac{d\Omega}{4\pi} (1 - \chi(n)) \left[ (nl)^2 l_i l_j 1V + \frac{1}{2} [n \times l]^2 l_i (\nabla_j - l_j \nabla) \right] \\ &\quad - \frac{3\tau\rho}{4m\chi_{\perp}} \int \frac{d\Omega}{4\pi} (1 - \chi(n)) (nl)^2 \left( 1 - \frac{5}{2} \frac{F_2 [n \times l]^2}{F_2 + 5} \right) [(\delta_{ij} - l_i l_j) 1V + l_j (\nabla_i - l_i \nabla)], \\ \gamma_{P, P_j} &= -\frac{9\rho^2 \tau}{m^2 N_F} \int \frac{d\Omega}{4\pi} (1 - \chi(n)) n_i n_j n_p n_m \nabla_p \nabla_m, \\ \gamma_{SS} &= \frac{3\tau\rho}{mm^2 T^2} \int_{-\infty}^{+\infty} d\xi \xi^2 \int \frac{d\Omega}{4\pi} \frac{\partial v^0}{\partial E} n_i n_j \nabla_i \nabla_j. \end{aligned} \quad (3.27)$$

The terms  $\gamma_{SL}$  and  $\gamma_{S\theta}$  can be neglected, since, compared with the other terms, they are of higher order in the gradients.

We shall turn our attention to the dissipative term

$$\gamma_{L,L_j} [\dot{\theta}_j^{-1/2} (\text{rot } \mathbf{v}^n)_j] \quad (3.28)$$

in the equation for the angular momentum. In the absence of flow of the normal component, it has been obtained by Cross and Anderson.<sup>[12]</sup> This term describes the frictional torque that arises when the angular velocity of rotation of the vector  $\mathbf{l}$  differs from the angular velocity  $\frac{1}{2} \text{curl } \mathbf{v}^n$  of the rotation of the normal component. Thus, if we rotate a vessel with  $A$  phase at a constant angular velocity, then, after the relaxation time  $\sim \gamma_{LL}/N_F$  has elapsed, the vector  $\mathbf{l}$  will rotate together with the vessel. The law of approach to the asymptotic behavior has the form

$$\dot{\theta}_\perp = \frac{1}{2} (\text{rot } \mathbf{v}^n)_\perp \left[ 1 - \exp \left( - \frac{8\gamma_{LL}t}{N_F \ln(t\Delta(T))} \right) \right]. \quad (3.29)$$

The nontrivial law of approach to the asymptote is explained by the presence of retardation in the coupling between the angular momentum  $\mathbf{L}$  and angular velocity  $\dot{\theta}$  (cf. (3.18))

$$\delta L_\perp(t) = \int_0^t d\tau \chi_\perp(\tau) \dot{\theta}_\perp(t-\tau), \quad (3.30)$$

$$\chi_\perp(\tau) = \int \frac{d\omega}{2\pi} \chi_{\perp\omega} e^{-i\omega\tau} \approx \frac{N_F}{8} \frac{1}{\tau + \Delta^{-1}(T)}.$$

#### 4. ORBITAL WAVES

The superfluid flow is described by the first two equations of the system (2.8). In the papers of other authors,<sup>[5,11-13]</sup> the equations corresponding to these are

$$\dot{\rho} + \nabla \mathbf{j} = 0, \quad \dot{\varphi} + 2 \frac{\delta F}{\delta \rho} = 0, \quad \dot{\tilde{\mathbf{L}}} + \frac{\delta F}{\delta \tilde{\mathbf{L}}} \Big|_P = 0, \quad (4.1)$$

where the angular momentum  $\tilde{\mathbf{L}}$  is equal to the spontaneous angular momentum  $\mathbf{L}^{\text{sp}}$ , i.e., its variation is

$$\delta \tilde{\mathbf{L}} = |\mathbf{L}^{\text{sp}}| \delta \mathbf{l} = [\delta \theta \times \mathbf{L}^{\text{sp}}]. \quad (4.2)$$

