Vortex-lattice vibrations in a rotating helium II

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The spectrum and attenuation of the vibrations of a vortex lattice in an unbounded rotating superfluid liquid are investigated with allowance for the rigidity of the lattice. The attenuation of waves propagating in a direction perpendicular (Tkachenko waves) or almost perpendicular to the vortices turns out to be quite weak because of strong dragging of the normal liquid by the vortices. The vibrational spectrum of the vortex lattice is also studied with allowance for compressibility. It is shown that the compressibility does not lead to the attenuation of the Tkachenko waves. The boundary-value problem of vortex-lattice vibrations in a cylindrical vessel is also solved. A relatively simple equation is derived for the eigenfrequencies of the cylinder with allowance for the pinning of the vortices to the bottom of the vessel. An analysis of Tsakadze's recent experiments on the basis of this equation shows that the eigenfrequency observed in these experiments is primarily determined by the pinning of the vortices, and not by the transverse lattice rigidity, which determines the velocity of the Tkachenko waves.

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The character of the vibrations of a vortex lattice in a rotating He II significantly depends on the direction of propagation of the vibrations. On the basis of this criterion, we can distinguish two types of vibrations. The first type is waves propagating along the vortices and connected with their flexural rigidity. They have been quite well studied theoretically and experimentally.[1–4] Their existence does not depend on whether or not the vortices form an ordered lattice. The second type is Tkachenko waves propagating in a direction perpendicular to the vortices. They are due to the existence of a regular triangular vortex lattice, and constitute transverse sound in such a lattice.[5]

The attenuation of Tkachenko waves owing to the frictional force acting on the normal component[6] has been considered by Stauffer.[7] He assumed the normal liquid to be stationary, and obtained appreciable attenuation. However, as has been shown by Tkachenko,[8] allowance for the dragging of the normal liquid by the vortices reduces this attenuation appreciably.

The Tkachenko waves were used to interpret the even periodic changes in the period of rotation of pulsars.[9] Furthermore, the experimental observation of these waves in rotating He II has recently been reported.[10,11]

In the present paper we consider the spectrum of the vortex-lattice vibrations during their attenuation for an arbitrary direction of the wave vector relative to the axis of rotation. Such a problem is of interest, since because of the interaction of the vortices with the bottom of the vessel, any attempt to excite Tkachenko waves in it is always accompanied by the flexure of the vortices. In determining the attenuation, the transverse frictional force between the vortices and the normal liquid was taken into account. This force, according to[12], significantly exceeds the longitudinal force, which was used in Tkachenko's paper,[8] and makes the adhesion of the normal liquid to the vortices even more effective and the attenuation of the Tkachenko waves weaker.

We investigate the effect of the compressibility of the liquid on the spectrum of the lattice vibrations. As has been pointed out by Reatto,[13] such an effect becomes appreciable for Tkachenko waves of very long wavelengths. However, the results obtained in the present paper differ from Reatto's results: the form of the spectrum is different and there is no damping. This difference is due to the imperfectness of the model used in[13] (see Sec. 2).

We also consider the boundary-value problem of the natural vibrations of a vortex lattice in a cylindrical vessel. We obtain effective boundary conditions at the bottom and at the wall of the vessel. The solution of the boundary-value problem enables us to derive for the determination of the eigenfrequencies of the vibrations an equation which is more general than Ruderman's equation[9] and which takes the pinning of the vortices to the bottom of the vessel and the related flexure of the vortices into account. A comparison of the eigenfrequencies obtained from this equation with the experimentally observed values[10] allows us to obtain the magnitude of the cohesive force that pins the vortices to the bottom of the vessel.

1. THE EQUATIONS OF TWO-VELOCITY HYDRODYNAMICS FOR A ROTATING SUPERFLUID. THE SPECTRUM OF THE LATTICE VIBRATIONS IN THE INCOMPRESSIBLE LIQUID

We shall use the following equations of the two-velocity hydrodynamics of a rotating fluid:

$$\frac{\partial \mathbf{v}_s}{\partial t} - \sigma\nabla T + \frac{1}{\rho}\nabla P + \varkappa\sum_j\int[d\mathbf{R}_j\times\mathbf{v}_L(\mathbf{R}_j)]\delta(\mathbf{R}-\mathbf{R}_j)=0, \qquad (1)$$

$$\frac{\partial \mathbf{v}_n}{\partial t} + \frac{\rho_s}{\rho_n}\sigma\nabla T + \frac{1}{\rho}\nabla P + \nu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{v}_n + [2\mathbf{\Omega}\times\mathbf{v}_n] + \frac{\rho_s}{\rho_n}\varkappa\sum_j\int[d\mathbf{R}_j(\mathbf{v}_s(\mathbf{R}_j)-\mathbf{v}_L(\mathbf{R}_j))]\delta(\mathbf{R}-\mathbf{R}_j)=0, \qquad (2)$$

where $\varkappa = h/m$ is the circulation quantum, $\mathbf{\Omega}$ is the angular velocity vector, $\mathbf{v}_L(\mathbf{R}_j)$ is the velocity of an element, $d\mathbf{R}_j$, of a vortex filament of radius vector $\mathbf{R}_j$, and the rest of the symbols have the same meanings as in[14]. The integration in (1) and (2) is performed along the vortex filaments of index j, over which the summation is then carried out.

