Dipole forces in two-dimensional and layered ferromagnets

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The role of long-range dipole forces in two-dimensional and quasi-two-dimensional (layered) ferromagnetic substances is considered. It is shown that in a two-dimensional ferromagnet the dipole forces stabilize the ferromagnetism, for in the presence of these forces, a term linear in the momentum appears in the Hamiltonian. This result is not inconsistent with the absence of long-range order in two-dimensional degenerate systems, since it is based on the assumption of a finite interaction range. In layered three-dimensional structures the dipole forces of all other planes acting on a selected crystal plane completely destroys (because of the long-range action) the stabilizing effect due to the dipole forces in the selected plane. As a result, long-range ferromagnetic order arises in the plane, as in the absence of dipole forces, only as a consequence of exchange between planes; dipole forces contribute a short-range correction which is related to the discreteness of the lattice and is antiferromagnetic for a tetragonal lattice. The spin-wave spectrum and temperture part of the magnetization of the plane are calculated for the case of anisotropy of the easy-plane type in the presence of dipole forces. The calculations are carried out for both ferromagnetic and antiferromagnetic ordering of the planes. The dipole interaction tensor is analyzed in the last section in greater detail than hitherto and a formula for the magnetic anisotropy energy is derived in the form of a rapidly converging sum.

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1. INTRODUCTION

It is well known that no long-range order can exist in two-dimensional degenerate systems at finite temperatures (see the books of Landau and Lifshitz[1] and of Patashinskiĭ and Pokrovskiĭ[2]). This means, in particular, that two-dimensional ferromagnetism is impossible in the case of isotropic exchange interaction. In fact, there is no two-dimensional long-range order because in the expression for the free energy the principal term that describes the spatial inhomogeneities is the square of the gradient of the order parameter. As a result, in two-dimensional space the mean squared fluctuation of this parameter is proportional to a logarithmically diverging integral and is therefore not small.

However, if the system contains unscreened long-range forces, then the situation may turn out to be essentially different. Indeed, if we trace, starting from the microscopic Hamiltonian, the appearance of the terms proportional to the square of the gradient (see, e.g., the book by Akhiezer, Bar'yakhtar, and Peletminskiĭ[3]), then it turns out that they are proportional to the integral $\int d^d\mathbf{r} r^2 U(\mathbf{r})$, where $U(\mathbf{r})$ is the initial energy of the interaction and d is the dimension of space. If this integral diverges, then the entire conclusion turns out to be incorrect and it is necessary to consider more accurately the role of the long-range part of $U(\mathbf{r})$. The dipole forces that are inevitably present in any real ferromagnet decrease like r^{-3}, and therefore do not satisfy the condition that this integral be finite, neither in the two-dimensional nor in the three-dimensional case.[1)] Therefore the entire investigation of the existence of long-range order in the presence of dipole forces must be carried out again. This is the subject of the present article. In Sec. 2 it is shown that the expression for the dipole energy of a two-dimensional ferromagnet contains a non-analytic term that is proportional to the first degree of the momentum and leads to stabilization of the long-range order. In other words, the dipole forces, owing to the long-range action, stabilize the two-dimensional ferromagnetism.[2)]

In Sec. 3 we consider dipole forces in real three-dimensional layered structures. It turns out that, owing to the long-range order, the action on a selected crystal plane by the dipole forces from all other planes, destroys completely the stabilizing effect of the dipole forces acting in the selected plane. As a result, the stabilization of the long-range ferromagnetic order in the plane is due only to the exchange interaction between the planes, to which is added a small short-range dipole increment due to the discrete character of the crystal lattice. For a tetragonal lattice, this increment is negative, so that if there is no exchange interaction between the planes the dipole forces stabilize the long-range order in the case of antiferromagnetic ordering of the planes. It appears, however, that in all real cases the exchange interaction between the planes is much larger than that part of the dipole forces. We calculate incidentally the low-temperature asymptotic value of the magnetization. The obtained formulas are a generalization of the corresponding results of Berezinskiĭ and Blank[4] to the case when both anisotropy and and dipole forces are present in the system.

In Sec. 4 of the paper, which is by way of a mathematical appendix, the tensor dipole forces are analyzed in greater detail than in the past, and, in particular, an expression is derived for the magnetic-anisotropy energy in the form of rapidly converging series.

2. TWO-DIMENSIONAL FERROMAGNET

We write down the Hamiltonian of the system in the usual manner:

$$H=-\frac{1}{2}\sum_{l\neq l'}(V_{ll'}\delta_{\alpha\beta}+Q_{ll'}^{\alpha\beta})S_l^{\alpha}S_{l'}^{\beta},$$
$$Q_{ll'}^{\alpha\beta}=(g\mu)^2(3R_{ll'}^{\alpha}R_{ll'}^{\beta}-\delta_{\alpha\beta}R_{ll'}^{2})R_{ll'}^{-5}, \quad (1)$$

where $\mathbf{S}_l$ is the spin of the atom at site l, $V_{ll'}$ is the exchange integral, and $Q_{ll'}$ is the dipole-interaction tensor. We assume that the exchange interaction is positive and is large in comparison with the characteristic dipole energy $(g\mu)^2a^{-3}$, where a is the lattice constant. We assume also that the atoms are located in the sites of a planar lattice, whose symmetry does not admit of uniaxial anisotropy in the plane.

Changing over in the usual manner to Fourier components, we obtain

$$H=-\frac{1}{2}\sum_{\mathbf{k}}(V_{\mathbf{k}}\delta_{\alpha\beta}+Q_{\mathbf{k}}^{\alpha\beta})S_{\mathbf{k}}^{\alpha}S_{-\mathbf{k}}^{\beta},$$
$$\mathbf{S}_{\mathbf{k}}=N_2^{-1/2}\sum_{\mathbf{R}}e^{-i\mathbf{k}\mathbf{R}}\mathbf{S}_{\mathbf{R}}, \tag{2}$$

where R and k are respectively the two-dimensional lattice vector and the two-dimensional momentum (both lie in the lattice plane, henceforth taken to be the xy plane), and N_2 is the number of lattice points.

We need to know $Q_{\mathbf{k}}^{\alpha\beta}$ at small k (the corresponding formulas for arbitrary k are given in Sec. 4). We write down $Q_{\mathbf{k}}^{\alpha\beta}$ in the form

$$Q_{\mathbf{k}}^{\alpha\beta}=Q_0^{\alpha\beta}+Q_{1\mathbf{k}}^{\alpha\beta},$$
$$Q_0^{\alpha\beta}=(g\mu)^2\sum_{\mathbf{R}\neq 0}(3R_\alpha R_\beta-\delta_{\alpha\beta}R^2)R^{-5}, \tag{3}$$
$$Q_{1\mathbf{k}}^{\alpha\beta}=(g\mu)^2\sum_{\mathbf{R}\neq 0}(e^{i\mathbf{k}\mathbf{R}}-1)(3R_\alpha R_\beta-\delta_{\alpha\beta}R^2)R^{-5}.$$

$Q_0^{\alpha\beta}$ is a symmetrical tensor of second rank with zero trace, for the construction of which two quantities are available, $\delta_{\alpha\beta}$ and $z_\alpha z_\beta$, where z is a unit vector normal to the plane. Therefore

$$Q_0^{\alpha\beta}=A^{(0)}(1/3\delta_{\alpha\beta}-z_\alpha z_\beta), \quad A^{(0)}=3/2(g\mu)^2\sum_{\mathbf{R}\neq 0}R^{-3}. \tag{4}$$

In the expression for $Q_{1\mathbf{k}}$, the restriction $\mathbf{R}\neq 0$ is inessential, at $ka\ll 1$ the sum can be replaced by an integral, and simple calculations yield

$$Q_{1\mathbf{k}}^{\alpha\beta}=\frac{2\pi(g\mu)^2}{v_2}k\left(z_\alpha z_\beta-\frac{k_\alpha k_\beta}{k^2}\right), \tag{5}$$

where $\mathbf{k}=(k_x, k_y, 0)$ and v_2 is the "volume" of the planar unit cell. This is in fact the interaction term that is linear in the momentum and stabilizes the paramagnetic order.