The difference between the system (4.1)–(4.2) and the first two equations of (2.8) is that the term containing the generalized susceptibility  $\chi_\perp$  (cf. (3.19)) is not taken into account in the expression (4.2). (In (4.1), the free energy  $F$  is assumed to be independent of  $\dot{\theta}$ , and therefore  $\tilde{\mathbf{L}}$  appears in place of the generalized angular momentum  $\mathbf{L}$ .) The absence in Eq. (4.2) of a term containing  $\chi_\perp$  has led to the conclusion that orbital waves are absent, since the mode corresponding to the dynamics of the vector  $\mathbf{l}$  was found to be purely diffusive by virtue of the smallness of  $|\mathbf{L}^{\text{sp}}| \ll \gamma_{LL}$ .<sup>[12,14]</sup>

We shall show that allowance for the term proportional to  $\chi_\perp$  leads to the possibility of propagation of weakly damped orbital waves for  $T \rightarrow T_c$ .<sup>[10]</sup> In fact, we shall consider Eq. (2.8) in the fourth-sound regime ( $\mathbf{v}^n = 0$ ), neglecting the temperature variation. Then, neglecting the term  $|\mathbf{L}^{\text{sp}}| \ll \gamma_{LL}$ , neglecting the term  $\gamma_{\theta\theta} \dot{\mathbf{L}}$  in com-

parison with  $\dot{\theta}$  by virtue of the smallness of  $\omega\tau \ll 1$ , and also neglecting the term  $\gamma_{\theta L} \ll 1$  for  $T \rightarrow T_c$ , we obtain the first two equations of (2.8) in the form

$$\dot{\tilde{\mathbf{L}}}_i - M_i^R = -\gamma_{L,L_j} \dot{\theta}_j, \quad \delta \tilde{\mathbf{L}} = \hat{\chi} \dot{\theta}.$$

Separating out the longitudinal and transverse components in these equations, we obtain

$$\chi_\perp \ddot{\varphi} = \frac{1}{2m} \nabla \mathbf{j}, \quad \chi_\perp \ddot{\theta}_\perp + \gamma_{LL} \dot{\theta}_\perp = \mathbf{M}_\perp^R. \quad (4.3)$$

From this it can be seen that for  $\omega \ll \gamma_{LL}/\chi_\perp$  the equation for the angle  $\theta_\perp$  has a purely diffusive character, since in it we can neglect the first term in comparison with the second, and, consequently, orbital waves are absent in this regime.

In this region of frequencies the orbital motion can be observed in an indirect way, e.g., from the influence it exerts on the propagation of ordinary sound. An analysis of Eqs. (2.8) shows that the orbital motion gives an additional contribution to the attenuation of first sound. Near  $T_c$ ,

$$\omega = cq(1 - i\gamma q), \quad (4.4)$$

$$\gamma = \gamma_0 + \frac{\rho}{8\gamma_{LL}} \frac{(q\mathbf{l})^2 [q \times \mathbf{l}]^2}{q^4},$$

where  $\gamma$  is the sound attenuation in the normal Fermi liquid.

When  $\gamma_{LL}/\chi_\perp \ll \omega \ll \tau^{-1}$ , which is possible only near  $T_c$  (cf. [10]), we can neglect the second dissipative term in comparison with the first dynamical term and we then obtain two sound-like branches of weakly damped orbital waves with the spectrum

$$\begin{aligned} \omega_\parallel^2 &= \frac{\rho_\parallel^*}{8m^2\chi_\perp} q^2 \frac{2q^2 + (q\mathbf{l})^2}{2q^2 - (q\mathbf{l})^2}, & \delta\theta_\parallel \parallel q_\perp, \\ \omega_\perp^2 &= \frac{\rho_\perp^*}{8m^2\chi_\perp} (q^2 + 2(q\mathbf{l})^2), & \delta\theta_\perp \perp q_\perp, \end{aligned} \quad (4.5)$$