Equations (1) and (2) can be derived in the following manner. From the standard equations of two-fluid hydrodynamics in the inertial frame of reference[14] we must go over to a rotating coordinate system and take into account the fact that in the rotating reference frame

$$\operatorname{rot} \mathbf{v}_s = -2\boldsymbol{\Omega} + \varkappa \sum_j \int d\mathbf{R}_j \delta(\mathbf{R}-\mathbf{R}_j),$$

that each vortex exerts on the superfluid component a frictional force

$$\varkappa \int [d\mathbf{R}_j \times (\mathbf{v}_s - \mathbf{v}_L)] \delta(\mathbf{R}-\mathbf{R}_j),$$

and that the same force, but with opposite sign, acts on the normal component. Then in the equations thus obtained we discard all the nonlinear terms, including all those that are quadratic in the velocities $\mathbf{v}_s$ and $\mathbf{v}_n$. These terms are either not at all important in the linear theory of the vibrations (for example, terms of the type $[\operatorname{curl}\mathbf{v}_n \times \mathbf{v}_n]$, $\mathbf{v}_n \cdot \operatorname{div}\mathbf{v}_n$), or they lead to weak scattering of the acoustic waves by the vortices (terms of the type $\frac{1}{2}\nabla \mathbf{v}_s^2$, $\mathbf{v}_s \cdot \operatorname{div}\mathbf{v}_s$, etc.), which scattering is, for the purpose of our paper, of no interest to us.

The Eqs. (1) and (2) have, of course, a definite physical meaning only outside the vicinity of the vortex lines, near which two-velocity hydrodynamics is, strictly speaking, inapplicable. On the other hand, the singular terms, which are different from zero along the vortex filaments, play the role of pseudopotentials whose magnitude is determined by the momentum fluxes from the vicinities of the vortex filaments. A more detailed computation of these fluxes confirms the correctness of the Eqs. (1) and (2).[12]

In order to obtain a closed system of equations, we must add an equation determining the vortex velocity $\mathbf{v}_L$. In[12], it is shown that in both the phonon and roton regions the vortex velocity is, to a high degree of accuracy, equal to the mean-mass velocity, i.e., that

$$\mathbf{v}_L(\mathbf{R}_j) = \frac{\rho_s}{\rho}\mathbf{v}_s(\mathbf{R}_j) + \frac{\rho_n}{\rho}\mathbf{v}_n(\mathbf{R}_j). \tag{3}$$

Such an approximation implies the neglect of the longitudinal frictional force, which is small compared to the transverse frictional force, i.e., to the Lifshitz–Pitaevskiĭ force for rotons[15] and the Iordanskiĭ force for phonons.[16]

For the subsequent analysis it is convenient to separate the superfluid- and normal-velocity fields into their longitudinal parts, satisfying the equations $\operatorname{curl}\mathbf{v}_{s\parallel} = 0$ and $\operatorname{curl}\mathbf{v}_{n\parallel} = 0$, and the transverse parts, for which $\operatorname{div}\mathbf{v}_{s\perp} = 0$ and $\operatorname{div}\mathbf{v}_{n\perp} = 0$. On account of the fact that the vortex lattice vibrations are fairly slow, the principal role is played by the transverse degrees of freedom. However, as the wavelength of the Tkachenko waves increases, the longitudinal degrees of freedom nevertheless become important (see the following section).

In this section we shall consider only the transverse motion, i.e., we shall consider the liquid—both the superfluid and the normal components—incompressible, and, for brevity, we shall henceforth drop the index $\perp$ on the velocity symbols. The superfluid velocity is determined in this case by the location of all the vortices at the moment of time under consideration and, in the rotating system, is equal to:

$$\mathbf{v}_s(\mathbf{R}_j) = -[\boldsymbol{\Omega}\times\mathbf{R}_j] + \sum_k \frac{\varkappa}{4\pi}\int \frac{[d\mathbf{R}_k(\mathbf{R}_j-\mathbf{R}_k)]}{|\mathbf{R}_j-\mathbf{R}_k|^3}. \tag{4}$$

Let the vortex lines execute small vibrations characterized by the two-dimensional displacement vector $\mathbf{r}_L(\mathbf{R}_j)$ lying in the xy plane (the z axis is directed along the axis of rotation). Let us consider a plane wave of wave vector $\mathbf{Q}$ propagating obliquely to the z axis:

$$\mathbf{r}_L = \mathbf{u}(\mathbf{Q}) \exp(i\mathbf{Q}\mathbf{R}_j) = \mathbf{u}(\mathbf{Q}) \exp(ipz_j + i\mathbf{q}\mathbf{r}_j), \tag{5}$$

where z_j and $\mathbf{r}_j$ are the components of the radius vector $\mathbf{R}_j$ along the z axis and in the xy plane, while p and $\mathbf{q}$ are the corresponding components of the vector $\mathbf{Q}$. Introducing the displacements u_q and u_t parallel and perpendicular to the two-dimensional vector $\mathbf{q}$, and linearizing the Eqs. (4), we obtain

$$\begin{aligned} v_{sq}(\mathbf{R}_j) &= (\alpha(\mathbf{Q})u_q + (\Omega+\beta(\mathbf{Q}))u_t)\exp(i\mathbf{Q}\mathbf{R}_j),\\ v_{st}(\mathbf{R}_j) &= (-(\Omega-\gamma(\mathbf{Q}))u_q - \alpha(\mathbf{Q})u_t)\exp(i\mathbf{Q}\mathbf{R}_j), \end{aligned} \tag{6}$$

