Spectra and excitation methods of turbulence in a compressible fluid

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We develop a random force method and apply it to the turbulence of a compressible fluid. We show that the invariance of the equation for the characteristic functional under the scale transformation group enables us to determine the turbulence spectrum in the inertial range both for incompressible and for compressible fluids. We evaluate in the inertial range the spectra of the two-point semi-invariants of arbitrary order, like the energy spectrum.

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INTRODUCTION

It is well known that the problem of the statistical description of turbulence can be formulated in terms of a characteristic functional (Hopf, [1] Novikov[2]). Novikov obtained the appropriate equation with energy pumping by an external force for an incompressible fluid, and we obtain it in the present paper for a compressible fluid. For a small-amplitude potential external force, acoustic turbulence is excited in a compressible fluid and this has been considered in papers by Zakharov and Sagdeev, [3] and by Kadomtsev and Petviashvili. [4] In the present paper we consider the case where the external force has large amplitude potential and rotational components.

It is well known (see, e.g., Monin and Yaglom[5]) that an application of perturbation theory to the equations of statistical hydrodynamics leads to difficulties connected with the large magnitude of the coupling constant. In a compressible fluid the situation is made complicated by the presence of, in general, several strongly coupled fields. Further progress has been connected with the presence of similarity properties for turbulent pulsations which are, apparently, characteristic for a wide class of strongly interacting systems (Kuz'min and Patashinskiĭ[6]). We show in the present paper that the Cauchy problem for the characteristic functionals of incompressible and compressible fluids with energy pumping by large amplitude external force has a group of invariants which leads to a similarity theorem for them. This fact combined with the assumption that there exists an equilibrium section in the spectrum (inertial range) allows us to find the spectral characteristics of turbulence in both incompressible and compressible fluids. For an incompressible fluid one obtains in that way the Kolmogorov law[7,8] and in a compressible fluid the spectrum

$$E(k)\sim c_0^{2/(1-3\gamma)}\bar{\varepsilon}^{2\gamma/(3\gamma-1)}k^{-(5\gamma-1)/(3\gamma-1)}$$

(c_0 is the sound velocity, $\bar{\varepsilon}$ the average rate of energy dissipation, and $\gamma=c_p/c_v$ the adiabatic index). In the limit of an incompressible fluid ($\gamma\to\infty$) the spectrum obtained changes to $E(k)\propto\bar{\varepsilon}^{2/3}k^{-5/3}$.

The action of the similarity group on the characteristic functional of a compressible fluid leads to an effective renormalization of the parameters determining turbulence in the inertial range. As a result of this the statistical characteristics turn out to depend on the renormalized "energy influx velocity" $\varepsilon^*=c_0^{2/(\gamma-1)}\bar{\varepsilon}$,

$$\varepsilon^*=\frac{\eta^*}{2}\left\langle\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2\right\rangle+\xi^*\langle(\operatorname{div}v)^2\rangle,$$

where η^* and ξ^* are the renormalized viscosity coefficients, $\eta^*=c_0^{2/(\gamma-1)}\eta$ and $\xi^*=c_0^{2/(\gamma-1)}\xi$. Apart from the turbulence energy spectra we find in the equilibrium range the spectral characteristics of the two-point semi-invariants of arbitrary order. We must emphasize that the approach developed in this paper may find applications also in other physical problems with a strong interaction.

§1. CHARACTERISTIC FUNCTIONAL OF A COMPRESSIBLE FLUID

We consider the spectral characteristics in a region of k-space where we can neglect the dissipative coefficients and, hence, the change of the entropy with time. (The entropy s of the unperturbed state of the fluid is constant.) For that reason we must expect that when finding the first approximation which does not contain dissipative factors one can use, as was done by Chandrasekhar, [9] a model equation of state

$$P=A_0\rho^\gamma,\quad A_0=\rho_0^{1-\gamma}c_0^2 \tag{1.1}$$

(c_0 is the sound velocity for $\rho=\rho_0$). Moreover, for instance in the case of acoustic turbulence, the change in entropy turns out to be a third-order quantity (see the book by Landau and Lifshitz[10]) and when considering quadratic effects we may assume that $s=\text{const}$. When using the model relation (1.1) the main set of equations is simplified and contains the Navier-Stokes equation and the continuity equation. It is convenient to write it in the following variables: v, $\tilde{\rho}=A_0^{1/(\gamma-1)}\rho$,

$$\begin{gathered}\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\frac{1}{\tilde{\rho}}\nabla\tilde{\rho}^\gamma+\frac{\mu_1}{\tilde{\rho}}\Delta\mathbf{v}+\frac{\mu_2}{\tilde{\rho}}\nabla(\nabla\mathbf{v})+\frac{c_0^{2/(\gamma-1)}}{\tilde{\rho}}\mathbf{F};\\ \partial\tilde{\rho}/\partial t+\nabla(\tilde{\rho}\mathbf{v})=0.\end{gathered} \tag{1.2}$$