which is obtained by substituting  $\mathbf{M}_i^R$  and  $\mathbf{j}$  from (3.23) and (3.14) into (4.3), using the expressions (3.26) for  $C_1$  and  $C_3$ . The spectrum of the longitudinal oscillations ( $\delta\theta_\parallel \parallel q_\perp$  or  $\delta\mathbf{l} \parallel q_\perp$ ) has been written in the approximation  $\chi_\perp \gg \chi_\parallel$ , which holds by virtue of the large value of  $F_0$  (cf. 3.18). These oscillations are coupled with ordinary sound (oscillations of  $\rho$  or  $\varphi$ ). Owing to this coupling (albeit small by virtue of the smallness of  $\chi_\parallel/\chi_\perp$ ), longitudinal orbital waves can be excited as ordinary sound. The transverse orbital waves ( $\delta\theta_\perp \perp q_\perp$ ) are not coupled with density oscillations.

In conclusion, for purposes of reference, we give the expressions for the coefficients appearing in the phenomenological hydrodynamics of Graham<sup>[5]</sup> (as throughout, we use the smallness of  $T_c/\varepsilon_F$ ):

$$\begin{aligned} \beta &= \alpha_1 = \alpha_2 = 0, \\ C_\mu &= \frac{1}{2} \rho (1 - \Phi_\parallel) \left( \frac{\delta_{ij} - l_i l_j}{1 + \gamma_{S F_1} \Phi_\perp} - \frac{l_i l_j}{1 + \gamma_{S F_1} \Phi_\parallel} \right), \\ K_1 &= 4m^2 C_1, \quad K_2 = 4m^2 (C_1 + C_2), \quad K_3 = 4m^2 (C_1 + C_3), \\ \eta &= \frac{1}{2\gamma_{LL}}, \quad \xi = -\frac{\rho}{4\gamma_{LL}}, \quad \zeta = \frac{\tau}{1 + F_0} \frac{\partial \mu}{\partial \rho} \int \frac{d\Omega}{4\pi} (1 - \chi(\mathbf{n})), \end{aligned}$$

$$\xi_{ij} = \frac{\tau}{1+F_0} \rho \frac{\partial \mu}{\partial \rho} \int \frac{d\Omega}{4\pi} (1-\chi(\mathbf{n})) (\delta_{ij}-3n_i n_j),$$

$$\nu_{ikl} = \frac{\tau \rho^2}{2N_F} \int \frac{d\Omega}{4\pi} (1-\chi(\mathbf{n})) (3n_i n_k - \delta_{ik}) (3n_j n_l - \delta_{jl}) + \frac{\rho^2}{32\gamma_{LL}}$$

$$\times [(\delta_{ij}-l_i l_j) l_k l_l + (\delta_{kl}-l_k l_l) l_i l_j + (\delta_{ij}-l_i l_j) l_k l_l + (\delta_{kl}-l_k l_l) l_i l_j],$$

$$\kappa_{ij} = -\frac{3\tau\rho}{mm^*T^2} \int_{-\infty}^{\infty} d\xi \int \frac{d\Omega}{4\pi} \xi^2 \frac{\partial v^0}{\partial E} n_i n_j.$$

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## APPENDIX A

For illustration we shall consider the first of Eqs. (2.3). The right-hand side of the first equation of (2.8) is the nonequilibrium part  $M^D$  of the generalized torque  $M$  (the index  $D$  signifies "dissipative"). The nonequilibrium part  $M^D$  arises because of the dependence of  $M$  on the quasi-particle distribution function  $\nu_{\mathbf{k}}$ , which contains the nonequilibrium correction  $\delta\nu_{\mathbf{k}}^D$ :

$$M^D = \sum_{\mathbf{k}} \frac{\delta M}{\delta \nu_{\mathbf{k}}} \delta \nu_{\mathbf{k}}^D = - \sum_{\mathbf{k}} \left. \frac{\partial E_{\mathbf{k}}}{\partial \theta} \right|_{\nu} \delta \nu_{\mathbf{k}}^D. \quad (\text{A.1})$$