Since $A^{(0)}>0$, the spins should lie in the lattice plane in the ground state. We assume the presence of ferromagnetic order and take the spontaneous-moment direction to be the x axis. In the lowest order in the value of the spin deviation we then have the standard formulas[3]

$$S_{\mathbf{k}}^{z}=-SN_2^{1/2}\delta_{\mathbf{k},0}+N_2^{-1/2}\sum_{\mathbf{k}_1}a_{\mathbf{k}_1+\mathbf{k}}^{+}a_{\mathbf{k}_1},$$
$$S_{\mathbf{k}}^{y}=(S/2)^{1/2}(a_{\mathbf{k}}^{+}+a_{-\mathbf{k}}), \quad S_{\mathbf{k}}^{z}=-i(S/2)^{1/2}(a_{\mathbf{k}}^{+}-a_{-\mathbf{k}}), \tag{6}$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{+}$ are the usual Holstein-Primakoff operators, and that part of the Hamiltonian (2) which is bilinear in $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{+}$ takes the form

$$H=\sum_{\mathbf{k}}[\xi_{\mathbf{k}}a_{\mathbf{k}}^{+}a_{\mathbf{k}}+1/2\,B_{\mathbf{k}}(a_{\mathbf{k}}a_{-\mathbf{k}}+a_{\mathbf{k}}^{+}a_{-\mathbf{k}}^{+})];$$
$$\xi_{\mathbf{k}}=Dk^2+1/2\Omega_0[\alpha-(ka)k_x^2k^{-2}],$$
$$B_{\mathbf{k}}=-1/2\Omega_0[\alpha-ka(1+k_y^2k^{-2})], \tag{7}$$

where $Dk^2=S(V_0-V_{\mathbf{k}})$ at $ka\ll 1$; $\Omega_0=2\pi S(g\mu)^2(v_2a)^{-1}\ll SV_0$ and $\alpha=SA^{(0)}\Omega_0^{-1}\sim 1$. As a result we obtain in the usual manner[3] for the retarded Green's function of the operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{+}$

$$G(\mathbf{k},\omega)=(\omega+\xi_{\mathbf{k}})(\omega^2-\varepsilon_{\mathbf{k}}^2+i\delta)^{-1},$$
$$\varepsilon_{\mathbf{k}}=[(Dk^2+\Omega_0\alpha)(Dk^2+\Omega_0ka\sin^2\varphi_{\mathbf{k}})]^{1/2}, \tag{8}$$

where $\varphi_{\mathbf{k}}$ is the polar angle of the vector $\mathbf{k}$ and is reckoned from the x axis.

We note that at sufficiently small k the energy of the spin wave has the unusual form $\varepsilon_{\mathbf{k}}\approx\Omega_0|\sin\varphi_{\mathbf{k}}|(\alpha ka)^{1/2}$, i.e., it is proportional to $k^{1/2}$.

By virtue of (6) and (8) we obtain for the relative deviation of the magnetization from the maximum value

$$\frac{\delta S}{S}=\frac{1}{N_2S}\sum_{\mathbf{k}}\langle a_{\mathbf{k}}^{+}a_{\mathbf{k}}\rangle=-\frac{v_2}{(2\pi)^2S}\int d^2\mathbf{k}\,d\omega\,\mathrm{Im}\,G(\mathbf{k},\omega)n(\omega)$$
$$=\frac{v_2}{(2\pi)^2S}\int d^2\mathbf{k}\left[\frac{\xi_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}}\left(\exp\left\{\frac{\varepsilon_{\mathbf{k}}}{T}\right\}-1\right)^{-1}+\frac{\xi_{\mathbf{k}}-\varepsilon_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}}\right]. \tag{9}$$

In the case of pure exchange interaction we have $\xi_{\mathbf{k}}=\varepsilon_{\mathbf{k}}\sim k^2$ and this expression diverges logarithmically at small k, thus indicating the absence of long range order. We shall show now that the dipole forces cut off this divergence and that the quantity $\delta S/S$ is small in a rather wide range of temperatures.

The second term in the right-hand side of (9) is the energy of the zero-point oscillations. Under the assumptions made above concerning the parameters of the problem ($SV_0\gg\Omega_0$) the corresponding integral can be easily evaluated and we obtain

$$\left(\frac{\delta S}{S}\right)_0=\frac{v_2\alpha\Omega_0}{16\pi DS}\sim\frac{\Omega_0}{S^2V_0}\ll 1. \tag{10}$$

From this formula it follows, in particular, that the zero-point oscillations are not small if the dipole forces become equalized with the exchange forces; but our theory does not hold in this region.

For the temperature contribution to $\delta S/D$ we have by virtue of (7) and (9)

$$\left(\frac{\delta S}{S}\right)_T=\frac{v_2}{2(2\pi)^2S}\int\frac{d^2\mathbf{k}[2Dk^2+\Omega_0(\alpha-ka\cos^2\varphi)]}{[(Dk^2+\alpha\Omega_0)(Dk^2+\Omega_0ka\sin^2\varphi)]^{1/2}}$$
$$\times[\exp\{T^{-1}[(Dk^2+\Omega_0\alpha)(Dk^2+\Omega_0ka\sin^2\varphi)]^{1/2}\}-1]^{-1}. \tag{11}$$

If we neglect in this equation the interaction-induced term $\Omega_0ka\sin^2\varphi$ in comparison with Dk^2, then we obtain a logarithmically diverging integral. The cause of this divergence is that in the analysis of the dipole forces we took into account only the fact that they lift partially the degeneracy (introduction of the easy-magnetization plane xy). Allowance for the terms that are linear in k and are connected with the long-range action makes the integral in (11) convergent. This integral must be cal-

culated differently in three temperature regions:

1) $T \gg \Omega_0\alpha$; 2) $\Omega_0\alpha \gg T \gg \Omega_0\alpha(\Omega_0a^2/D\alpha)^{1/2}$;
3) $\Omega_0\alpha(\Omega_0a^2/D\alpha)^{1/2} \gg T$.

In the first two cases the k-space must be broken up into two partially overlapping regions, in one of which the terms with $\sin^2\varphi$ can be neglected, and in the second the argument of the exponential is small. The resultant integrals are easily evaluated, and the final result for the two first temperature regions takes the same analytic form. In the third temperature region, the integral is calculated because the principal role is played by small φ.

The final results for all three regions can consequently be represented in the form

$$\left(\frac{\delta S}{S}\right)_T = \frac{Tv_2}{4\pi DS}\ln\left[\frac{4T}{\Omega_0}\left(\frac{D}{\Omega_0\alpha a^2}\right)^{1/2}\right] \sim \frac{T}{4\pi S^2V_0}\ln\left[\frac{T}{\Omega_0}\left(\frac{SV_0}{\Omega_0}\right)^{1/2}\right], \quad T \gg \Omega_0\alpha(\Omega_0a^2/D\alpha)^{1/2}; \tag{12a}$$

$$\left(\frac{\delta S}{S}\right)_T = \frac{\xi(3/2)\Gamma(3/4)\alpha^2}{8\pi\Gamma(5/4)S}\left(\frac{T}{\Omega_0\alpha}\right)^{3/4}\left(\frac{D\alpha}{\Omega_0a^2}\right)^{1/4} \sim \left(\frac{T}{V_0S^2}\right)^{3/4}\left(\frac{V_0S}{\Omega}\right)^{1/4}, \quad T \ll \Omega_0\alpha(\Omega_0a^2/D\alpha)^{1/2}. \tag{12b}$$

It follows from these formulas that the relative deviation of the magnetization from saturation is small if $T \ll V_0S^2[\ln(V_0S/\Omega_0)]^{-1} = T_1$. On the other hand, as is well known (see, e.g., [5], and also[2]), in the case of pure exchange interaction in a planar spin system, a phase transition to a state with short-range order takes place at $T \sim V_0S$. It is not clear as yet whether the dipole forces stabilize the ferromagnetic order all the way to temperatures of order SV_0, or whether there exists also another phase transition to the ferromagnetic state at a temperature $T \sim T_1$.

We note in conclusion that the dipole stabilization of the ferromagnetism has a simple physical cause. The magnetic energy of an infinite plane is minimal if all the spins are parallel and lie in this plane; the magnetic field in the space near the plane is then equal to zero. Any local violation of this order, with dimension R, produces a nonzero magnetic field in a volume on the order of R^3 and increases by the same token the energy of the system.