Here

$$\mu_1=\left[\frac{\eta}{\rho_0}\right]c_0^{2/(\gamma-1)},\quad \mu_2=\left[\frac{\xi+{}^1/{}_3\eta}{\rho_0}\right]c_0^{2/(\gamma-1)},\quad \langle\mathbf{F}\rangle=0,$$

$\mathbf{F}=\mathbf{f}/\rho_0$ is a random force which excites turbulence and realizes the energy functions of the coarse-grained motions. In a statistical description of turbulence $\mathbf{v}$ and $\tilde{\rho}$ must be assumed to be random fields. We shall describe the statistical properties of the fluid and the random force by a characteristic functional Φ:

$$\Phi=\left\langle \exp i\int\limits_{-\infty}^{+\infty}\left\{(\mathbf{v}\mathbf{y})+(\tilde{\rho}y_\rho)+\int\limits_0^\infty(\mathbf{F}\mathbf{y}_f)\,dt\right\}dx^3\right\rangle, \tag{1.3}$$

$\mathbf{y}(\mathbf{x})$, $y_\rho(\mathbf{x})$, and $\mathbf{y}_f(\mathbf{x}, t)$ are arbitrary fields which decrease sufficiently fast at infinity. The averaging in (1.3) is over the probability for the distribution of the random force. (We assume that the force is "switched on" at time $t=0$.) Apart from Φ we introduce yet another spatial functional φ which describes the statistical properties of only the fluid:

$$\varphi\{[\mathbf{y}(\mathbf{x})],\,[y_\rho(\mathbf{x})];\,t\}=\Phi|_{y_f=0}. \tag{1.4}$$

Using (1.2) we can easily obtain the evolution equation for Φ. For the sake of simplicity we consider the case where γ takes on discrete values: $\gamma=1, 2, 3, \ldots$. We shall show in what follows that the results remain valid also for continuous values $\gamma\geqslant 1$. (In view of the fact that γ does not lead to physical singularities this fact is rather obvious.) Differentiating (1.3) with respect to the time and taking into account the equations of motion we get for Φ a linear equation in the variational derivatives:

$$\frac{\partial\Phi}{\partial t}=\hat{L}\Phi+ic_0^{2/(\gamma-1)}\int\limits_{-\infty}^{+\infty}\mathbf{y}(\mathbf{x})D_\rho^{-1}\mathbf{D}_f\Phi\,dx^3, \tag{1.5}$$

where $\hat{L}$ is a linear operator:

$$\hat{L}\Phi=i\int\limits_{-\infty}^{+\infty}\mathbf{y}(\mathbf{x})\,\{\mathbf{D}\nabla\mathbf{D}-(i)^{1-\gamma}D_\rho^{-1}\nabla D_\rho^{\gamma}+\mu_1 D_\rho\Delta\mathbf{D}$$
$$+\mu_2 D_\rho^{-1}\nabla(\nabla\mathbf{D})\}\,\Phi\,dx^3+i\int\limits_{-\infty}^{+\infty}y_\rho(x)\,\nabla D_\rho\mathbf{D}\Phi\,dx^3. \tag{1.6}$$

Here $\mathbf{D}$ and D_ρ are variational differentiation operators with respect to the variables $\mathbf{y}(\mathbf{x})$, $y_\rho(\mathbf{x})$:

$$\mathbf{D}_f=\frac{\delta}{\delta\mathbf{y}_f\,dx^3\,dt}.$$

To simplify the notation of the equation for Φ we have introduced the operator D_ρ^{-1}:

$$D_\rho^{-1}\Phi=\left\langle\frac{1}{i\rho(\mathbf{x},t)}\exp i\int\limits_{-\infty}^{+\infty}\left\{(\mathbf{v}\mathbf{y})+(\tilde{\rho}y_\rho)+\int\limits_0^\infty(\mathbf{F}\mathbf{y}_f)\,dt\right\}dx^3\right\rangle.$$

We can dispense with it if we use the dynamic equations in the form of the continuity equations for the momentum and mass densities. Below we shall be interested in stationary distributions of $\mathbf{v}$ and ρ which arise in a fluid when oscillations are exicted in it by means of a random force, while the fluid is assumed to be at rest at $t=0$:

$$\mathbf{v}|_{t=0}=0,\qquad s=\text{const.}\quad \rho|_{t=0}=\text{const.}$$

We consider the case of homogeneous and isotropic turbulence. Apparently in such a turbulence the parameters of the initial unperturbed state do not affect the statistical properties of the fine-grained pulsations. The combined characteristic functional of a compressible fluid and the force accomplishes a too detailed statistical description of a system.