where

$$\alpha(\mathbf{Q}) = \sum_j \frac{\varkappa}{4\pi}\left(\frac{2}{r_j^2} - p^2K_2(pr_j)\exp(iqr_j\cos\varphi_j)\right)\sin 2\varphi_j, \tag{7}$$

$$\beta(\mathbf{Q}) = \sum_j \frac{\varkappa}{4\pi}\left(-\frac{2\cos 2\varphi_j}{r_j^2} + (p^2K_0(pr_j) + \cos 2\varphi_j p^2K_2(pr_j))\exp(iqr_j\cos\varphi_j)\right),$$

$$\gamma(\mathbf{Q}) = \sum_j \frac{\varkappa}{4\pi}\left(-\frac{2\cos 2\varphi_j}{r_j^2} + (-p^2K_0(pr_j) + \cos 2\varphi_j p^2K_2(pr_j))\exp(iqr_j\cos\varphi_j)\right).$$

Here φ_j is the angle between $\mathbf{r}_j$ and $\mathbf{q}$; $K_n(pr_j)$ are Macdonald functions; $v_{sq}(\mathbf{R}_j)$ and $v_{st}(\mathbf{R}_j)$ are the components of $\mathbf{v}_s(\mathbf{R}_j)$ parallel and perpendicular to $\mathbf{q}$ in the xy plane.

At $T=0$ there is no normal component, and the vortices move with the superfluid velocity, i.e.,

$$\mathbf{v}_L(\mathbf{R}_j) = \mathbf{v}_s(\mathbf{R}_j) = \frac{\partial \mathbf{u}(\mathbf{Q})}{\partial t}\exp(i\mathbf{Q}\mathbf{R}), \tag{8}$$

and the Eqs. (6) and (9) constitute a closed system of equations describing vortex-lattice vibrations with the spectrum

$$\omega^2 = (\Omega+\beta)(\Omega-\gamma) - \alpha^2. \tag{9}$$

For the case of long waves, i.e., for $qa \ll 1$, where $a = (\varkappa/\Omega\sqrt{3})^{1/2}$ is the distance between nearest vortices in the triangular lattice, we obtain

$$\begin{aligned} \alpha = 0, \quad \beta &= \Omega\frac{p^2-q^2}{p^2+q^2} + \frac{\varkappa}{4\pi}p^2\ln\frac{r_n}{r_c} + \frac{\varkappa q^2}{16\pi},\\ \gamma &= -\Omega - \frac{\varkappa}{4\pi}p^2\ln\frac{r_n}{r_c} + \frac{\varkappa q^2}{16\pi}, \end{aligned} \tag{10}$$

where r_c is the radius of the vortex core, while r_n is the smaller of the two lengths: a and $1/p$. The first terms in the formulas (10) for β and γ are obtained on replacing the sums in the formulas (7) by integrals, i.e., on replacing the vortex lattice by a continuous distribution of curl$\mathbf{v}_s$.[17]

From the Eqs. (9) and (10) we can obtain the spectrum of the vibrations for $p \gg q$[1-4]

$$\omega = 2\Omega + \frac{\varkappa p^2}{4\pi}\ln\frac{r_n}{r_c}, \tag{11}$$

which we shall call longitudinal vortex waves, as well as the spectrum of the waves for $p \ll 1/a$,

$$\omega^2 = (2\Omega)^2\frac{p^2}{p^2+q^2} + c_T^2 q^2, \tag{12}$$

which we shall henceforth call transverse vortex waves.[1)] Here $c_T = \sqrt{\varkappa\Omega/8\pi}$ is the velocity of a Tkachenko wave, which is a particular case of the transverse vortex wave for $p=0$.

In order to determine the corrections due to the presence of a normal component at $T>0$, let us derive from (2) equations for the components v_{nq} and v_{nt} of the transverse normal velocity $\mathbf{v}_n$ averaged over the unit cell:

$$\begin{gathered}\frac{\partial v_{nq}}{\partial t} + \nu Q^2 v_{nq} - 2\Omega\frac{\rho_s}{\rho_n}\frac{p^2}{Q^2}(v_{st}-v_{Lt}) - 2\Omega\frac{p^2}{Q^2}v_{nt} = 0,\\ \frac{\partial v_{nt}}{\partial t} + \nu Q^2 v_{nt} + 2\Omega\frac{\rho_s}{\rho_n}(v_{sq}-v_{Lq}) + 2\Omega v_{nq} = 0.\end{gathered} \tag{13}$$

However, the normal velocity $\mathbf{v}_n(\mathbf{R}_j)$ in the vicinity of a vortex, which enters into Eq. (3), can be appreciably different from the mean velocity $\mathbf{v}_n$, owing to the viscous drift.[6] The relation between them for the incompressible normal liquid can also be obtained from (2):

$$\mathbf{v}_n(\mathbf{R}_j) = \mathbf{v}_n - \frac{[\varkappa(\mathbf{v}_s(\mathbf{R}_j)-\mathbf{v}_L(\mathbf{R}_j))]}{4\pi\nu}\ln\frac{r_m}{l}, \tag{14}$$

where l is the mean free path of the quasiparticles, r_m is the smaller of the two lengths: the lattice constant and the viscous length.[12]