3. LAYERED MAGNETS

Thus, dipole forces stabilize two-dimensional ferromagnetism on account of the terms linear in the momentum in expression (5) for the dipole tensor. At the same time, as will be shown in the next section of this paper, in the three-dimensional case the long-range effect produced on a selected layer by the remaining layers is such that the terms linear in the momentum vanish completely from the dipole tensor.[3)] This tensor takes therefore, the usual three-dimensional form (see, e.g., [3]) that follows from macroscopic considerations (the magnetic anisotropy energy plus a part that depends on the momentum direction), with an exponentially small increment due to the discrete arrangement of the atoms in the layer and taking the form of anisotropic antiferromagnetic exchange between the layers. It will be shown below that in this case the stabilization of the long-range order is due only to exchange interaction of the layers.

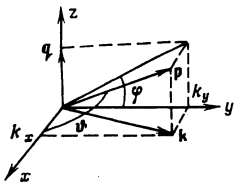

FIG. 1.

Let us make one qualitative remark. The system of ferromagnetic layers must be oriented in accordance with the character of the interaction between them (with allowance for the short-range part of the dipole forces). In the case of antiferromagnetic interactions, this is obvious, since antiferromagnetic order corresponds to a minimum of the magnetic energy. For a very small ferromagnetic exchange this is no longer so obvious, since the ferromagnetic order is accompanied by an increase of the magnetic energy of the system. But the magnetic dipole interaction of the layers is proportional to the linear dimensions, [4)] and the exchange interaction to their area. Therefore, if the layers are large enough the exchange interaction overcomes the dipole interaction, and the result is a normal three-dimensional ferromagnet in which the domains are already a secondary phenomenon. Both types of interaction are encountered in nature (see the review of de Jongh and Miedema[6]).

We start with the simpler case of ferromagnetic interaction between the layers and consider only easy-plane anisotropy, the easy plane coinciding with the plane of the layers. This anisotropy does not lead to a gap in the spin-wave spectrum and is the result of dipole forces if the distance between the layers is large enough. We assume for simplicity that the spins are located at the sites of a primitive tetragonal lattice, with a the lattice constant in the layer and $b > a$ the distance between layers. We then obtain for the Hamiltonian, with the aid of formula (37) of Sec. 4,

$$H = -\tfrac{1}{2}\sum_{\mathbf{p}}\{V_{\mathbf{p}}\delta_{\alpha\beta} + \omega_0(\tfrac{1}{3}\delta_{\alpha\beta} - p_\alpha p_\beta/p^2) + (B + A_q - A_0)(\tfrac{1}{3}\delta_{\alpha\beta} - z_\alpha z_\beta)\} S_{\mathbf{p}}^{\alpha}S_{-\mathbf{p}}^{\beta} - g\mu N^{1/2}S_0^xH_0^x,$$
$$S_{\mathbf{p}} = N^{-1/2}\sum_{\mathbf{r}} e^{-i\mathbf{pr}}\mathbf{S}_{\mathbf{r}} = N^{-1/2}\sum_{\mathbf{R},\rho} e^{-i\mathbf{kR}-iq\rho}\mathbf{S}_{\mathbf{R},\rho}, \tag{13}$$
$$A_q = -\frac{3\pi(g\mu)^2}{v_2}\sum_{\tau_2,\rho}\exp(-|\tau_2||\rho| + iq\rho)|\tau_2|,$$

where N is the total number of spins; $\mathbf{p} = (\mathbf{k}, q)$ and $\mathbf{r} = (\mathbf{R}, \rho)$ the three-dimensional momentrum and lattice vectors; $\omega_0 = 4\pi(g\mu)^2v_0^{-1}$, where v_0 is the volume of the unit cell; B is the anisotropy constant and τ_2 is the layer reciprocal-lattice vector multiplied by 2π; A_q is the already mentioned layer dipole-interaction component due to the discrete character of the lattice.

Choosing the coordinate system illustrated in the figure, and directing the spontaneous moment along the x axis, we obtain with the aid of formulas similar to (6) the Hamiltonian of the free spin waves in the form

$$H=\sum_{\mathbf{p}}\{\xi_{\mathbf{p}}a_{\mathbf{p}}^{+}a_{\mathbf{p}}+{}^{1}/_{2}(B_{\mathbf{p}}^{*}a_{\mathbf{p}}^{+}a_{-\mathbf{p}}^{+}+B_{\mathbf{p}}a_{\mathbf{p}}a_{-\mathbf{p}})\},$$

$$\xi_{\mathbf{p}}=S(V_0-V_{\mathbf{p}})+{}^{1}/_{6}S(A_q-A_0)+{}^{1}/_{2}S(B+\omega_0\sin^2\vartheta_{\mathbf{p}})+|g\mu|H_i, \quad (14)$$

$$B_{\mathbf{p}}={}^{1}/_{2}S[\omega_0\sin^2\vartheta_{\mathbf{p}}\exp(2i\varphi_{\mathbf{p}})-B+A_0-A_q],$$

where H_i is the internal field; we shall henceforth put $H_i = 0$. The energy of the spin waves then takes the form

$$\varepsilon_{\mathbf{p}}=(\xi_{\mathbf{p}}^{2}-|B_{\mathbf{p}}|^{2})^{1/2}=\{[S(V_0-V_{\mathbf{p}})-{}^{1}/_{3}S(A_0-A_q)][SB+S\omega_0\sin^2\vartheta$$
$$+S(V_0-V_{\mathbf{p}})+{}^{1}/_{3}S(A_q-A_0)]+S^2[B+{}^{1}/_{2}(A_q-A_0)]\omega_0\sin^2\vartheta\cos^2\varphi\}^{1/2}$$
$$\approx[(Dk^2+d_1)(Dk^2+d_q+SB+S\omega_0\sin^2\vartheta)+S^2B\omega_0\sin^2\vartheta\cos^2\varphi]^{1/2}, \quad (15)$$

where $Dk^2 + d_q \approx S(V_0 - V_{\mathbf{p}})$; the approximate equality on the right-hand side takes place at small k and if the difference $A_0 - A_q$ is neglected. In this case the spectrum has a well-known form.[3] It is interesting to note that, inasmuch as $A_q < 0$, ferromagnetic ordering of the planes takes place only if ferromagnetic exchange between the planes is strong enough and exceeds the short-range part A_q of the dipole interaction of the planes. In particular, at zero exchange between planes, since A_q is negative, the planes should become antiferromagnetically ordered.

The expressions for the Green's function and $\delta S/S$ are direct generalizations of (8) and (9), and will not be presented here.

We assume that the strongest is the exchange interaction inside the layer ($Da^{-2} \gg d_q$, B, ω_0) and neglect A_q. It is seen from the expression for $\delta S/S$, which is analogous to (9), that if d_q is neglected then the integral with respect to k diverges logarithmically at the lower limit in the region ($\cos^2\vartheta \sim \cos^2\varphi \approx 0$). This means that the long-range order becomes stabilized in the plane only in the presence of exchange between planes, just as in the absence of dipole forces. In the case of greatest interest, that of not too low temperatures $T \gg B$, ω_0, $\max d_q$), the main contribution to $\delta S/S$ comes from the region of large q, the corresponding integral is easily calculated, and the final result is

$$\frac{\delta S}{S}=\frac{v_2T}{4\pi DS}\left\{\ln\frac{T}{[d_1(d_1+BS+\omega_0S)]}+\frac{b}{2\pi}\int_0^{\pi/b}dq\ln\frac{d_1(d_1+BS+\omega_0S)}{d_q(d_q+BS+\omega_0S)}\right\}, \quad (16)$$

where $d_1 = \max d_q$. This formula generalizes the result obtained by Berezinskiĭ and Blank[4] for pure exchange interaction. It is obvious that the heat capacity, just as in[4], is proportional to T.