As in the present paper we shall be interested only in the turbulence spectra it will be necessary for us to know the characteristic functional of one fluid. The equation for φ follows from (1.5):

$$\partial\varphi/\partial t=L\varphi+I. \tag{1.7}$$

Here I is the source describing the action of the external force on the fluid:

$$I=-c_0^{2/(\gamma-1)}\int\limits_{-\infty}^{+\infty}\mathbf{y}(\mathbf{x})D_\rho^{-1}\left\langle\mathbf{F}\exp i\int\limits_{-\infty}^{+\infty}\{(\mathbf{v}\mathbf{y})+(\tilde{\rho}y_\rho)\}\,dx^3\right\rangle dx^3. \tag{1.8}$$

A closed description of the statistical properties of the fluid in terms of the functional φ is possible only in the case when the source can be expressed in terms of φ.

§2. EXCITATION OF TURBULENCE BY A RANDOM FORCE

We have already mentioned that a stationary turbulence regime is maintained in the model considered due to the work done by the external force. We average the continuity equation for the energy of the fluid in an external force:

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2}+\rho\varepsilon\right)=-\operatorname{div}\left\{\rho\mathbf{v}\left(\frac{v^2}{2}+w\right)\right\}+v_i\frac{\partial\sigma_{ik}}{\partial x_k}+v_if_i, \tag{2.1}$$

where

$$\varepsilon=\int\limits_{\rho_0}\frac{P}{\rho^2}d\rho+\varepsilon_0,\qquad w=\int\limits_{P_0}^{P}\frac{dP}{\rho}+w_0,$$
$$\sigma_{ik}=\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)+\xi\delta_{ik}\frac{\partial v_l}{\partial x_l}.$$

In a stationary homogeneous case we get

$$\langle v_if_i\rangle=\frac{\eta}{2}\left\langle\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2\right\rangle+\xi\left\langle\left(\frac{\partial v_l}{\partial x_l}\right)^2\right\rangle\equiv\bar{\varepsilon}, \tag{2.2}$$

$\bar{\varepsilon}$ is the average rate of energy dissipation of a compressible fluid. We determine the restrictions which arise on the statistical properties of an external perturbation in connection with the presence of the condition (2.2). We write (2.2) in the form

$$-i\left\{D_i\left\langle f_i\exp i\int\limits_{-\infty}^{+\infty}\mathbf{y}\mathbf{v}\,dx^3\right\rangle\right\}\bigg|_{\mathbf{y}=0}=\bar{\varepsilon}. \tag{2.3}$$

For the evaluation of the mean value which occurs here we use the method expounded in[5]. In that case the velocity is expressed identically by the relation as follows:

$$\mathbf{v}(\mathbf{x},t)=\mathbf{v}(\mathbf{x},t-\delta)+\int\limits_{t-\delta}^{t}\frac{\partial\mathbf{v}(\mathbf{x},\tau)}{\partial\tau}d\tau,\quad\delta\geqslant 0. \tag{2.4}$$

We consider the mean value

$$\left\langle f_i \exp i\int\limits_{-\infty}^{+\infty}\left(\mathbf{y}\cdot\frac{\partial\mathbf{v}}{\partial\tau}\right)d\tau\right\rangle$$

$$=\sum_{n=1}\frac{(i)^n}{n!}\int\limits_{t-\delta}\cdot\cdot\int\limits_{t-\delta}\left\langle f_i\left(\mathbf{y}\cdot\frac{\partial v}{\partial\tau}\right)\ldots\left(\mathbf{y}\cdot\frac{\partial\mathbf{v}}{\partial\tau}\right)\right\rangle d\tau_1\ldots d\tau_n. \tag{2.5}$$

$(\mathbf{p}\cdot\mathbf{q})$ is here a scalar product:

$$(\mathbf{p}\cdot\mathbf{q})=\int\limits_{-\infty}^{+\infty}\mathbf{p}(\mathbf{x})\mathbf{q}(\mathbf{x})\,dx^3.$$

The averaging in (2.5) is over the random force under the condition that the value of the exponent $\exp i(\mathbf{y}\cdot\mathbf{v}(t-\delta))$ is fixed. We can easily go over from the average (2.5) to the average occurring in (2.3). To do this we must multiply (2.5) by $\exp i(\mathbf{y}\cdot\mathbf{v}(t-\delta))$, average over the possible values of this exponent, and take the limit as $\delta\to 0$. One sees easily that in Eq. (2.3) we get only a contribution from the first term in the expansion (2.5):

$$i\int\limits_{-\infty}^{+\infty}y_i(\mathbf{x}_1)\int\limits_{t-\delta}^{t}\left\langle f_i(\mathbf{x}_1,t)\frac{\partial v_j}{\partial\tau_1}\right\rangle d\tau_1\,dx_1^3.$$

As a result (2.2) takes the form

$$\lim_{\delta\to 0}\int\limits_0^{\delta}K_{ii}(0,\xi)\,d\xi=\varepsilon; \tag{2.6}$$

here

$$K_{ij}(\mathbf{r},\xi)=\left\langle f_i(\mathbf{x},t)\frac{\partial v_j(\mathbf{x}_1,\tau_1)}{\partial\tau_1}\right\rangle$$
$$\mathbf{r}=\mathbf{x}_1-\mathbf{x},\quad \xi=t-\tau_1.$$