The quantity  $\delta\nu_{\mathbf{k}}^D$  does not coincide, generally speaking, with the quantity  $g_{\mathbf{k}}$  introduced in (2.5), which is found from the kinetic equation and, in the  $\tau$ -approximation, is equal to

$$g_{\mathbf{k}} = -\tau \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} \left[ \frac{\partial E_{\mathbf{k}}}{\partial t} - \left( \frac{\partial}{\partial t} + \frac{\partial E}{\partial \mathbf{k}} \nabla \right) \left( \mathbf{k} \mathbf{v}^n + E_{\mathbf{k}} \frac{\delta T}{T} \right) \right]. \quad (\text{A.2})$$

This is because, in the expression (2.5), the term  $\delta E_{\mathbf{k}}$  itself depends on  $\delta\nu_{\mathbf{k}}^D$  through the equality (3.6). Using (2.5) and (3.6), we find the relation between  $\delta\nu_{\mathbf{k}}^D$  and  $g_{\mathbf{k}}$ :

$$\delta \nu_{\mathbf{k}}^D = \sum_{\mathbf{k}'} \beta_{\mathbf{k}, \mathbf{k}'} g_{\mathbf{k}'}. \quad (\text{A.3})$$

where the matrix  $\beta_{\mathbf{k}, \mathbf{k}'}$  satisfies the equation

$$\beta_{\mathbf{k}, \mathbf{k}'} = \delta_{\mathbf{k}, \mathbf{k}'} + \sum_{\mathbf{k}''} \frac{\partial \nu_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}} g_{\mathbf{k}, \mathbf{k}''} \beta_{\mathbf{k}'', \mathbf{k}'}. \quad (\text{A.4})$$

On the other hand, noting that this same matrix  $\beta_{\mathbf{k}, \mathbf{k}'}$  appears in the relation between  $\partial E_{\mathbf{k}}/\partial \theta$  for fixed  $\nu_{\mathbf{k}}$  and  $\partial E_{\mathbf{k}}/\partial \theta$  for fixed  $\mathbf{v}^n$  and  $T$ , we obtain

$$M^D = - \sum_{\mathbf{k}} \left. \frac{\partial E_{\mathbf{k}}}{\partial \theta} \right|_{\mathbf{v}^n, T} g_{\mathbf{k}}.$$

The time derivative of the energy  $E_{\mathbf{k}}$  can be expressed in the following way:

$$\frac{\partial E_{\mathbf{k}}}{\partial t} = \left. \frac{\partial E_{\mathbf{k}}}{\partial \theta} \right|_{\mathbf{v}^n, T} \dot{\theta} + \left. \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{L}} \right|_{\mathbf{v}^n, T} \dot{\mathbf{L}} + \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{v}^n} \dot{\mathbf{v}}^n + \frac{\partial E_{\mathbf{k}}}{\partial T} \dot{T}.$$

Having convinced ourselves that the time derivatives of  $\mathbf{v}^n$  and  $T$  can be neglected, inasmuch as they appear in terms that are small in the parameter  $T_c/\epsilon_F$ , we obtain the expressions (2.5) for the kinetic coefficients

$\gamma_{LL}$ ,  $\gamma_{L\theta}$  and  $\gamma_{LP}$ . The remaining coefficients are obtained either in an analogous way or by using the right-hand sides of Eqs. (2.6). Of course, the  $\tau$ -approximation gives only a crude estimate of the kinetic coefficients, although it does enable us to obtain their orders of magnitude. For an exact calculation of each of the kinetic coefficients it is necessary each time to solve the kinetic equation with a collision term, as is done, e.g., in the papers of Soda and Fujiki<sup>[15]</sup> and Valls and Houghton,<sup>[16]</sup> in which the viscosity and thermal-conductivity coefficients are calculated.