We proceed now to the case of antiferromagnetic interaction of the layers. Assuming the same lattice geometry as before, we break up the system of layers in the usual manner into two sublattices and introduce the Holstein-Primakoff operators for the sublattice spins:

$$S_{x\mathbf{p}}^{(1)}=-\left(\frac{N}{2}\right)^{1/2}S\delta_{\mathbf{p},0}+\left(\frac{2}{N}\right)^{1/2}\sum_{\mathbf{p}_1}a_{\mathbf{p}_1+\mathbf{p}}^{+}a_{\mathbf{p}_1},$$

$$S_{x\mathbf{p}}^{(2)}=\left(\frac{N}{2}\right)^{1/2}S\delta_{\mathbf{p},0}-\left(\frac{2}{N}\right)^{1/2}\sum_{\mathbf{p}_1}b_{\mathbf{p}_1+\mathbf{p}}^{+}b_{\mathbf{p}_1}, \quad (17)$$

$$S_{y\mathbf{p}}^{(1)}=(S/2)^{1/2}(a_{\mathbf{p}}^{+}+a_{-\mathbf{p}}), \quad S_{y\mathbf{p}}^{(2)}=(S/2)^{1/2}(b_{\mathbf{p}}^{+}+b_{-\mathbf{p}}),$$

$$S_{z\mathbf{p}}^{(1)}=-i(S/2)^{1/2}(a_{\mathbf{p}}^{+}-a_{-\mathbf{p}}), \quad S_{z\mathbf{p}}^{(2)}=i(S/2)^{1/2}(b_{\mathbf{p}}^{+}-b_{-\mathbf{p}}).$$

With the aid of these operators, after laborious but straightforward calculations, taking into account the results of Sec. 4 below with respect to the properties of the dipole tensor, we obtain for the spin-wave part of the Hamiltonian

$$H=\sum_{\mathbf{p}}\{\xi_{\mathbf{p}}(a_{\mathbf{p}}^{+}a_{\mathbf{p}}+b_{\mathbf{p}}^{+}b_{\mathbf{p}})+{}^{1}/_{2}B_{\mathbf{p}}(a_{\mathbf{p}}a_{-\mathbf{p}}+b_{\mathbf{p}}^{+}b_{-\mathbf{p}}^{+})$$
$$+{}^{1}/_{2}B_{\mathbf{p}}^{*}(a_{\mathbf{p}}^{+}a_{-\mathbf{p}}^{+}+b_{\mathbf{p}}b_{-\mathbf{p}})+\Delta_{\mathbf{p}}b_{\mathbf{p}}a_{-\mathbf{p}}$$
$$+\Delta_{\mathbf{p}}^{*}b_{\mathbf{p}}^{+}a_{-\mathbf{p}}^{+}+C_{\mathbf{p}}a_{\mathbf{p}}^{+}b_{\mathbf{p}}+C_{\mathbf{p}}^{*}a_{\mathbf{p}}b_{\mathbf{p}}^{+}\}. \quad (18)$$

$$\xi_{\mathbf{p}}=S\{(v_{\mathbf{p}}^{(0)}-v_0^{(0)})+\omega_1\sin^2\vartheta_{\mathbf{p}}+B-v_0-{}^{1}/_{3}\alpha_0+{}^{1}/_{3}(\alpha_{\mathbf{p}}^{(0)}-\alpha_0^{(0)})\},$$

$$B_{\mathbf{p}}=S\{{}^{1}/_{2}(\alpha_{\mathbf{p}}^{(0)}-\alpha_0^{(0)})+\omega_1\sin^2\vartheta_{\mathbf{p}}\exp(2i\varphi_{\mathbf{p}})-B\},$$

$$\Delta_{\mathbf{p}}=e^{iqb}\Delta_{\mathbf{p}}^{(0)}=e^{iqb}S\{-\omega_1\cos^2\vartheta_{\mathbf{p}}-v_{\mathbf{p}}+{}^{1}/_{6}\alpha_{\mathbf{p}}\},$$

$$C_{\mathbf{p}}=e^{iqb}C_{\mathbf{p}}^{(0)}=e^{iqb}S\{\omega_1[1+\sin^2\vartheta_{\mathbf{p}}\exp(-2i\varphi_{\mathbf{p}})]-{}^{1}/_{2}\alpha_{\mathbf{p}}\},$$

where the angles $\vartheta_{\mathbf{p}}$ and $\varphi_{\mathbf{p}}$ are shown in the figure and the following notation is introduced:

$$v_{\mathbf{p}}^{(0)}=\sum_{\mathbf{R},\rho_0}V_{\mathbf{R},\rho_0}\exp(i\mathbf{kR}+iq\rho_0), \quad \alpha_{\mathbf{p}}^{(0)}=\sum_{\rho_0\neq0}A(\rho_0)\exp(iq\rho_0),$$

$$v_{\mathbf{p}}=\sum_{\mathbf{R},\rho_0}V_{\mathbf{R},\rho_0+b}\exp(i\mathbf{kR}+iq(\rho_0+b)), \quad \alpha_{\mathbf{p}}=\sum_{\rho_0}A(\rho_0+b)\exp(iq(\rho_0+b)), \quad (19)$$

$$A(\rho_0)=-\frac{3\pi(g\mu)^2}{v_2}\sum_{\tau_2}|\tau_2|\exp(-|\tau_2||\rho_0|),$$

$$B={}^{1}/_{2}\left[A^{(0)}+\sum_{\rho_0\neq0}A(\rho_0)\right]-\omega_1+B', \quad \omega_1=\omega_0/4.$$

Here $A^{(0)}$ is defined in (4); $\rho_0 = 2nb$ ($n = 0, \pm 1, \ldots$); B is the anisotropy constant with B' the contribution made to this constant by all but the dipole interactions; we are interested in anisotropy of the easy-plane type, so that $B > 0$. The quantity $v_{\mathbf{p}}^{(0)}$ describes both the exchange interaction inside the layer and the exchange interaction with the layers of its own sublattice, while $v_{\mathbf{p}}$ is the exchange interaction of the sublattices. The quantities $\alpha_{\mathbf{p}}^{(0)}$ and $\alpha_{\mathbf{p}}$ correspond to the short-range part of the dipole forces in one sublattice and between sublattices. From the definitions of the introduced quantities it follows that

$$B\gtrsim\omega_1\gg|\alpha_0|\gg|\alpha_0^{(0)}|, \quad |v_0^{(0)}|\gg|v_0|. \quad (20)$$

Introducing in the usual manner the Green's functions

$$G_{\mathbf{p}}(t)=-i\vartheta(t)\langle[a_{\mathbf{p}}(t),\ a_{\mathbf{p}}^{+}(0)]\rangle, \quad F_{\mathbf{p}}^{+}=-\vartheta(t)\langle[a_{-\mathbf{p}}^{+}(t),\ a_{\mathbf{p}}^{+}(0)]\rangle, \quad (21)$$

$$H_{\mathbf{p}}^{+}(t)=-\vartheta(t)\langle[b_{-\mathbf{p}}^{+}(t),\ a_{\mathbf{p}}^{+}(0)]\rangle, \quad Q_{\mathbf{p}}(t)=-i\vartheta(t)\langle[b_{\mathbf{p}}(t),\ a_{\mathbf{p}}^{+}(0)]\rangle,$$

we obtain for them in the ω representation the system of equations

$$\begin{aligned}
&(\omega-\xi_{\mathbf{p}})G_{\mathbf{p}}-iB_{\mathbf{p}}^{*}F_{\mathbf{p}}^{+}-C_{\mathbf{p}}Q_{\mathbf{p}}-i\Delta_{\mathbf{p}}H_{\mathbf{p}}^{+}=1,\\
&-iB_{\mathbf{p}}G_{\mathbf{p}}+(\omega+\xi_{\mathbf{p}})F_{\mathbf{p}}^{+}-i\Delta_{\mathbf{p}}Q_{\mathbf{p}}+C_{-\mathbf{p}}^{*}H_{\mathbf{p}}^{+}=0,\\
&-C_{\mathbf{p}}^{*}G_{\mathbf{p}}-i\Delta_{\mathbf{p}}^{*}F_{\mathbf{p}}^{+}+(\omega-\xi_{\mathbf{p}})Q_{\mathbf{p}}-iB_{\mathbf{p}}H_{\mathbf{p}}^{+}=0,\\
&-i\Delta_{\mathbf{p}}^{*}G_{\mathbf{p}}+C_{-\mathbf{p}}F_{\mathbf{p}}^{+}-iB_{\mathbf{p}}^{*}Q_{\mathbf{p}}+(\omega+\xi_{\mathbf{p}})H_{\mathbf{p}}^{+}=0.
\end{aligned} \quad (22)$$

The determinant of this system takes the form

$$Z(\omega)=(\omega^2-\varepsilon_p^{(+)2})(\omega^2-\varepsilon_p^{(-)2}), \tag{23}$$

where $\varepsilon_p^{(\pm)}$ are the energies of the two spin-wave branches.