Expressions (2.6) shows that the correlator K_{ij} must contain a δ-function in the time $K_{ij}\sim\delta(\xi)$, i.e., K_{ij} is given by the formula

$$K_{ij}=B_{ij}\delta(\xi),$$

B_{ij} is the spatial part of the correlation tensor. The condition that the energy of the fluid be a stationary leads thus to the following restriction on the pair correlator B_{ij} of the force and the fluid:

$$\tfrac{1}{2}B_{ii}(0)=\varepsilon. \tag{2.7}$$

The condition (2.7) is valid both for an external force of the form $\mathbf{f}(\mathbf{x},t)$ and also for a force which may "adjust itself" to the state of the fluid, being a functional $\mathbf{f}(\mathbf{x},t;[\mathbf{v}],[\rho])$. It would appear that a force of the form $\mathbf{f}(\mathbf{x},t)$ is more natural, as in the case of an incompressible fluid. However, we shall show below that such a force in a compressible fluid leads to a non-local energy pumping which may lead to an appreciable perturbation of the statistical properties of the fluid. Since the role of the force can be reduced to energy pumping it is natural to consider such forces which are completely determined by condition (2.7) and lead to a local influx of energy into the fluid near small k. The simplest case of such a force, which allows a description of the state of the fluid in terms of the functional φ, is

$$\mathbf{f}(\mathbf{x},t,[\rho])=\rho(\mathbf{x},t)\mathbf{g}(\mathbf{x},t),$$

where $\mathbf{g}(\mathbf{x},t)$ has a Gaussian probability distribution law and is δ-correlated in the time

$$\langle g_i(\mathbf{x},t)g_j(\mathbf{x}_1,t_1)\rangle=B_{ij}(\mathbf{x}_1-\mathbf{x})\delta(t-t_1).$$

Using the Furutsa–Novikov formula[2]

$$\langle R[\mathbf{g}]\mathbf{g}(s)\rangle=\int\langle\mathbf{g}(s)\mathbf{g}(s')\rangle\left\langle\frac{\delta R[g]}{\delta\mathbf{g}(s')ds'}\right\rangle ds' \tag{2.8}$$

($R[\mathbf{g}]$ is a functional of $\mathbf{g}$ and s' a complete set of independent variables) and the formula

$$\delta v_i(\mathbf{x},t)/\delta g_k(\mathbf{x}_1,t)=\tfrac{1}{2}\delta_{ik}\delta(\mathbf{x}-\mathbf{x}_1),$$

which follows from the Navier–Stokes equation we get

$$\langle\rho\rangle B_{ii}(0)=\varepsilon.$$

If the turbulence is isotropic the tensor B_{ij} can be expressed solely in terms of the longitudinal B_{LL} and transverse B_{NN} correlation functions:

$$B_{ij}=(B_{LL}-B_{NN})\frac{r_ir_j}{r^2}+B_{NN}\delta_{ij},\quad B_{LL}(0)=B_{NN}(0).$$

From (2.8) we get

$$B_{LL}(0)=B_{NN}(0)=2\varepsilon^*/3\langle\bar\rho\rangle. \tag{2.9}$$

Formula (2.9) shows that one can write the longitudinal and transverse functions in the form

$$B_{LL}(r)=\frac{2}{3}\frac{\varepsilon^*}{\langle\bar\rho\rangle}\psi_{LL}\left(\frac{r}{L_L}\right),$$
$$B_{NN}(r)=\frac{2}{3}\frac{\varepsilon^*}{\langle\bar\rho\rangle}\psi_{NN}\left(\frac{r}{L_N}\right). \tag{2.10}$$

Here ψ_{LL} and ψ_{NN} are dimensionless functions while

$$\psi_{LL}(0)=\psi_{NN}(0)=1,$$

L_L and L_N are the corresponding characteristic length scales.

We shall in what follows consider the case where the scales L_L and L_N are of the same order. We introduce an over-all length scale for the turbulence:

$$L=\min(L_L,L_N). \tag{2.11}$$

In the limit as $L\to\infty$

$$B_{LL}=B_{NN}\to 2\varepsilon^*/3\langle\bar\rho\rangle. \tag{2.12}$$

If $\mathbf{g}$ is Gaussian the condition that the energy density in the fluid is stationary allows us thus to determine the form of the pair correlation tensor B_{ij}. Using Eq. (2.8) we get an equation for the functional φ in closed form with a source of the form

$$I\{\varphi\}=-\frac{1}{2}\iint_{-\infty}^{+\infty} y_i(\mathbf{x})\,y_j(\mathbf{x}_1)\,B_{ij}(\mathbf{x}-\mathbf{x}_1)\,dx^3\,dx_1^3\,\varphi, \qquad (2.13)$$

the term $I\{\varphi\}$ in (1.7) can be interpreted as being responsible for "diffusion in velocity space." In the limiting case as $L\to\infty$ the diffusion coefficient is constant $D\sim\bar{\varepsilon}$, which corresponds to the energy pumping in k-space being local.