## APPENDIX B

Under a gauge transformation, the term (3.5) is added to the Hamiltonian of the system. Inasmuch as  $-M$  is the variation of the free energy with respect to  $\theta$ ,

$$M = - \sum_{\mathbf{k}} n_{\mathbf{k}} \frac{\partial}{\partial \theta} (\mathbf{k} \nabla) \frac{\phi_{\mathbf{k}}}{2m^*} = \sum_{\mathbf{k}} \frac{\partial \phi_{\mathbf{k}}}{\partial \theta} \frac{(\mathbf{k} \nabla)}{2m^*} n_{\mathbf{k}}(\mathbf{x}). \quad (\text{B.1})$$

We substitute  $n_{\mathbf{k}}(\mathbf{x})$  from (3.3) into (B.1) and obtain for  $M$  the following expression:

$$- \frac{1}{2m^2} \sum_{\mathbf{k}} \left( \frac{\partial n}{\partial \xi} - \frac{\partial \nu}{\partial E} \right) l_{\mathbf{k}} (\mathbf{k} \nabla)^2 \phi_{\mathbf{k}} \quad (\text{B.2})$$

(for simplicity we neglect the Fermi-liquid corrections). As it should,  $M$  depends on the second derivatives of  $\theta$  with respect to the coordinates. We note that (B.2) does not give the full value of the quantity  $M$ . In fact,  $n_{\mathbf{k}}$  depends not only on the gradients of  $\theta$ , but also on  $\theta$  itself, through

$$\delta |\Delta_{\mathbf{k}}|^2 = - \left( \delta \theta \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] \right) |\Delta_{\mathbf{k}}|^2 = 2\Delta^2(T) (n\mathbf{l}) (\delta \theta [n \times \mathbf{l}]).$$

Therefore, in order that all terms of the form  $\nabla_i \nabla_j \theta_m$  be taken fully into account in  $M$ , we must find the phase  $\phi_{\mathbf{k}}$  of the order parameter to terms linear in the gradients. For this we shall perform a rotation transformation, through an angle  $\delta \theta$  independent of the coordinates and time. The rotation operator has the form

$$\hat{U} = \exp(i\hat{S}), \quad \hat{S} = \frac{1}{2i} \int d^3r [\delta \theta(\mathbf{r}, t) \times \mathbf{r}] (\hat{\Psi} + \nabla \hat{\Psi} - (\nabla \hat{\Psi}^+) \hat{\Psi}^+). \quad (\text{B.3})$$

Under this transformation the Hamiltonian is supplemented by the term

$$- \int d^3r \sum_{\mathbf{k}} \left( \dot{\theta} + \frac{(\mathbf{k} \nabla)}{m} \theta \right) \hat{L}_{\mathbf{k}}(\mathbf{r}), \quad (\text{B.4})$$

$$\hat{L}_{\mathbf{k}}(\mathbf{r}) = \frac{1}{2i} \sum_q e^{i\mathbf{q} \cdot \mathbf{r}} \left( a_{\mathbf{k}+\mathbf{q}/2}^+ \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] a_{\mathbf{k}-\mathbf{q}/2} - \left( \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] a_{\mathbf{k}+\mathbf{q}/2}^+ \right) a_{\mathbf{k}-\mathbf{q}/2} \right).$$

We now perform the Bogolyubov transformation

$$a_{\mathbf{k}} = u_{\mathbf{k}} b_{\mathbf{k}} + v_{\mathbf{k}}^* b_{-\mathbf{k}}^+$$

on the whole Hamiltonian, including (B.4), and require that the coefficients of the nondiagonal terms  $b_{-\mathbf{k}} b_{\mathbf{k}}$  and  $b_{\mathbf{k}}^* b_{-\mathbf{k}}^*$  vanish. Here it is not difficult to see that the change of  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  on account of (B.4) can be compensated by a gauge transformation defined by the operator

$$\hat{U}_s = \exp(i\hat{S}_s), \quad \hat{S}_s = \int d^3r \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}}(\mathbf{r}, t) \hat{n}_{\mathbf{k}}(\mathbf{r}),$$

$$\delta\phi_{\mathbf{k}}(\mathbf{r}, t) = -2\delta\theta(\mathbf{r}, t) \mathbf{l}_{\mathbf{k}} + \frac{\xi_{\mathbf{k}}}{4E_{\mathbf{k}}^2 |\Delta_{\mathbf{k}}|^2} \left( \frac{\partial}{\partial t} + \frac{\mathbf{k}\nabla}{m} \right) \delta|\Delta_{\mathbf{k}}|^2, \quad (\text{B.5})$$

$$\delta|\Delta_{\mathbf{k}}|^2 = -\delta\theta(\mathbf{r}, t) \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] |\Delta_{\mathbf{k}}|^2.$$

Substituting (B.5) into (B.1), we obtain the full value of the torque  $\mathbf{M}$ , coinciding with the result of the microscopic derivation of<sup>[9,11]</sup>.