We are interested in the stability of the spectrum and in the gap produced in it as $p \to 0$; this forces us to present the cumbersome formulas for $\varepsilon_p^{(\pm)}$, obtained only by neglecting the very small difference $\alpha_p^{(0)} - \alpha_0^{(0)}$, allowance for which does not affect the result:

$$\begin{aligned}
\varepsilon_p^{(\pm)2} &= \Omega^2 + XD_1 + D_1^2 \pm [\Omega^4 + Y_1 D_1 + Y_2 D_1^2]^{1/2}, \\
D_1 &= S[v_p^{(0)} - v_0^{(0)} + v_p - v_0 + {}^1/_3(\alpha_p - \alpha_0)] \approx Dk^2 + \delta_q, \\
\Omega^2 &= 2S^2(B+\omega_1)[2\omega_1 \sin^2\vartheta\cos^2\varphi + u_p] \\
&\quad + 2S^2(-\alpha_p)\omega_1 \sin^2\vartheta\cos^2\varphi + S^2(-\alpha_p)u_p \\
&\approx 2S^2(B+\omega_1)[2\omega_1 \sin^2\vartheta\cos^2\varphi + u_p], \\
X &= 2S(B+\omega_1\sin^2\vartheta + u_p), \\
Y_1 &= 8S^3(B+\omega_1)\{[2\omega_1(1+\sin^2\vartheta)+u_p]\sin^2\vartheta\cos^2\varphi \\
&\quad u_p\cos^2\vartheta\} + 4S^3(-\alpha_p)\{2\omega_1[\omega_1(2+\sin^2\vartheta)+B+u_p]\sin^2\vartheta\cos^2\varphi \\
&\quad + u_p[B+\omega_1(1+\cos^2\vartheta)]\} + 2S^3\alpha_p^2[2\omega_1\sin^2\vartheta\cos^2\varphi + u_p] \\
&\approx 8S^3(B+\omega_1)\{[2\omega_1(1+\sin^2\vartheta)+u_p]\sin^2\vartheta\cos^2\varphi + u_p\cos^2\vartheta\}, \\
Y_2 &= 4S^2\omega_1^2(\cos^4\vartheta + 4\sin^2\vartheta\cos^2\varphi) + 4S^2(-\alpha_p)\omega_1(\cos^2\vartheta + 2\sin^2\vartheta\cos^2\varphi) \\
&\quad + S^2\alpha_p^2 \approx 4S^2\omega_1^2(\cos^4\vartheta + 4\sin^2\vartheta\cos^2\varphi).
\end{aligned} \tag{24}$$

Here $u_p = -v_p - \alpha_p/3$; the approximate equalities in these formulas correspond to small k and to neglecting, in accordance with (20), of the terms containing α_p.

If $B = \omega_1 = 0$, then the formula for $\varepsilon_p^{(\pm)}$ goes over into the usual expression for the spin-wave energy in an antiferromagnet.[3] For the spectrum to be stable it is obviously necessary that Ω^2, X, and $Y_{1,2}$ be positive for all ϑ and φ. Since $\alpha_p > 0$, the stability condition is the inequality

$$u_p = (-v_p - \alpha_p/3) > 0, \tag{25}$$

which means that the complete interaction between planes, which takes into account the short-range part of the dipole forces, must be antiferromagnetic. In particular, the spectrum is stable at $v_p = 0$. This conclusion agrees with the analysis presented above for the condition for the existence of ferromagnetism.

From the requirement $\varepsilon_p^{(-)2} > 0$ follows one more stability condition which takes upon satisfaction of (20) the form

$$B + u_p > \omega_1. \tag{26}$$

In the case of pure dipole anisotropy this condition is satisfied automatically, inasmuch as in expression (19) for B, as shown below, we have $A^{(0)} \approx 4\pi(g\mu)^2 a^{-3}$, $\omega_1 = 4\pi(g\mu)^2(a^2 2b)^{-1}$, and b is always larger than a in layered magnets.

It follows from (24) that the $\varepsilon_p^{(-)}$ branch has a zero gap and at sufficiently small p we have

$$\begin{aligned}
\varepsilon_p^{(+)} &\approx (2\Omega)^{1/2} = 2S[(B+\omega_1)(2\omega_1\sin^2\vartheta\cos^2\varphi + u_p)]^{1/2}, \\
\varepsilon_p^{(-)} &\approx \{(Dk^2+\delta_q)[2\omega_1 S(B-\omega_1+{}^1/_2 u_p) \\
&\quad + Su_p(B+u_p+2\omega_1\sin^2\vartheta-\omega_1)]\}^{1/2}(2\omega_1\sin^2\vartheta\cos^2\varphi + u_p)^{-1/2}.
\end{aligned} \tag{27}$$

The relative deviation $\delta S/S$ of the sublattice magnetization is expressed in terms of the Green's function G, for which we obtain from (22), taking (18), (20), (23), and (24) into account

$$\begin{aligned}
G(\mathbf{p},\omega) &= \gamma(\mathbf{p},\omega)[(\omega^2-\varepsilon_p^{(+)2})(\omega^2-\varepsilon_p^{(-)2})]^{-1}, \\
\gamma(\mathbf{p},\omega) &= (\omega + D_1 + S\omega_1\sin^2\vartheta + SB + Su_p)(\omega^2 - \Omega^2 - XD_1 - D_1^2) \\
&\quad + 2S^2\omega_1^2 D_1(\cos^4\vartheta + 4\sin^2\vartheta\cos^2\varphi) + \\
&\quad + 4S^3\omega_1(B+\omega_1)[\omega_1(1+\sin^2\vartheta)+u_p]\sin^2\vartheta\cos^2\varphi + 2S^3\omega_1 u_p(B+\omega_1)\cos^2\vartheta.
\end{aligned} \tag{28}$$

The expression analogous to (9) for $\delta S/S$ in terms of ImG, takes, as can be easily verified, the form

$$\frac{\delta S}{S} = \frac{2bv_2}{S(2\pi)^3}\int_{-\pi/2b}^{\pi/2b} dq \int d^2\mathbf{k}\left\{\frac{[\gamma(\varepsilon_+)+\gamma(-\varepsilon_+)]n(\varepsilon_+)+\gamma(-\varepsilon_+)}{2\varepsilon_+(\varepsilon_+^2-\varepsilon_-^2)} + \frac{[\gamma(\varepsilon_-)+\gamma(-\varepsilon_-)]n(\varepsilon_-)+\gamma(-\varepsilon_-)}{2\varepsilon_-(\varepsilon_-^2-\varepsilon_+^2)}\right\}. \tag{29}$$

That part of $\delta S/S$ which is connected with the zero-point oscillations is small and vanishes when B, ω_1, and the exchange between planes tend to zero, and will therefore not be calculated.

The temperature contribution to $\delta S/S$ diverges logarithmically if we neglect in expression (24) for D_1 that part of the dispersion δ_q which depends on the interaction between planes. This follows directly from the fact that in the region $\varepsilon_\pm < T$ the integrand in (24), as can be easily verified, is proportional to D_1^{-1}. At temperatures $T \gg B$, ω_1, u_p the quantity $\delta S/S$ can be easily calculated. It is necessary for this purpose to break up the region of integration with respect to k in (29) into two parts: in the first $Dk^2 \gg B$, ω_1, u_p and $\varepsilon_\pm \approx Dk^2$, and in the second $\varepsilon_\pm \ll T$ and $n(\varepsilon_\pm) \approx T\varepsilon_\pm^{-1}$. As a result, the main contribution is made by $q \ll k$, so that we can make the substitutions $\sin^2\vartheta = 1$, $\cos^2\vartheta = \cos^2\varphi = 0$, after which the integration becomes elementary and the final expression takes the form

$$\begin{aligned}
\frac{\delta S}{S} &= \frac{v_2 T}{4\pi DS}\left(\ln\frac{T}{[S(B+\omega_1+u_1)\delta_1+\delta_1^2/2]^{1/2}}\right. \\
&\quad + \frac{b}{2\pi}\int_0^{\pi/2b} dq \ln\left[\left(S(B+\omega_1+u_1)\delta_1 + \frac{\delta_1^2}{2}\right)\right. \\
&\quad \left.\left.\times\left\{\left[S(B+\omega_1+u_q)\delta_q + \frac{\delta_q^2}{2}\right]\left[2S^2(B+\omega_1)u_q + S(B+\omega_1+u_q)\delta_q + \frac{\delta_q^2}{2}\right]\right\}^{-1}\right]\right),
\end{aligned} \tag{30}$$

where $\delta_1 = \max\delta_q$ and u_1 is the value of u_q at the maximum of δ_q. Formula (30) is also a generalization of the corresponding result of Berezinskiĭ and Blank.[4]

Comparing (12), (16), and (30) we see that all these expressions differ, as expected, only by a dimension-dependent quantity that cuts off the logarithmic divergence, and by an inessential constant of the order of unity. In real three-dimensional layered systems, the final result is due only to the existence of exchange interaction between the planes (in which we include also the short-range part of the dipole forces).