We now use for the energy pumping a Gaussian force $\mathbf{f}(\mathbf{x}, t)$ with a correlator

$$\langle f_i(\mathbf{x}, t)\,f_j(\mathbf{x}_1, t_1)\rangle=B_{ij}(\mathbf{x}-\mathbf{x}_1)\,\delta(t-t_1).$$

In that case (2.2) leads to the condition

$$\frac{1}{2}B_{ii}(0)\left\langle\frac{1}{\rho}\right\rangle=\varepsilon.$$

For the longitudinal and transverse correlation functions we get

$$B_{LL}(0)=B_{NN}(0)=\frac{2}{3}\varepsilon\left\langle\frac{1}{\rho}\right\rangle^{-1}.$$

The fact that the force is Gaussian and δ-correlated again enables us to close the equation for the characteristic functional of the fluid.

Applying Eq. (2.8) we get

$$I\{\varphi\}=\frac{c_0^{4/(\gamma-1)}}{2\rho_0^2}\iint_{-\infty}^{+\infty} B_{ij}(\mathbf{x}-\mathbf{x}_1)\,\mathbf{y}(\mathbf{x})\,\mathbf{y}(\mathbf{x}_1)\,D_\rho^{-1}D_{\rho'}^{-1}\varphi\,dx^3\,dx_1^3. \qquad (2.14)$$

The Gaussian nature of the random accelerations and of the external force leads to the diffusion approximation, but (2.14) show that in the last case the diffusion coefficient is not constant as $L\to\infty$

$$D\sim\left\langle\frac{1}{\rho\rho'}\right\rangle\varepsilon. \qquad (2.15)$$

In agreement with (2.15) the energy influx into any turbulence scale turns out to depend on the density correlations at those scales and this leads to a non-local energy pumping. The "diffusion in velocity space" due to the energy pumping can most simply be interpreted in the case of small density pulsations, if $\delta\rho/\rho\ll 1$. In that case density perturbations propagate in the form of sawtooth waves. As to order of magnitude we get from (2.15) for the diffusion coefficient

$$D\sim\text{const}\left(1+\left\langle\frac{\delta\rho}{\rho_0}\frac{\delta\rho'}{\rho_0}\right\rangle\right). \qquad (2.16)$$

Equation (2.16) shows that the diffusion coefficient is larger wherever the density pulsations are more strongly correlated. Since sawtooth waves are regions of strong correlations between harmonics, energy pumping in the crest proceeds faster than in the region where the correlations are small. Since the external force is not a small perturbation and has zero time correlations, in the process of the relaxation of the turbulence to the stationary state the correlations in sawtooth waves are destroyed. This method of excitation is more complex and will be considered in what follows.

§3. SPECTRAL CHARACTERISTICS OF THE TURBULENCE

The characteristic functional for the excitation of turbulence by random Gaussian accelerations satisfies Eq. (1.7) with the source (2.13) and an initial condition corresponding to a state of rest of the fluid at $t=0$, $\tilde{\rho}\,|_{t=0}=\langle\tilde{\rho}\rangle=\tilde{\rho}_0$:

$$\varphi|_{t=0}=\exp i\tilde{\rho}_0\int_{-\infty}^{+\infty} y_\rho(\mathbf{x})\,dx^3. \qquad (3.1)$$

In an incompressible fluid we would have instead of (3.1) simply

$$\varphi|_{t=0}=1. \qquad (3.2)$$

To find the spectral characteristics of the turbulence the invariance properties of the Cauchy problem for the functional φ are essential. We consider the following group of transformations:

$$\begin{gathered}\alpha\mathbf{x}=\mathbf{x}',\quad \alpha^{1-\beta}t=t',\\ \alpha^{-(\beta+3)}\mathbf{y}(\mathbf{x})=\mathbf{y}'(\mathbf{x}'),\quad \alpha^{-\beta/(\gamma-1)-3}y_\rho(\mathbf{x})=y_\rho'(\mathbf{x}'),\\ \alpha^{1+\beta+2\beta/(\gamma-1)}\mu_{1,2}=\mu_{1,2}',\quad \alpha^{2\beta/(\gamma-1)}\tilde{\rho}_0=\tilde{\rho}_0',\\ \alpha L_L=L_L',\quad \alpha L_N=L_N',\\ \alpha^{3\beta-1+2\beta/(\gamma-1)}\varepsilon^*=\varepsilon^{*\prime}\quad (\varepsilon^*=c_0^{2/(\gamma-1)}\,\varepsilon),\end{gathered} \qquad (3.3)$$

α, β are arbitrary parameters ($0<\alpha<\infty$, $\beta<\infty$).