The Eqs. (3), (6), (13), and (14) constitute a closed system describing all possible vortex vibrations in an incompressible liquid.[2)] We shall not give the complete dispersion equation for this system, because of its unwieldiness. Among its solutions is a solution corresponding to a transverse vortex wave whose spectrum now has, without allowance for damping, the form

$$\omega^2 = (2\Omega)^2\frac{p^2}{p^2+q^2} + \frac{\rho_s}{\rho}c_T^2 q^2. \tag{15}$$

In such a wave, in the limit as $\omega/2\Omega \to 0$, the normal and superfluid components, as well as the vortices, move with approximately the same velocity $\mathbf{v}_L$ that is transverse with respect to the wave vector $\mathbf{Q}$ and that lies in the xy plane. The small relative velocity between the vortices and the normal component gives rise to weak damping of the vibrations. For a Tkachenko wave ($p=0$) and small values of the parameter $E\omega/2\Omega$, we obtain

$$\frac{\mathrm{Im}\,\omega}{\omega} = -\frac{1}{2}\frac{\rho_n}{\rho}\left(\frac{\nu q^2}{\omega} + \frac{\omega}{E2\Omega}\right), \tag{16}$$

where

$$E = \frac{4\pi\nu}{\varkappa\ln(r_m/l)}.$$

Under normal conditions this attenuation is weak.

Vortex lattice vibrations giving rise to vortex displacement and flexure could, in principle, lead to an appreciable destruction of the order of the vortex lattice. But this does not occur, since, according to estimates based on the vibration spectrum (12), the ratio of the root-mean-square displacement of the points of the vortex filaments, $\overline{\Delta r} \sim \sqrt{aT/\rho\varkappa^2}$, to the intervortex distance a is sufficiently small.

2. VORTEX-LATTICE VIBRATIONS IN A COMPRESSIBLE LIQUID

Let us consider a compressible rotating superfluid at $T=0$ ($\rho_n=0$). Allowance for the compressibility implies allowance for the longitudinal degrees of freedom of the liquid. Let us retain in Eq. (1), averaged over the unit cell of the vortex lattice, only the longitudinal—to the vector $\mathbf{Q}$—terms. Solving the equation thus obtained simultaneously with the equation of continuity for the liquid, we find the values of the components of the superfluid velocity $\mathbf{v}_\parallel$ connected with the longitudinal degrees of freedom of the liquid:

$$v_{\parallel t} = 0, \quad v_{\parallel q} = \frac{i\omega 2\Omega}{\omega^2 - c^2Q^2}\frac{q^2}{Q^2}v_{Lt}, \quad v_{\parallel z} = \frac{i\omega 2\Omega}{\omega^2 - c^2Q^2}\frac{pq}{Q^2}v_{Lt}, \tag{17}$$

where the indices z, q, and t correspond respectively to the components along the z axis, the vector $\mathbf{q}$, and the vector $[\mathbf{\Omega}\times\mathbf{q}]$; and c is the velocity of sound.

The velocity $\mathbf{v}_\parallel$ should be added to the right-hand side of Eq. (3) for the vortex velocity, i.e.,

$$\mathbf{v}_L(\mathbf{R}_j) = \mathbf{v}_s(\mathbf{R}_j) + \mathbf{v}_\parallel, \tag{18}$$

where $\mathbf{v}_s(\mathbf{R}_j)$ is determined, as before, by the Eqs. (6), (7), and (10) for the incompressible liquid. Finally, we obtain for the two-dimensional vector u (see (5)) an equation that has solutions at frequencies satisfying the following dispersion equation:

$$\omega^2 - \left(2\Omega + \frac{\varkappa p^2}{4\pi}\ln\frac{r_n}{r_c}\right)\left(2\Omega\frac{\omega^2 - c^2p^2}{\omega^2 - c^2Q^2} + \frac{\varkappa p^2}{4\pi}\ln\frac{r_n}{r_c} + \frac{c_T^2q^2}{2\Omega}\right) = 0. \tag{19}$$

This equation has two solutions for ω^2, one of which corresponds at large Q to ordinary sound. Let us consider the case when $p=0$. One of the solutions to Eq. (19) corresponds to the Tkachenko wave:

$$\omega = c_T q[1 + (2\Omega)^2/c^2q^2]^{-1/2}. \tag{20}$$

Equation (20) shows that the compressibility essentially alters the spectrum of this wave at small $q \ll 2\Omega/c$, making it parabolic: $\omega = c_T c q^2/2\Omega$. The spectrum (20) for the Tkachenko waves differs from the spectrum obtained by Reatto.[13] In particular, according to Reatto, the compressibility leads to the damping of the Tkachenko waves. The incorrect results obtained by Reatto are due to the imperfectness of the model used by him. In this model the compressibility was allowed for by introducing a time lag between the displacement of some vortex and the corresponding—to the displacement—change in the superfluid velocity at some point of the liquid. In other words, in the formula (4) for $\mathbf{v}_s(\mathbf{R}_j)$ the value of $\mathbf{R}_k$ was taken not at the moment of time t, but at the moment $t - |\mathbf{R}_j - \mathbf{R}_k|/c$.