We note also that the dependence of the phase on  $\theta$  changes the magnitude of the generalized susceptibility  $\chi_{\perp}$ . Indeed, substituting (B.5) into the expression (3.20) for the angular momentum  $\tilde{\mathbf{L}}$ , we obtain additionally the term  $\chi_{\perp}' \theta_{\perp}$ , where

$$\chi_{\perp}' = -\frac{N_F}{16} \sum_{\mathbf{k}} \left( \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] |\Delta_{\mathbf{k}}| \right)^2 \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^3} \frac{\partial}{\partial E} \left( E^{-1} \text{th} \frac{E}{2T} \right). \quad (\text{B.6})$$

In the region of existence of the  $A$  phase this quantity is small compared with (3.18), inasmuch as it does not contain the factor  $\ln(\Delta(T)/\omega)$ , and, in addition, is proportional to  $\rho^s$  near  $T_c$ .

The formulas (3.18) and (B.6) for the generalized susceptibility  $\chi_{\perp}$  are valid in the hydrodynamic limit  $\omega\tau \ll 1$ . In the high-frequency, zero-sound limit, when  $\Delta(T) \gg \omega \gg 1/\tau$ , the distribution function does not have time to adjust to the change of  $\theta$ , i.e.,  $\nu_{\mathbf{k}}$  must be regarded as independent of  $\theta$ . For the total  $\chi_{\perp}$  we obtain the following expression:

$$\chi_{\perp} = \frac{N_F}{16} \sum_{\mathbf{k}} E^{-3} \text{th} \frac{E}{2T} \left[ |\Delta_{\mathbf{k}}|^2 \left( \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] \phi_{\mathbf{k}} \right)^2 + \frac{\xi_{\mathbf{k}}^2}{E^2} \left( \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] |\Delta_{\mathbf{k}}| \right)^2 \right]. \quad (\text{B.7})$$

This expression coincides with the result of Leggett and Takagi,<sup>[17]</sup> which is thereby valid only in the high-frequency limit.

We note also that in the high-frequency limit a term proportional to the vector  $\theta_{\perp}$  appears in  $\mathbf{M}_{\perp}$  (cf. <sup>[17]</sup>). This is connected with the dependence of  $\mathbf{M}_{\perp}$  on  $\nu_{\mathbf{k}}$ . In the low-frequency limit, when  $\nu_{\mathbf{k}}$  is the local-equilibrium distribution function,  $\mathbf{M}_{\perp}$  does not depend on  $\theta$  but depends only on the gradients of  $\theta$ , inasmuch as the energy of the system does not change on rotation of the entire system through angle  $\theta$ . In the high-frequency limit the distribution function does not have time to change with change of  $\theta$ , and therefore it is necessary to eliminate from  $\mathbf{M}_{\perp}$  the terms connected with the variation of  $\nu_{\mathbf{k}}$ :

$$\sum_{\mathbf{k}} \frac{\delta \mathbf{M}_{\perp}}{\delta \nu_{\mathbf{k}}} \delta \nu_{\mathbf{k}} = -\frac{1}{2} \delta \theta_{\perp} \sum_{\mathbf{k}} \left( \frac{\partial E}{\partial \theta_{\perp}} \right)^2 \frac{\partial \nu_{\mathbf{k}}}{\partial E}$$

$$= -\frac{1}{2} \delta \theta_{\perp} \sum_{\mathbf{k}} \frac{|\Delta_{\mathbf{k}}|^2}{E^2} \frac{\partial \nu_{\mathbf{k}}}{\partial E} \left( \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] |\Delta_{\mathbf{k}}| \right)^2 = -\mathbf{M}_{\perp}^0. \quad (\text{B.8})$$

Consequently, inasmuch as there was no term proportional to  $\theta_{\perp}$  in the hydrodynamic limit, the term  $\mathbf{M}_{\perp}^0$  appears in the high-frequency (zero-sound) limit.