Equation (1.7) with the source (2.13) and the initial condition (3.1) are invariant under the transformation (3.3). Hence follows the relation[1)]

$$\begin{aligned}&\varphi([\mathbf{y}(\mathbf{x})];\,[y_\rho(\mathbf{x})];\,\mu_1,\mu_2,\varepsilon^*,\tilde{\rho}_0,L_L,L_N,t)\\ &=\varphi([\mathbf{y}'(\mathbf{x}')];\,[y_\rho'(\mathbf{x}')];\,\mu_1',\mu_2',\varepsilon^{*\prime},\tilde{\rho}_0',L_L',L_N',t').\end{aligned} \qquad (3.4)$$

We note that in this case the invariance property (3.4) is not a form of the similarity hypothesis but is an exact relation. In the case of an incompressible fluid the characteristic functional satisfies a generalized Hopf equation[1]

$$\begin{aligned}\frac{\partial\varphi}{\partial t}&=i\left(\mathbf{y}\cdot\frac{\partial}{\partial x_\alpha}\mathbf{D}D_\alpha\varphi\right)+\nu(\mathbf{y}\cdot\Delta\mathbf{D}\varphi)\\ &-\frac{1}{2}\iint_{-\infty}^{+\infty} y_i(\mathbf{x})\,y_j(\mathbf{x}_1)\,B_{ij}(\mathbf{x}-\mathbf{x}_1)\,dx^3\,dx_1^3\,\varphi,\end{aligned} \qquad (3.5)$$

where B_{ij} is the spatial part of the correlation tensor of the Gaussian external force. Equation (3.5) and the initial condition (3.2) are invariant under the group

$$\begin{gathered}\alpha\mathbf{x}=\mathbf{x}',\quad \alpha^{1-\beta}t=t',\quad \alpha^{-(\beta+3)}\mathbf{y}(\mathbf{x})=\mathbf{y}'(\mathbf{x}'),\\ \alpha L=L',\quad \alpha^{1+\beta}\nu=\nu',\quad \alpha^{3\beta-1}\varepsilon=\varepsilon',\end{gathered} \qquad (3.6)$$

where L is the external scale length of the turbulence and $\bar{\varepsilon}$ the average rate of energy dissipation in the incompressible fluid. It follows from the above-mentioned invariance that

$$\varphi([\mathbf{y}(\mathbf{x})],\nu,\varepsilon,L,t)=\varphi([\mathbf{y}'(\mathbf{x}')],\nu',\varepsilon',L',t'). \qquad (3.7)$$

Equations (3.7) and (3.4) enable us to determine the spectral characteristics of the turbulence in the inertial range.

We consider firstly the turbulence spectrum in an incompressible fluid:

$$E(k)=-\frac{1}{(2\pi)^3}\iint\limits_{|\mathbf{k}|=k}\int\limits_{-\infty}^{+\infty}e^{-ik\mathbf{hr}}\{D_iD_i'\varphi\}|_{\gamma=0}\,dr^3\,dS(\mathbf{k}), \tag{3.8}$$

$dS(\mathbf{k})$ is an element of area of the surface of the sphere $|\mathbf{k}|=k$, while $h=k/|\mathbf{k}|$. Using Eq. (3.7) we go over in (3.8) to a functional depending on the dashed variables. Putting $\alpha\equiv k$ and changing in (3.8) to an integration over $k\mathbf{r}$, using the fact that the integration limits are infinite, we get

$$E(k,t)=\frac{f(k^{1-\beta}t,\ k^{1+\beta}\nu,\ k^{3\beta-1}\varepsilon,\ kL)}{k^{2\beta+1}}. \tag{3.9}$$

In the stationary case $\partial E/\partial t=0$. From the fact that β is arbitrary it follows that $\partial E/\partial\beta=0$. As a result we get the equation

$$\xi_1\frac{\partial f}{\partial\xi_1}+3\xi_2\frac{\partial f}{\partial\xi_2}=2f,$$
$$\xi_1=k^{1+\beta}\nu,\quad \xi_2=k^{3\beta-1}\varepsilon,$$

the solution of which we can write in the form

$$f=\xi_2^{2/3}f_1(\xi_1/\xi_2^{1/3})$$

(f_1 is a dimensionless function). As a result the spectral density turns out to be the following:

$$E(k)=\varepsilon^{2/3}k^{-5/3}f(kL,\ kl),$$

where $l=\nu^{3/4}\bar{\varepsilon}^{-1/4}$ is the Kolmogorov scale length. Assuming that the inertial range exists, i.e., that we can have an asymptotic expansion of $f(kL, kl)$ with

$$kl\ll 1,\quad (kL)^{-1}\ll 1,$$

we get to a first approximation the Kolmogorov–Obukhov law

$$E(k)\sim\varepsilon^{2/3}k^{-5/3}. \tag{3.10}$$

We note that the requirement that the turbulence is isotropic is not necessary for obtaining the spectrum (3.10). Similarly the similarity theory (3.4) can be applied to evaluate the spectral characteristics of developed turbulence in a compressible fluid when there is an equilibrium range in the spectrum. We start with the spectrum of the kinetic energy of the turbulence. We consider the correlator

$$K=\langle\tilde{\rho}(\mathbf{x})\mathbf{v}(\mathbf{x})\mathbf{v}(\mathbf{x}_1)\rangle.$$