The appearance of the logarithmic divergence in $\delta S/S$ at zero exchange between the planes is connected with

the fact that actually the dipole forces introduce an anisotropy of the easy-plane type and in practice they do not lift the degeneracy in the xy plane at $q \gg k$. Indeed, it is seen from the figure that at $q \gg k$

$$\sin^2\vartheta \approx 1, \quad \sin^2\vartheta\cos^2\varphi \approx k_y^2 q^{-2}, \quad \cos^2\vartheta \approx k_x^2 q^{-2}$$

and, as can be easily verified, the terms dependent on the direction of the vector k in the expressions for the spin-wave energies (15) and (24) are small, while the energies themselves are proportional to k. The combinations of the residues of the Green's functions (of the coefficients of the (u, v) transformation), [3] which enter in (9) and (29), turn out to be proportional to $\varepsilon_{\mathbf{p}}^{-1}$, so that the produced integral is of the same type as in the case of short-range action.

It is interesting to note that in both ferromagnetic and antiferromagnetic ordering of the planes the situation is in fact the same, namely, there are no terms linear in k and the quadratic combinations of the trigonometric functions become proportional to k^2 as $k \to 0$.

4. PROPERTIES OF THE DIPOLE TENSOR

In this section, which is by way of a mathematical appendix, we investigate in greater detail than before the properties of the dipole tensor in momentum space:

$$Q_{\mathbf{p}}^{\alpha\beta} = (g\mu)^2 \sum_{\mathbf{r}\neq 0} e^{i\mathbf{pr}}(3r_\alpha r_\beta - r^2\delta_{\alpha\beta})r^{-5} = (g\mu)^2(-3\nabla_{\mathbf{p}}^\alpha\nabla_{\mathbf{p}}^\beta + \delta_{\alpha\beta}\nabla_{\mathbf{p}}^2)F_{\mathbf{p}},$$
$$F_{\mathbf{p}} = \sum_{\mathbf{r}\neq 0} e^{i\mathbf{pr}} r^{-3}. \tag{31}$$

This tensor was previously investigated many times, and its part connected with large r has been thoroughly studied; to analyze this part it is best to use a macroscopic approach (see, e.g., [3]). As to the contribution from small r, it has been investigated so far only by numerical calculations, the most complete of which have been made by Colpa. [7,8] We derive here analytic expressions for the different contributions made to Q, and in particular, we express the magnetic-anisotropy energy in the form of rapidly converging sums.

We regard an arbitrary Bravais lattice as a system of crystalline planes and write down the vector $\mathbf{r}$ in the form

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}, \quad \mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2, \quad \boldsymbol{\rho} = n_3\mathbf{a}_3,$$

where $\mathbf{a}_i$ are the principal lattice periods and $\mathbf{R}$ is a planar vector. The quantity F in (31) can be represented in the form

$$F_{\mathbf{p}} = \Phi_0 + \sum_{\rho\neq 0} e^{i\mathbf{p}\boldsymbol{\rho}}\Phi_\rho,$$
$$\Phi_0 = \sum_{\mathbf{R}\neq 0} e^{i\mathbf{pR}} R^{-3}, \quad \Phi_\rho = \sum_{\mathbf{R}} e^{i\mathbf{pR}}|\boldsymbol{\rho}+\mathbf{R}|^{-3}. \tag{32}$$

We consider first Φ_ρ. Using the known procedure for transforming lattice sums, [8] we write

$$\Phi_\rho = \frac{4}{3\sqrt{\pi}}\int_0^\infty dt\, t^{1/2}\exp(-\rho^2 t)\sum_{\mathbf{R}}\exp[i(\mathbf{p}_\parallel + 2i\boldsymbol{\rho}_\parallel t)\mathbf{R} - R^2 t], \tag{33}$$

where the symbol $\parallel$ denotes the projection of the vector on the R plane. We use the well known formula [8]

$$\sum_{\mathbf{R}_d}\exp(i\mathbf{R}_d\mathbf{p}_d - R_d^2 t) = \frac{1}{v_d}\left(\frac{\pi}{t}\right)^{d/2}\sum_{\boldsymbol{\tau}_d}\exp\left[\frac{-(\mathbf{p}+\boldsymbol{\tau}_d)^2}{4t}\right], \tag{34}$$

where d is the dimensionality of space, v_d is the corresponding unit-cell volume, and $\boldsymbol{\tau}_d$ is the reciprocal-lattice vector multiplied by 2π.

Putting $d = 2$, we can easily transform (33) into

$$\Phi_\rho = \frac{2\pi}{3v_2}\sum_{\boldsymbol{\tau}_2}\frac{1+\rho_\perp|\mathbf{p}_\parallel+\boldsymbol{\tau}_2|}{\rho_\perp^3}\exp[-\rho_\perp|\mathbf{p}_\parallel+\boldsymbol{\tau}_2| - i\boldsymbol{\rho}_\parallel(\mathbf{p}_\parallel+\boldsymbol{\tau}_2)], \tag{35}$$

where the symbol $\perp$ denotes that part of the vector which is perpendicular to the $\mathbf{R}$ plane.

For a tetragonal lattice this formula was obtained by Benson and Mills. [9]

Substituting (35) in (31) we obtain for the contribution of the ρ plane to $Q_{\mathbf{p}}$

$$Q_\rho^{\alpha\beta} = \frac{2\pi(g\mu)^2}{v_2}e^{i\mathbf{p}\boldsymbol{\rho}}\sum_{\boldsymbol{\tau}_2}\exp[-\rho_\perp|\mathbf{p}_\parallel+\boldsymbol{\tau}_2| - i\boldsymbol{\rho}_\parallel(\mathbf{p}_\parallel+\boldsymbol{\tau}_2)]$$
$$\times\left\{-\frac{(\mathbf{p}_\parallel+\boldsymbol{\tau}_2)^\alpha(\mathbf{p}_\parallel+\boldsymbol{\tau}_2)^\beta}{|\mathbf{p}_\parallel+\boldsymbol{\tau}_2|} + |\mathbf{p}_\parallel+\boldsymbol{\tau}_2|\frac{\rho_\perp^\alpha\rho_\perp^\beta}{\rho_\perp^2} + i\frac{\rho_\perp^\alpha}{\rho_\perp}(\mathbf{p}_\parallel+\boldsymbol{\tau}_2)^\beta\right.$$
$$\left. + i\frac{\rho_\perp^\beta}{\rho_\perp}(\mathbf{p}_\parallel+\boldsymbol{\tau}_2)^\alpha\right\}. \tag{36}$$

At small p this formula can be written in the form

$$Q_\rho^{\alpha\beta} = Q_{\rho 1}^{\alpha\beta} + Q_{\rho 2}^{\alpha\beta}; \tag{37}$$
$$Q_{\rho 1}^{\alpha\beta} = \frac{2\pi(g\mu)^2}{v_2}\sum_{\boldsymbol{\tau}_2\neq 0}\exp(-\rho_\perp\tau_2 - i\boldsymbol{\rho}_\parallel\boldsymbol{\tau}_2)$$
$$\times\left(-\frac{\tau_2^\alpha\tau_2^\beta}{\tau_2} + \tau_2\frac{\rho_\perp^\alpha\rho_\perp^\beta}{\rho_\perp^2} + i\frac{\rho_\perp^\alpha}{\rho_\perp}\tau_2^\beta + i\frac{\rho_\perp^\beta}{\rho_\perp}\tau_2^\alpha\right),$$
$$Q_{\rho 2}^{\alpha\beta} = \frac{2\pi(g\mu)^2}{v_2}\exp(i\mathbf{p}\boldsymbol{\rho} - \rho_\perp p_\parallel - i\boldsymbol{\rho}_\parallel\boldsymbol{\tau}_2)$$
$$\times\left(-\frac{p_\parallel^\alpha p_\parallel^\beta}{p_\parallel} + p_\parallel\frac{\rho_\perp^\alpha\rho_\perp^\beta}{\rho_\perp^2} + i\frac{\rho_\perp^\alpha}{\rho_\perp}p_\parallel^\beta + i\frac{\rho_\perp^\beta}{\rho_\perp}p_\parallel^\alpha\right).$$