As already noted in the text, the quantities  $\mathbf{L}$  and  $\tilde{\mathbf{L}}$  appearing in the present paper are the generalized and total angular momentum of the superfluid flow, since they arise from the superfluid flux  $\mathbf{j}_0$  (cf. (3.20)) and do not contain the angular momentum  $\mathbf{r} \times \mathbf{P}$  of the excita-

tions. The total angular momentum of the liquid is equal to

$$\sum_{\mathbf{k}} \langle \hat{\mathbf{L}}_{\mathbf{k}}(\mathbf{r}) \rangle = \tilde{\mathbf{L}}(\mathbf{r}) + [\mathbf{r} \times \mathbf{P}]. \quad (\text{B.9})$$

The equation for the total angular momentum of the system can be obtained by varying the expression (B.4) with respect to  $\theta$ :

$$\frac{\partial}{\partial t} \sum_{\mathbf{k}} \langle \hat{\mathbf{L}}_{\mathbf{k}}(\mathbf{r}) \rangle = -\frac{\nabla_{\perp}}{m} \sum_{\mathbf{k}} k_{\perp} \langle \hat{\mathbf{L}}_{\mathbf{k}}(\mathbf{r}) \rangle, \quad (\text{B.10})$$

i.e., as we should expect, the total angular momentum of the liquid is conserved, which cannot be said of the quantities  $\tilde{\mathbf{L}}$  and  $\mathbf{r} \times \mathbf{P}$  separately. The equation for the angular momentum  $\mathbf{r} \times \mathbf{P}$  of the excitations can be found from the kinetic equation (2.2). Subtracting it from (B.10), we obtain the equation for the superfluid angular momentum  $\tilde{\mathbf{L}}$  in the form

$$\frac{\partial \tilde{\mathbf{L}}}{\partial t} = \frac{1}{2m} \sum_{\mathbf{k}} \frac{\partial \phi_{\mathbf{k}}}{\partial \theta} (k\nabla) n_{\mathbf{k}}(\mathbf{r}) + \mathbf{M}_{\text{fr}},$$

where  $\mathbf{M}_{\text{fr}}$  is the frictional torque due to the excitations, equal to the transfer of angular momentum per unit time from the excitations to the superfluid flow. The principal contribution to  $\mathbf{M}_{\text{fr}}$  is given by the term (3.28);

$$\gamma_{\nu, \nu} [\theta_{\perp}^{-1/2} (\text{rot } \mathbf{v}^n)],$$

The presence of such a term is characteristic for the hydrodynamics of any liquid having an internal rotational degree of freedom (see the book by Gyarmati<sup>[18]</sup>).

(Note added in proof (July 23, 1976). As was first noted in a paper by Saslow (W. M. Saslow, J. Low Temp. Phys. **23**, 495 (1976)), the entropy-density tensor (27) is isotropic. It is easy to verify this by integrating (2.7) by parts.)

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# Microscopic scaling theory in the percolation problem

S. L. Ginzburg

Leningrad Nuclear Physics Institute, USSR Academy of Sciences  
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 Zh. Eksp. Teor. Fiz. **71**, 1145-1158 (September 1976)

By means of field-theoretical methods, the critical indices and correlation functions of percolation theory are calculated and the analog of the equation of state is constructed. For  $3 < d < 6$  space dimensions the indices are equal to  $\gamma = 8/(2+d)$ ,  $\nu = 4/(2+d)$ ,  $\beta = 2(d-2)/(d+2)$  and  $\Delta = \beta + \gamma = 2$ . This is in agreement with computer calculations, which for  $d=3$  give the following values of the indices:  $\beta = 0.35 \pm 0.05$ ,  $\gamma = 1.69 \pm 0.05$ ,  $\nu = 0.9 \pm 0.05$  and  $\Delta = 2.2 \pm 0.3$ .