Its spectral density, integrated over the angles, equals

$$E(k)=\frac{1}{(2\pi)^3}\iint\limits_{|\mathbf{k}|=k}\int\limits_{-\infty}^{+\infty}e^{-ik\mathbf{hr}}K(\mathbf{r})\,dr^3\,dS(\mathbf{k}),\quad \mathbf{r}=\mathbf{x}_1-\mathbf{x}.$$

In that case

$$\left\langle\frac{\rho v^2}{2}\right\rangle=\frac{1}{2}\rho_0c_0^{-2/(\gamma-1)}K(0),\quad K(0)=\int\limits_0^\infty E(k)\,dk,$$

i.e., $E(k)$ differs only by a factor from the spectral density of the kinetic energy of the fluid. We express $E(k)$ in terms of φ:

$$E(k)=\frac{i}{(2\pi)^3}\iint\limits_{|\mathbf{k}|=k}\int\limits_{-\infty}^{+\infty}e^{-ik\mathbf{hr}}\{D_\rho\cdot\mathbf{DD}\}|_{\gamma=\nu_\rho=0}\,dr^3\,dS(\mathbf{k}).$$

Using (3.4) as when we obtained the spectrum of an incompressible fluid, we get

$$E(k,t)=f(k^{1-\beta}t,\ k^{1+\beta+2\beta/(\gamma-1)}\mu_1,\ k^{1+\beta+2\beta/(\gamma-1)}\mu_2,\ k^{3\beta-1+2\beta/(\gamma-1)}\varepsilon.$$
$$k^{\beta/(\gamma-1)}\tilde{\rho}_0,\ kL_L,\ kL_N)\,k^{-1-2\beta\gamma/(\gamma-1)}.$$

It is natural to assume that for sufficiently fine-grained pulsations there exists a range of statistical equilibrium in which the probability distribution for turbulent pulsations is independent of the viscosity and the initial characteristics of the flow. The above-mentioned assumption means that in the stationary case the spectral density for the fine-grained pulsations from the equilibrium range has the form

$$E(k)=\frac{f_1(k^{3\beta-1+2\beta/(\gamma-1)}\varepsilon)}{k^{1+2\beta\gamma/(\gamma-1)}},$$

as $\partial E/\partial\beta=0$,

$$f_1(k^{3\beta-1+2\beta/(\gamma-1)}\varepsilon)=(k^{3\beta-1+2\beta/(\gamma-1)}\varepsilon)^{2\gamma/(3\gamma-1)},$$

i.e., the spectrum has in that range the form

$$E(k)=\mathrm{const}\,\rho_0c_0^{2/(1-3\gamma)}\varepsilon^{2\gamma/(3\gamma-1)}k^{-(5\gamma-1)/(3\gamma-1)}. \tag{3.11}$$

In the limit of an incompressible fluid ($\gamma\to\infty$) the spectrum (3.11) changes to the Kolmogorov–Obukhov law. We note that for arbitrary γ the spectra $E(k)$ lie in a rather narrow range between k^{-2} and $k^{-5/3}$.

In analogy with the spectrum of the kinetic energy we can obtain also other spectral characteristics of the turbulence. For instance, the spectrum of the pressure pulsations

$$\langle P(\mathbf{x})P(\mathbf{x}')\rangle-\langle P\rangle^2=\int\limits_0^\infty E^P(k)\,dk$$

has in the equilibrium range the form

$$E^P(k)\sim\rho_0^2c_0^{-4/(3\gamma-1)}\varepsilon^{4\gamma/(3\gamma-1)}k^{-(7\gamma-1)/(3\gamma-1)}, \tag{3.12}$$

or the expression for the correlator, equivalent to (3.12):

$$\langle P(\mathbf{x})P(\mathbf{x}')\rangle-\langle P\rangle^2\sim\rho_0^2c_0^{-4/(3\gamma-1)}\varepsilon^{4\gamma/(3\gamma-1)}r^{4\gamma/(3\gamma-1)}.$$

It is clear that the corresponding proportionality constants are functions of γ. We give also the expression for the density correlator and the vorticity correlator in the equilibrium range:

$$\langle\rho(\mathbf{x})\rho(\mathbf{x}')\rangle-\langle\rho\rangle^2\sim\rho_0^2c_0^{-12/(3\gamma-1)}\varepsilon^{4/(3\gamma-1)}r^{4/(3\gamma-1)}.$$

The longitudinal and transverse vorticity correlation functions B^{ω}_{LL} and B^{ω}_{NN} have the form

$$B_{LL}{}^{\omega}\sim B_{NN}{}^{\omega}\sim c_0^{4/(3\gamma-1)}\varepsilon^{2(\gamma-1)/(3\gamma-1)}r^{-4\gamma/(3\gamma-1)}.$$