The wavelengths $\sim c/2\Omega$ at which the compressibility changes the spectrum of the Tkachenko waves are very long, and can be realized only in astrophysical objects. However, in this case another effect arising from the finite compressibility—a change in the density of the liquid due to the centrifugal forces—may turn out to be important, since at a distance $c/2\Omega$ from the axis of rotation the velocity of the liquid becomes equal to the velocity of sound c.

3. VORTEX-LATTICE VIBRATIONS IN A CYLINDRICAL VESSEL

Let us now proceed to the consideration of the boundary-value problem of vortex lattice vibrations for a superfluid contained in a cylindrical vessel of radius R and height L. We consider low frequencies $\omega \ll 2\Omega$. Therefore, only transverse vortex waves for $p \ll q$ can be excited. The oscillations of the liquid are assumed to be axisymmetric, i.e., all the variables depend only on r and z. Then to the dispersion equation (15) corresponds a velocity field in the form of a sum of the solutions $J_1(qr)e^{ipz}$ over all possible q and p, where the $J_1(qr)$ are Bessel functions. The indices q, t, and z used above now correspond to the radial and azimuthal components of the velocities, as well as to the components along the z axis. In the limit as $\omega/2\Omega \to 0$ the principal component of all the velocities is the azimuthal component. It is the same for the normal and superfluid velocities $\mathbf{v}_n$ and $\mathbf{v}_s$, as well as for the vortex velocity $\mathbf{v}_L$, and will be denoted by the quantity v_t, which is equal to $v_t = v_{nt} = v_{st} = v_{Lt}$. The remaining components are small in the limit as $\omega/2\Omega \to 0$.

Let us now derive the boundary conditions for the transverse vortex wave.

At the boundary of the liquid we have for the superfluid and normal components and the vortex displacements a whole set of boundary conditions that cannot be simultaneously satisfied by one transverse vortex wave. This means that along with it should arise other types of waves allowed by the dispersion relation for the vibration equations for a given vibration frequency ω. However, all the types of waves, except the transverse vortex wave, have complex wave vectors at low frequencies $\omega \ll 2\Omega$ and attenuate rapidly with distance from the boundaries. Therefore, their presence can be allowed for by the proper choice of an effective boundary condition for the transverse vortex wave. Let us derive such a boundary condition first for the bottom of the vessel (the plane $z = 0$). There the following boundary conditions, which we give for the case $\rho_n \ll \rho$, should be fulfilled:

$$v_{sz} = -\frac{2\Omega}{i\omega}\frac{1}{\hat{Q}^2}\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial}{\partial r}(r v_{Lt}) = 0, \tag{21}$$

$$\frac{\partial \mathbf{v}_L}{\partial z} - b(\mathbf{v}_L - [\mathbf{\Omega}_c \times \mathbf{r}]) + b'\left[\frac{\mathbf{\Omega}}{\Omega} \times (\mathbf{v}_L - [\mathbf{\Omega}_c \times \mathbf{r}])\right] = 0, \tag{22}$$

$$\mathbf{v}_n|_T - [\mathbf{\Omega}_c \times \mathbf{r}] = 0, \tag{23}$$

where Ω_c is the amplitude of the oscillations of the angular velocity of the vessel, $\mathbf{v}_n|_T$ is the tangential component of the normal velocity at the bottom of the vessel, and the action of the operator $\hat{Q}^2 = \hat{p}^2 + \hat{q}^2$ amounts to the multiplication of each of the terms of $J_0(qr)e^{ipz}$ in the expansion in terms of cylindrical waves by the number $p^2 + q^2$. Therefore, the operator $\hat{Q}^{-2}\partial/\partial z$ is the operator $\hat{q}^{-2}\partial/\partial z$ for the transverse vortex wave if $p \ll q$ and the operator $-\int dz$ for the rest of the waves arising at the bottom of the vessel and for which $p \gg q$. The relation, used in (21), between the normal—to the surface of the bottom—component, v_{sz}, of the superfluid velocity and the azimuthal velocity v_{Lt} follows from two equations: the equation of continuity for the superfluid, connecting v_{sz} and v_{sq}, and the Eq. (1), averaged over the unit cell of the vortex lattice, in which only the transverse—to the wave vector Q—terms are retained.

The coefficients b and b' in Eq. (22) are different from zero if the vortices interact with the irregularities of the surface, i.e., if there is vortex pinning.[18] Such an interaction does not vanish as $\rho_n \to 0$.

In all, we have five boundary conditions, since in Eqs. (22) and (23) figure two-dimensional vectors in the plane perpendicular to the vortices. The dispersion equation for the lattice vibrations for given q and ω has five roots for p^2. One of them corresponds to a wave with the spectrum (15). Two roots, which are identical in the limit as $\omega/2\Omega \to 0$, correspond to two damped—in the liquid—longitudinal vortex waves:

$$p^2 = -k_l^2 = -\frac{8\pi\Omega}{\varkappa \ln(r_n/r_c)}. \tag{24}$$

They can be obtained from the dispersion equation (11).[3)] Finally, two more values of p^2 correspond to two viscous waves in the normal liquid:

$$p^2 = -\frac{2\Omega}{\nu}\frac{1}{E \pm i}. \tag{25}$$

Constructing from the five indicated waves a linear combination that satisfies all the five boundary conditions (21)–(23), we find that the transverse vortex wave, which is the only wave that is not appreciably damped as it propagates away from the surface, should satisfy the following effective boundary condition:

$$\alpha(q)\frac{\partial v_t}{\partial z} - (v_t - \Omega_c r) = 0, \tag{26}$$

where

$$\alpha(q)=\frac{k_t}{q^2}\left(b+k_t+\frac{b'^2}{b+k_t}\right)$$
$$\times\left[b+\frac{b'^2}{b+k_t}-\frac{\rho_n}{\rho_s}\frac{i\omega}{2\Omega}\left(\frac{\Omega}{\nu}\right)^{1/2}E^2\left(\frac{\sqrt{E^2+1}-E}{E^2+1}\right)^{1/2}\right]^{-1}. \quad (27)$$

On the free surface (the plane $z=L$) the two factors responsible for the pinning of the vortices to the surface (pinning proper for $b, b'>0$ and the interaction with the normal component, which are responsible for the appearance of terms $\sim\rho_n/\rho$ in (27)) are absent, and, therefore, the effective boundary condition has a simpler form:

$$\partial v_t/\partial z=0. \quad (28)$$

Let us now derive the boundary condition at the lateral walls. In order to consider all the types of waves arising at the walls, we must find all the solutions to the dispersion equation, treating it as an equation for q^2 for given ω and p. There are four of such solutions: two of them are transverse vortex waves (see Eq. (12)) and the other two are viscous waves with wave numbers

$$q_1^2=-\frac{2\Omega}{\nu}\frac{E}{E^2+1},\quad q_2^2=\frac{i\omega}{\nu}. \quad (29)$$

There exist four boundary conditions for these solutions. Two conditions for the adhesion of the normal component to the walls are similar to (23). The remaining two conditions, which are for the superfluid component, are:

$$v_{sq}=\frac{2\Omega}{i\omega}\frac{1}{q^2}\frac{\partial^2 v_{Lt}}{\partial z^2}=0, \quad (30)$$
$$\frac{\partial v_{Lt}}{\partial r}-\frac{v_{Lt}}{R}=0. \quad (31)$$

Undamped in the linear combination constructed from the four waves are the two transverse vortex waves, whose sum satisfies two boundary conditions, one of which has the same form as (30), while the second is:

$$\alpha_c\left(\frac{\partial v_t}{\partial r}-\frac{v_t}{R}\right)-(v_t-\Omega_cR)=0, \quad (32)$$

where

$$\alpha_c=\begin{cases}\dfrac{\varkappa}{16\pi}\dfrac{\rho}{\rho_n}\dfrac{1}{i\omega}\left[\dfrac{2\Omega(E^2+1)}{\nu E}\right]^{1/2}, & E\dfrac{\omega}{2\Omega}\ll 1,\\[2ex] \dfrac{\varkappa}{8\pi(1+i)}\dfrac{\rho}{\rho_n}\dfrac{\Omega}{\omega^2}\left[\dfrac{2\omega}{\nu}\right]^{1/2}, & E\dfrac{\omega}{2\Omega}\gg 1.\end{cases}$$

Below we shall solve the problem for the case when $\alpha_c=0$. The transverse-vortex-wave velocity field that satisfies the boundary conditions at the free surface and at the walls (Eqs. (26), (28), (30), and (32)) can be represented in the form

$$v_t(r,z)=\Omega_cR\left\{\frac{J_1(qr)}{J_1(qR)}+\sum_{n=1}^{\infty}A_nJ_1(q_nr)\frac{\cos(p_n(z-L))}{\cos(p_nL)}\right\}. \quad (33)$$

The first term, which is a Tkachenko wave, satisfies the boundary condition (32) for $\alpha_c=0$. The condition (30) is satisfied identically at $p=0$. The sum over n in (33) is a general solution to the problem with homogeneous boundary conditions (i.e., in (32), $\Omega_cR=0$). Therefore, the wave numbers q_n are determined from the condition $J_1(q_nR)=0$.[4)] The numbers p_n are chosen such that all the terms in (33) correspond to waves of one and the same frequency, i.e.,

$$\omega^2=c_T^2q^2=(2\Omega)^2\frac{p_n^2}{q_n^2}+c_T^2q_n^2. \quad (34)$$

The coefficients A_n in (33) are chosen so as to also satisfy the effective boundary condition (27) at the bottom of the vessel. Using the orthogonality condition for the functions $J_1(q_nr)$, we obtain that

$$A_n=\frac{2q^2}{q^2-q_n^2}\frac{1}{q_nRJ_2(q_nR)}\frac{1}{1-\alpha(q_n)p_n\operatorname{tg}p_nL}. \quad (35)$$

The equation determining the frequencies of the lattice vibrations in the vessel is found from the condition for the conservation of angular momentum during the period of the vibrations:

$$\frac{J_2(qR)}{qRJ_1(qR)}+\sum_{n=1}^{\infty}\left(\frac{2\operatorname{tg}(p_nL)}{p_nL(1-\alpha(q_n)p_n\operatorname{tg}(p_nL))}\frac{1}{(q_nR)^2}\frac{q^2}{q^2-q_n^2}\right)+\frac{\beta}{4}=0, \quad (36)$$

where β is the ratio of the moment of inertia of the vessel to the moment of inertia, $\pi\rho LR^4/2$, of the liquid.