The quantity $Q_{\rho 1}^{\alpha\beta}$ describes the short-range part of the dipole interaction of the planes; this part decreases exponentially with increasing distance. Since $\tau_2 \gtrsim 2\pi a_{1,2}^{-1}$, this interaction is very small for layered magnets in which $a_3 > a_{1,2}$. In the case of a rectangular lattice, Q_1 leads to the short-range dipole terms written out above in (13) and (19). Q_2 is linear in $p_\parallel$ and in the limit as $p_\parallel \to 0$ it makes a finite contribution to $Q_{\mathbf{p}}$, inasmuch as the terms with $\rho_\perp \sim p_\parallel^{-1}$ play the important role in the summation over ρ. In particular, if p^{-1} is of the order of the system dimensions, it is precisely Q_2 which leads to the appearance of the demagnetization tensor in $Q_{\mathbf{p}}$, as can be easily verified with a sufficiently thick infinite plate as an example.

We now calculate the contribution made to $Q_{\mathbf{p}}$ at small p by the terms with $\rho \neq 0$. If we recognize that $\rho_\perp^\alpha\rho_\perp^{-1} = z_\alpha\varepsilon(\rho_\perp)$ and $\rho_\perp^\alpha\rho_\perp^\beta = \rho_\perp^2 z_\alpha z_\beta$, where ε is the sign function, then all the sums over ρ turn out to be geometric progressions, and we have

$$\sum_{\rho\neq 0}\exp(i\mathbf{p}\boldsymbol{\rho} - \rho_\perp p_\parallel) \approx 2p_\parallel(a_{3\perp}p^2)^{-1} - 1,$$
$$\sum_{\rho\neq 0}\exp(i\mathbf{p}\boldsymbol{\rho} - \rho_\perp p_\parallel)\rho_\perp^\alpha\rho_\perp^{-1} \approx 2iz_\alpha(p_\perp a_3)(a_{3\perp}p)^{-2}. \tag{38}$$

We have retained in these formulas all the terms that do not vanish in the limit as $p \to 0$; substitution of these terms in (37) leads to the following expression for the contribution of the planes with $\rho \neq 0$ to the dipole tensor:

$$Q_{p_1}^{\alpha\beta} = \frac{4\pi(g\mu)^2}{v_0}\left(\frac{1}{3}\delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2}\right) + \frac{2\pi(g\mu)^2}{v_2}\left(\frac{p_{\parallel\alpha}p_{\parallel\beta}}{p_\parallel} - p_\parallel z_\alpha z_\beta\right) + B_1^{\alpha\beta},$$

$$B_1^{\alpha\beta} = \frac{2\pi(g\mu)^2}{v_2}\sum_{\substack{\tau_2\neq 0;\\ \rho=0}} \exp(-\rho_\perp\tau_2 - i\rho_\parallel\tau_2)\left\{-\frac{\tau_2^\alpha\tau_2^\beta}{\tau_2} + \tau_2\frac{\rho_\perp^\alpha\rho_\perp^\beta}{\rho_\perp^2} + i\frac{\rho_\perp^\alpha}{\rho_\perp}\tau_2^\beta + i\frac{\rho_\perp^\beta}{\rho_\perp}\tau_2^\alpha\right\} - \frac{4\pi(g\mu)^2}{v_0}\left(\frac{1}{3}\delta_{\alpha\beta} - z_\alpha z_\beta\right). \quad (39)$$

Here $v_0 = v_2 a_{3\perp}$ is the volume of the three-dimensional unit cell.

The first term in Q_{p1} is that part of the dipole tensor which follows from macroscopic considerations; the second term is not a three-dimensional tensor, it differs in sign from the dipole-sum part linear in the momentum for one plane (5) and vanishes in the complete expression for $Q_p^{\alpha\beta}$;[5] finally, B_1 is the contribution made to the magnetic anisotropy by the planes with $\rho \neq 0$.

We proceed now to the sum with $\rho = 0$. Breaking up the crystal plane into lines, we have

$$\Phi_0 = \sum_{\mathbf{R}_1\neq 0}\frac{\exp(i\mathbf{pR}_1)}{R_1^5} + \sum_{\mathbf{R}_2\neq 0}\exp(i\mathbf{pR}_2)\sum_{\mathbf{R}_1}\frac{\exp(i\mathbf{pR}_1)}{|\mathbf{R}_1+\mathbf{R}_2|^5}, \quad (40)$$

where $\mathbf{R}_i = \mathbf{a}_i n_i$. With the aid of transformations that are perfectly analogous to those given above, we can rewrite (40) in the form

$$\Phi_0 = \sum_{\mathbf{R}_1\neq 0}\frac{\exp(i\mathbf{pR}_1)}{R_1^5} + \frac{2}{3a_1}\sum_{\mathbf{R}_2\neq 0}\exp(i\mathbf{p}_2\mathbf{R}_{22}) \times \sum_{\tau_1}\exp(-i\mathbf{R}_{21}\tau_1)\frac{(\mathbf{p}_1+\tau_1)^2}{R_{22}^2}K_2(R_{22}|\mathbf{p}_1+\tau_1|), \quad (41)$$

where $\tau_1 = 2\pi a_1^{-1} m_1$ is the reciprocal-lattice vector for the line R_1 and is directed along this line, $\mathbf{p}_1$ and $\mathbf{R}_{21}$ are the projections of the corresponding vectors on this line, $\mathbf{R}_{22}$ is the component of $\mathbf{R}_2$ perpendicular to the line R_1, $\mathbf{p}_2$ is the component of the vector $\mathbf{p}_\parallel$ perpendicular to R_1, and K_2 is a Macdonald function.

From (41) we obtain for the dipole tensor of the plane $\rho = 0$

$$Q_{p_2}^{\alpha\beta} = 6(x_\alpha x_\beta - {}^1/_3\delta_{\alpha\beta})(g\mu)^2\sum_{R_1>0}\frac{1}{R_1^3}\cos p_1 R_1 + (g\mu)^2\sum_{\mathbf{R}_2}\exp(i\mathbf{p}_2\mathbf{R}_2)\frac{2}{a_1 R_{22}^2}\sum_{\tau_1}\exp[-i\mathbf{R}_{12}(\tau_1+\mathbf{p}_1)] \times\left\{\left(R_{22}^\alpha R_{22}^\beta - \frac{\delta_{\alpha\beta}R_{22}^2}{3}\right)(\mathbf{p}_1+\tau_1)^2 K_2(R_{22}|\mathbf{p}_1+\tau_1|) + \left(\frac{\delta_{\alpha\beta}}{3} - \frac{(\mathbf{p}_1+\tau_1)_\alpha(\mathbf{p}_1+\tau_1)_\beta}{|\mathbf{p}_1+\tau_1|^2}\right)|\mathbf{p}_1+\tau_1|^2 R_{22}^2 K_0(|\mathbf{p}_1+\tau_1|R_{22}) + \left[i(\mathbf{p}_1+\tau_1)_\alpha R_{22}^\beta + i(\mathbf{p}_1+\tau_1)_\beta R_{22}^\alpha + \left(x_\alpha x_\beta - \frac{\delta_{\alpha\beta}}{3}\right)\right] \times|\mathbf{p}_1+\tau_1|R_{22}K_1(|\mathbf{p}_1+\tau_1|R_{22})\right\}, \quad (42)$$

where x_α is a unit vector along the R_1 axis.

At $p = 0$ the sum over $\mathbf{R}_2$ is the dipole potential of a chain of magnetic atoms. At large x we have $K_n(x) \propto e^{-x}$ and therefore only the first terms in the sum over $\mathbf{R}_2$ are significant at $\tau_1 \neq 0$. The terms with $\tau_1 = 0$ require at small p a more accurate analysis. Since K_2 and K_1 are infinite at zero, the sum over $\mathbf{R}_2$ cannot be replaced by an integral even in the limit as $p \to 0$. Expressing K_2 in terms of K_1 and using the well known presentation (see[10])

$$K_1(x) = x\int_0^\infty dt\, e^{-xt}(t^2-1)^{1/2}, \quad (43)$$

we can carry out the summation over $\mathbf{R}_2$ and then calculate the corresponding expression at small p.