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## 1. INTRODUCTION

It is well known that the problem of percolation very much resembles the problem of second-order phase transitions (cf., e.g., the reviews<sup>[1,2]</sup>). In percolation theory, the analog of the order parameter is the "power" of an infinite cluster. This analogy was formulated mathematically rigorously by Kasteleyn and Fortuin,<sup>[3]</sup> who showed that the percolation problem is the limiting case of the so-called S-model for  $S \rightarrow 1$ . A scaling hypothesis for the percolation problem has been formulated using this analogy.<sup>[4-6]</sup>

On the other hand, the theory of second-order phase transitions can be formulated from a microscopic point of view (cf., e.g.,<sup>[7,8]</sup>), using field theory. One can use either the  $\epsilon$ -expansion method,<sup>[7]</sup> or renormalization-group theory directly in three-dimensional space.<sup>[9,10]</sup> It is therefore natural to attempt to construct, using the field theory for the S-model, a microscopic scaling theory for the percolation problem too. The first attempt in this direction was made by Harris *et al.*,<sup>[11]</sup> who carried out the  $\epsilon$ -continuation from six-dimensional space (since the corresponding field theory is logarithmic in six-dimensional space). In<sup>[11]</sup>, however, the Hamiltonian of the S-model was replaced by another model Hamiltonian, introduced by Golner.<sup>[12]</sup> This replacement is, generally speaking, not justified, and this is obviously why the Fisher parameter  $\eta$  calculated by Harris *et al.*<sup>[11]</sup> turned out to be negative, which would be completely incomprehensible.

In the present paper we shall consider the bond problem in percolation theory, using the field theory for the S-model directly in the three-dimensional case, analogously to the way in which this was done earlier<sup>[9,10]</sup> for the theory of second-order phase transitions. We shall calculate the critical indices, correlation functions and analog of the free energy for the percolation problem, as functions of the concentration  $q$  of broken bonds and

of the fictitious magnetic field  $x$  introduced by Kasteleyn and Fortuin.<sup>[3]</sup> Using three results, we shall then calculate the thermodynamic functions of a disordered Ising ferromagnet as functions of the temperature and the real magnetic field at low temperatures near the percolation threshold.

## 2. CONNECTION BETWEEN THE S-MODEL AND THE PERCOLATION PROBLEM

Since the analogy between the percolation problem and the partition function of the S-model has been formulated in the language of the mathematical theory of linear finite graphs,<sup>[3]</sup> we shall briefly derive the principal results of this work in the language of statistical physics. The S-model (the Ashkin-Teller-Potts model<sup>[13,14]</sup>) is a generalization of the Ising model. Suppose that at each site of the lattice there is a certain object which can be in one of  $S$  possible states, while the energy of the interaction between sites depends only on whether the objects at neighboring sites are in the same states or in different states. A solid solution of  $S$  components with equal concentrations can serve as one of the realizations of this model (just as one realization of the Ising model is a solution with two components). The S-model with  $S=2$  corresponds to the Ising model. We shall write the Hamiltonian of the S-model in two physically equivalent forms:

$$H = -J \sum_{i,k} (\delta_{\sigma_i \sigma_k} - 1) - 2\omega_0 \sum_i (\delta_{\sigma_i 1} - 1), \quad (1a)$$

$$H = -J_1 \sum_{i,k} P_{\sigma_i \sigma_k} - 2\omega_1 \sum_i P_{\sigma_i 1}, \quad (1b)$$

$$P_{\sigma_i \sigma_i} = S \delta_{\sigma_i \sigma_i} - 1.$$

Here  $J$  is the exchange integral,  $J_1 = JS^{-1}$ ,  $\omega_0 = \mu H$ ,  $H$  is the magnetic field,  $\mu$  is the magnetic moment,  $\omega_1 = \omega_0 S^{-1}$ ,  $S$  is the number of components, and  $\sigma_i$  is the index labeling the components of the S-model.