Ruderman's formula is obtainable from (36) if $\beta=0$ and either $L\to\infty$, or $\alpha\to\infty$, i.e., either the vessel is very long, or there is no cohesive force that pins the vortices to the bottom of the vessel. In the other limiting case, when $L\to 0$, the velocity ceases to vary along the z axis, and the sum over n in (33) is the expansion of the velocity $\Omega_c(r-RJ_1(qr)/J_1(qR))$ in terms of the orthogonal functions $J_1(q_nr)$. Determining the angular momentum for such a velocity field, and using its expansion in terms of the Bessel functions $J_1(q_nr)$, we obtain the exact relation:

$$\frac{1}{4}-\frac{J_2(qR)}{qRJ_1(qR)}=\sum_n\frac{2q^2}{(q_nR)^2(q^2-q_n^2)}. \quad (37)$$

With the aid of this relation, the condition (36) can be rewritten in the following form:

$$\frac{1+\beta}{4}+\sum_{n=1}^{\infty}\gamma_n\frac{2q^2}{(q_nR)^2(q^2-q_n^2)}=0, \quad (38)$$

where

$$\gamma_n=\frac{\operatorname{tg}(p_nL)}{p_nL(1-\alpha(q_n)p_n\operatorname{tg}(p_nL))}-1, \quad (39)$$

or, if $p_nL\ll 1$,

$$\gamma_n=\frac{\alpha(q_n)p_n^2L}{1-\alpha(q_n)p_n^2L}. \quad (40)$$

According to (27), $\alpha(q)=A/q^2$, where A does not depend on q. Then the relation (37) allows us to sum the series in (38) for $p_nL\ll 1$ and obtain, as a result, the following condition for the eigenfrequencies:

$$\frac{1+\beta}{4}-\frac{q^2}{q^2-u^2}\left(\frac{1}{4}-\frac{J_2(R\sqrt{q^2-u^2})}{R\sqrt{q^2-u^2}J_1(R\sqrt{q^2-u^2})}\right)=0, \quad (41)$$

where

$$u=2\Omega/c_T\sqrt{AL}.$$

For $p_n L \ll 1$, we can also sum the series in the formula (33), so that in the volume the velocity of the liquid is determined by the expression

$$v_t(r,z)=\Omega_c\left[r-\frac{q^2}{q^2-u^2}\left(r-R\frac{J_1(r\sqrt{q^2-u^2})}{J_1(R\sqrt{q^2-u^2})}\right)\right]. \quad (42)$$

It follows from (41) that, as AL decreases in the case when the inequality $u \gg 1/R$ is fulfilled, the fundamental frequency approaches from above the value

$$\omega_0=c_T u=2\Omega/\sqrt{AL}. \quad (43)$$

Simultaneously with this decreases the spacing between the low-lying eigenfrequencies, so that the difference between the fundamental and the next eigenfrequency becomes equal to

$$\Delta\omega=\frac{c_T}{2uR^2}(x_2^2-x_1^2)=3{,}7\frac{c_T^2}{\omega_0 R^2}, \quad (44)$$

where $x_1 = 6.38$ and $x_2 = 9.76$ are the two smallest roots of the equation

$$^1/_4-J_2(x)/xJ_1(x)=0.$$

Another distinctive feature of the $u \gg 1/R$ case is the fact that, according to (42), the velocity in the volume can by far exceed the liquid velocity at the lateral walls, this velocity coinciding with the linear velocity of the walls themselves.

The formula (43) for the fundamental frequency remains valid in the limit as $ALa/R \to 0$ and for nonzero values of α_c in the boundary condition (32) at the walls. Into the equation, which is a generalization of Eq. (38) to the $\alpha_c \neq 0$ case, also enter the coefficients γ_n. For $AL \to 0$, only a fairly high frequency, at which $\gamma_n \to 1$, can be a root of such an equation. This condition leads to (43).

Let us now turn to Tsakadze's experiments[10] ($T = 1.46$ °K, $\Omega = 6$ rad/sec, $R = 3.2$ cm, and $L = 5$ cm). In this case the error introduced by the assumptions, $p_n L \ll 1$ and $\alpha_c = 0$, made in deriving (41) and (42) is small, since, according to estimates,

$$\frac{(p_n L)^2}{3}<\frac{1}{12}\frac{\omega^4 L^2}{c_T^2(2\Omega)^2}=0.12, \quad \alpha_c R=0.05.$$

In the experiment were observed damped vibrations corresponding to the complex frequency $\omega = (0.2 - 0.025i)$ sec^{-1}, which significantly exceeds in absolute value the theoretical value given by Ruderman's formula.[9] Besides this, as Dzh. S. and S. Dzh. Tsakadze have informed us, it was observed in the experiment that the vibration frequency is independent of the radius of the cylinder, which is in agreement with the formula (43); in a number of cases there also arose in the vibrations beats with frequency close in order of magnitude to the values given by the formula (44). All this indicates that the $u \gg 1/R$ (i.e., the small AL) case was realized in the experiments in question, and with the aid of the formula (43) we can determine the values of A corresponding to the experimentally observed frequencies. Equation (27) determines the connection between the quantity $A = \alpha(q)q^2$ and the coefficients b and b', which characterize the interaction of the vortices with the bottom of the vessel. It can be verified that the contribution made by the normal component to this effect is small (the terms $\sim \rho_n/\rho$ in the formula (27)); furthermore, it is usually assumed in the interpretation of experiments on the interaction of vortices with the surface of a vessel that $b' = 0$.[1,19] Then we obtain from the formula (27) that the value $b \approx (25 - 6i)$ cm^{-1} corresponds to the frequency observed by Tsakadze,[10] whereas according to Hall's experiments[1] $b \approx -100i$ cm^{-1}. Gamtsemlidze *et al.*[19] have determined the imaginary contribution to the coefficient b that is proportional to ω, i.e., according to their experiments, $b = -i\omega/a$, where $a = 0.1$ cm/sec, which gives for the vibration frequency in Tsakadze's experiments the value $b = -2i$ cm^{-1}.