We do not present here these simple calculations, but formulate the result directly: the terms linear in $p_\parallel$ coincide exactly with (5) and cancel out the corresponding term in (39), while the final expression for the tensor Q_p becomes

$$Q_p^{\alpha\beta} = \frac{4\pi(g\mu)^2}{v_0}\left(\frac{\delta_{\alpha\beta}}{3} - \frac{p_\alpha p_\beta}{p^2}\right) + B_{\alpha\beta}; \quad (44)$$

$$B_{\alpha\beta} = (g\mu)^2\left\{\left(\frac{6\xi(3)}{a_1^3} + \frac{2\pi^2}{3a_1a_{22}}\right)\left(x_\alpha x_\beta - \frac{\delta_{\alpha\beta}}{3}\right) + \frac{4\pi^2}{3a_1a_{22}^2}\left(y_\alpha y_\beta - \frac{\delta_{\alpha\beta}}{3}\right) + \frac{4\pi}{v_0}\left(z_\alpha z_\beta - \frac{\delta_{\alpha\beta}}{3}\right)\right\} + \frac{2(g\mu)^2}{a_1}\sum_{\substack{\mathbf{R}_2\neq 0\\ \tau_1\neq 0}}\exp(-i\mathbf{R}_{21}\tau_1)R_{22}^{-2}\left\{\left(x_\alpha x_\beta - \frac{\delta_{\alpha\beta}}{3}\right) \times[\tau_1 R_{22}K_1(\tau_1 R_{22}) - \tau_1^2 R_{22}^2 K_0(R_{22}\tau_1)] + \left(y_\alpha y_\beta - \frac{\delta_{\alpha\beta}}{3}\right)R_{22}^2\tau_1^2 K_2(R_{22}\tau_1) + i(\tau_1^\alpha R_{22}^\beta + \tau_1^\beta R_{22}^\alpha)\tau_1 R_{22}K_1(\tau_1 R_{22})\right\} + \frac{2\pi(g\mu)^2}{a_1 a_{22}}\sum_{\substack{\tau_2\neq 0\\ \rho\neq 0}}\exp(-\rho_\perp\tau_2 - i\rho_\perp\tau_2) \times\left(-\frac{\tau_2^\alpha\tau_2^\beta}{\tau_2} + \tau_2\frac{\rho_\perp^\alpha\rho_\perp^\beta}{\rho_\perp^2} + i\tau_2^\alpha\frac{\rho_\perp^\beta}{\rho_\perp} + i\tau_2^\beta\frac{\rho_\perp^\alpha}{\rho_\perp}\right).$$

Here y_α is a unit vector in the R plane and is perpendicular to x_α.

The tensor $B_{\alpha\beta}$ is the magnetic dipole anisotropy; despite its cumbersome form, the expression for $B_{\alpha\beta}$ is convenient for the calculation of the anisotropy, since the sums contained in it converge very rapidly and only the first terms are significant. If we neglect these sums completely, then we obtain formulas somewhat more accurate than those of Colpa.[7] The accuracy of such an approximation can be assessed by starting from the requirement that $B_{\alpha\beta}$ vanish for the primitive cubic lattice. The equality $B_{\alpha\beta} = 0$ is ensured by the equality of the numerical coefficients of all three tensors ($x_\alpha x_\beta - \delta_{\alpha\beta}/3$ etc.). The approximate values of these coefficients are -13.8, 13.2, and 12.6, i.e., the scatter is ~10%. With the same degree of accuracy, $A^{(0)}$ in (19) has a value $4\pi(g\mu)^2 a^{-3}$.

In conclusion, I wish to thank V. A. Ruban and B. N. Fillippov for interesting discussions.

[1)] By two-dimensional system we mean here a situation wherein the spins lie in on a plane located in a real three-dimensional space.

[2)] The reason why the stabilization of the ferromagnetic order is actually the result of long-range action and not of the lifting of the degeneracy is that, as we shall show, 1) the homogeneous part of the energy is invariant to planar rotations, 2) the stabilization stems from Hamiltonian terms that are linear in the momentum.

[3)]If the number of layers $N_\perp$ is large, then the cancellation of the linear terms is complete at $qb \ll 1$ (q is the momentum perpendicular to the layers and b is the distance between the layers). On the other hand if $N_\perp \sim 2$ or 3, then there is no complete cancellation and the dipole forces should stabilize the long-range order, but an investigation of this question is beyond the scope of the present article.

[4)]Thus, the magnetic interaction between two uniformly magnetized closely-located monatomic layers with sides l and m along the axes x and y is described by the formula

$$H_d = 2(g\mu)^2 (v_2^2 d)^{-1} \{(m^2+ld) S_x^{(1)} S_x^{(2)} + (l^2+md) S_y^{(1)} S_y^{(2)} - d(l+m+d) S_z^{(1)} S_z^{(2)}\},$$

where $d^2 = l^2 + m^2$, v_2 is the "volume" of the planar unit cell, and $\mathbf{S}^{(1,2)}$ is the average spin of the atom in the layer.

[5)]It is interesting to note that if we consider the region of the crystal near the boundary, then the summation over ρ has a finite limit on one side, and as a result the linear terms are not completely cancelled out in the expression for Q_p. The terms linear in $p_\parallel$ also remain if $p_\perp \sim a_3^{-1}$.

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Contribution to the theory of liquid ^{3}He

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We show that the anomalous properties of liquid ^{3}He can be explained by assuming that the fluctuation spectrum of its spin density has a deep roton minimum. This means that ^{3}He is close to a phase transition to an antiferromagnetic state. Comparison between theory and experiment confirms this assumption. We find the spin-roton parameters: $\Delta = 0.09$ K, $k_0 = 0.7\ p_F$, $M = 0.06\ m$. We determine the temperature dependence of the specific heat up to $T \approx 1.5$ K. For $0.05 < T < 0.5$ K the main term in the specific heat is $\propto \sqrt{T}$. A temperature of $T \approx 0.5$ K has the meaning of the degeneracy temperature of the spin-roton gas. We find the quasi-particle spectrum and the Landau Fermi-liquid theory parameters. We determine the wavevector dependence of the magnetic susceptibility χ. At $k = k_0$ the value of χ is 50 times larger than the susceptibility of a perfect Fermi gas of the same density.

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1. PHYSICAL PICTURE

Liquid ^{3}He can be satisfactorily described by Landau theory only for $T < 0.1$ K. At $T > 0.1$ K the specific heat, viscosity, and other physical characteristics of ^{3}He have a different order of magnitude than the values predicted by Landau and Pomeranchuk.[1–3] The strong difference between the properties of ^{3}He and those of a gas of quasi-particles can be explained if we assume that the liquid is close to a phase transition. Four types of instability are possible in a Fermi liquid which are connected with the two forms of its excitations—zero and spin sound. The first kind of instability is connected with long-wavelength density fluctuations. This instability arises when the velocity of the virtual zero sound is much smaller than the quasi-particle velocity on the Fermi surface. It is clear that such an instability can not be realized in ^{3}He as real zero sound can propagate in it, which has been observed experimentally.

The second type of instability is connected with short-wavelength zero sound which has a roton gap Δ for $k \approx 2p_F$. Such an instability is destroyed when the liquid goes over into the solid state. As the zero sound spectrum is unknown for $k \approx 2p_F$ it is necessary to consider the theoretical arguments for and against the existence of soft zero sound rotons in ^{3}He. The arguments for such a possibility are based upon the fact that ^{3}He under pressure becomes a solid. Moreover, in liquid ^{4}He, which differs from ^{3}He only by the statistics of the particles and an unimportant difference in the mass of the atoms, there are sound excitations with a roton gap. The argument against consists of the fact that a phase transition into the solid state is always a first order one and it seems doubtful that it takes place when $\Delta \ll \varepsilon_F$, when the rotons strongly affect the properties of the liquid phase. In particular, ^{4}He which in the solid state is very much like ^{3}He undergoes a transition into the solid state before the roton gap decreases so much that it becomes necessary to take the effect of the rotons on the nature of the transition into account. If, nevertheless, it turns out that when the density changes Δ becomes much less than ε_F prior to the occurrence of the phase transition, the exchange scattering ampli-