One can also easily get the general form for the two-point semi-invariants of arbitrary order when one assumes that they equal neither zero nor infinity, $S_{i_1\ldots i_m}\times(\mathbf{x}-\mathbf{x}_1)=S_{i_1\ldots i_m}(v_{i_1}\ldots v_{i_m}\rho_1\ldots\rho_n)$:

$$S_{i_1\ldots i_m}(r)=(-i)^{m+n}\rho_0{}^n c_0^{-2n/(\gamma-1)}D_{i_1}\ldots D_{i_m}D_{\rho_1}\ldots D_{\rho_n}\ln\varphi|_{y=y_\rho=0}.$$

The semi-invariant corresponds to the spectral tensor

$$S_{i_1\ldots i_m}(k)=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}e^{-i\mathbf{kr}}S_{i_1\ldots i_m}(r)\,dr^3,$$

which in the equilibrium range equals

$$S_{i_1\ldots i_m}(\mathbf{k})=\mathrm{const}\cdot\rho_0{}^n c_0^{2(m-3n)/(3\gamma-1)}\varepsilon^{(m\gamma-m+2n)/(3\gamma-1)}\times|\mathbf{k}|^{-[\gamma(m+9)-m+2n-3]/(3\gamma-1)}\hat{S}_{i_1\ldots i_m}(\mathbf{h}), \qquad (3.13)$$

$\hat{S}_{i_1\ldots i_m}(\mathrm{h})$ is the angular part of the spectral tensor. Equations (3.13) show that in the equilibrium range the statistical characteristics of the turbulence are functions of k and $\bar{\varepsilon}^*$, i.e., they are determined by the total "energy flux"

$$\varepsilon^*=\frac{\eta^*}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2+\xi^*(\operatorname{div}\mathbf{v})^2,$$

where η^* and ξ^* are the effective viscosity coefficients.

We consider the problem of the neglect of the damping in the equilibrium range. When the dissipative coefficients η, ξ, and $\varkappa$ are present one can form three quantitites with the dimensions of length:

$$l_1=\frac{(\eta/\rho_0)^{(3\gamma-1)/4\gamma}c_0^{1/\gamma}}{\varepsilon^{(\gamma+1)/4\gamma}},\quad l_2=\frac{[(\xi+{}^1/_3\eta)/\rho_0]^{(3\gamma-1)/4\gamma}c_0^{1/\gamma}}{\varepsilon^{(\gamma+1)/4\gamma}},\quad l_3=\frac{(\varkappa/\rho_0c_p)^{(3\gamma-1)/4\gamma}c_0^{1/\gamma}}{\varepsilon^{(\gamma+1)/4\gamma}} \qquad (3.14)$$

In the limit of an incompressible fluid, the length scales (3.14) go over into the Kolmogorov ones:

$$\left(\frac{\eta}{\rho_0}\right)^{3/4}\varepsilon^{-1/4},\quad\left(\frac{\xi+{}^1/_3\eta}{\rho_0}\right)^{3/4}\varepsilon^{-1/4},\quad\left(\frac{\varkappa}{\rho_0c_p}\right)^{3/4}\varepsilon^{-1/4}.$$

Here $\varkappa$ is the thermal conductivity coefficient, and c_p the specific heat at constant pressure. From the "renormalized" dissipative coefficients μ_1, μ_2 and the "renormalized" dissipation rate $\bar{\varepsilon}^*$ we can form characteristic quantities with the dimensions of velocity and density:

$$v_\nu=(\varepsilon^*\mu)^{(\gamma-1)/4\gamma}=c_0^{1/\gamma}\varepsilon^{(\gamma-1)/4\gamma}\nu^{(\gamma-1)/4\gamma},$$

$$\rho_\nu=(\varepsilon^*\mu)^{1/2\gamma}\rho_0c_0^{-2/(\gamma-1)}=\rho_0c_0^{-2/\gamma}(\varepsilon\nu)^{1/2\gamma};$$

as $\gamma\to\infty$

$$v_\nu\to(\nu\varepsilon)^{1/4},\qquad\rho_\nu\to\rho_0;$$

the Reynolds number, formed from the quantities v_ν, l_ν, ρ_ν, μ,

$$\mathrm{Re}=v_\nu l_\nu\rho_\nu/\mu=1,$$

i.e., pulsations with characteristic $v_l\sim v_\nu$, $\rho_l\sim\rho_\nu$, $l\sim l_\nu$ are effectively damped. This means that the neglect of dissipative factors is legitimate for pulsations with

$$v\gg v_\nu,\qquad\rho\gg\rho_\nu. \qquad (3.15)$$

As

$$v_l\sim(\varepsilon^*k^{-1})^{(\gamma-1)/(3\gamma-1)},\qquad\rho_l\sim\rho_0c_0^{-2/(\gamma-1)}(\varepsilon^*k^{-1})^{2/(3\gamma-1)}$$

condition (3.15) is equivalent to

$$k\ll\frac{1}{l_1}\ \frac{1}{l_2}\ \frac{1}{l_3}\ .$$

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