Thus, the previously observed values of the coefficient b can fully explain the vibration frequencies of the cylinder that were found to obtain in Tsakadze's experiments. On the other hand, the measurement of the eigenfrequency of the vibrations at small values of AL is of little use for the determination of the rigidity of the vortex lattice, i.e., for the determination of the velocity, c_T, of the Tkachenko waves, since for $AL \to 0$ the fundamental frequency does not, according to (43), depend on c_T at all. This means that in the spectrum, (12), of the transverse vortex waves excitable in the vessel, the first term is more important that the second, which is connected with the transverse rigidity of the triangular vortex lattice. A judgment can be made about the magnitude of the rigidity of the vortex lattice only from the frequency of the beats that can arise in an experiment (the formula (44)), or from the results of measurements of the liquid-velocity field inside the vessel (the formula (42)).

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1) Such a wave without allowance for the contribution made by the rigidity of the vortex lattice (i.e., with $c_T \approx 0$) has been considered by Hall.[1]

2) For $q = 0$ this system is equivalent to the set of hydrodynamic equations given in the paper[3] by Andronikashvili *et al.* if the coefficients α_n and β_n in them are chosen in accordance with the assertion made above about the magnitude and direction of the frictional force exerted by the normal component on the vortices (see the text after Eq. (3)).

3) The indicated three solutions for p^2 for given ω and q (one transverse and two longitudinal vortex waves) exhaust all the possible oscillations of a superfluid with $\rho_n = 0$ and at low ω. The more general spectrum (19), which allows for the compressibility of the liquid, does not give rise to new solutions. One of the two longitudinal vortex waves continuously joins

the acoustic branch $p^2=\omega^2/c^2$ as the frequency ω increases.

4)In the more general boundary-value problem in which $\alpha_c \neq 0$, we should have in Eq. (33) in place of the single Bessel function $J_1(q_n r)$ the combination of two Bessel functions $AJ_1(q_n r) + BJ_1(q_n' r)$, where $q_n'^2 = q^2 - q_n^2$ is the second root of the biquadratic equation (34) for q_n^2. However, for $\alpha_c = 0$ such a combination satisfies the homogeneous boundary conditions (30) and (32) for $\Omega_c R = 0$ only if A or B vanishes.

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Electric and magnetic properties of the organic metal TSeF-TCNQ

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The temperature dependence of the conductivity in the microwave region and the magnetic susceptibility in the 1.5–300°K temperature interval are studied for the organic metal tetraselenafulvalene-tetracyanoquinodimethan (TSeF-TCNQ). The conductivity is maximal at temperatures 35–40°K at which it is from 7 to 20 times greater than at 300°K. The dielectric constant is $3\cdot 10^4$ at $T=4.2$°K. The gap calculated from the conductivity at low temperatures is identical with that derived from the susceptibility data. At high temperatures the resistivity can satisfactorily be described by a quadratic function of T.

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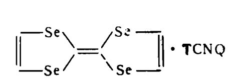
Great interest has been advanced recently in the study of organic salts based on tetracyanoquinodimethan (TCNQ), which have properties of quasi-one-dimensional metals.[1,2] These properties are determined principally by the composition and symmetry of the cation. It can now be regarded as established that such metallic properties are possessed by TCNQ salts of two types: the first type[3–7] are TCNQ salts with asymmetric cation, in which the Peierls instability of the one-dimensional metallic state is suppressed by the structural disorder, and the second type[8–11] comprises TCNQ salts with symmetrical cation, having conductivity along the cation and anion stacks. The suppression of the Peierls instability in salts of this type is apparently greatly weakened by the fact that the Peierls distortions are not commensurate with the period of the initial lattice, a situation resulting from the unequal distribution of the electrons among the cation and anion stacks.

Studies of TCNQ salts of the second type are presently diligently pursued, the symmetrical cation employed being tetrathiofulvalene (TTF)[10,11] and tetrathiatetracene.[8,9,12] In these complexes, the metallic state is stabilized in a wide temperature interval. The dc conductivity of the salt tetraselenafulvalene–tetracyanoquinodimethan (TSeF–TCNQ), which is isostructural with the salt TTF–TCNQ,[13] was recently investigated. In this study we investigate the temperature dependence of the conductivity of the same salt in the microwave band, as well as its magnetic susceptibility. The chemical formula of the salt is



The temperature dependence of the conductivity was measured at a frequency 10^{10} Hz by the contactless method described in[14], in the temperature interval 4.2–300 °K. For the conductivity measurements we