Diffusion-like processes in the quantum theory of scattering

A. I. Baz'

(Submitted July 25, 1975)
Zh. Eksp. Teor. Fiz. **70**, 397–410 (February 1976)

A model quantum system with a very large number of decay channels is considered. The concept of diffusion in channel space is introduced and the conditions under which the concept can be valid are formulated. Expressions are obtained for the wave function of the considered system and for the S matrix. The S matrix is represented as a product of factors, one of which describes diffusion in channel space and the others are of a kinematic origin (penetrability in the entrance and exit channels). Quasistationary states of the system are investigated. It is shown that in addition to the well known types of resonance there exist states of a completely new type in which the system spends most of its time by wandering from one channel to another.

PACS numbers: 03.65.Nk, 03.80.+r

In a number of problems of nuclear physics, such as nuclear fission and collision of medium and heavy particles with one another, we encounter the case of quantum systems with a very large number of decay channels, many of which, to be sure, may be closed at a given system energy. When such systems are analyzed, it turns out that under sufficiently general assumptions it is possible to introduce the concept of "diffusion in channel space." It is then possible to separate in the fundamental characteristic of such a system—the S matrix—factors, some of which are of purely kinematic origins (penetrabilities), and others can be calculated with the aid of a certain integral equation, which in a number of cases can be reduced to a differential equation of the diffusion type. We consider below a model problem the solution of which, first, explains the exact meaning of what was called "diffusion in channel space" above, and second, it spells out the sufficient conditions under which the diffusion concept can be used.

Equations (1a) and (1b) of our model are a simplified variant of the system of equations describing a system of A nucleons and obtained in[1] from an analysis of the many-nucleon Schrödinger equation. The mathematical formalism that must be used, and certain general results of the work, are unusual from the point of view of standard and (or) atomic physics. We therefore present in this article only a general treatment. The application of the general results obtained below to concrete problems, such as fission or collision of heavy nuclei, is the subject of a separate investigation.

1. EQUATIONS OF THE MODEL PROBLEM

We consider a system with a large number of bound one-dimensional equations of the following type:

$$(\hat{h}_0-E)\psi(\rho)=\sum_{\varkappa}D_{\varkappa}(\rho)\varphi_{\varkappa}(\rho), \tag{1a}$$

$$(\hat{h}_{\varkappa}-\varepsilon_{\varkappa})\varphi_{\varkappa}(\rho)=\sum_{\varkappa'}V_{\varkappa\varkappa'}(\rho)\varphi_{\varkappa'}(\rho)+D_{\varkappa}(\rho)\psi(\rho). \tag{1b}$$

The subscripts $\varkappa$ and $\varkappa'$ are used here to number the channels,

$$\hat{h}_0\equiv-\frac{h^2}{2m_0}\frac{d^2}{d\rho^2}+V_0(\rho),\quad \hat{h}_{\varkappa}\equiv-\frac{h^2}{2m_{\varkappa}}\frac{d^2}{d\rho^2}+V_{\varkappa}(\rho). \tag{2}$$

The potential $V_0(\rho)$ will be assumed to be such that the homogeneous equation (1a) has only a discrete spectrum:

$$(\hat{h}_0-E_\lambda)\psi_\lambda(\rho)=0;\quad \int d\rho\psi_\lambda(\rho)\psi_{\lambda'}(\rho)=\delta_{\lambda\lambda'}. \tag{3}$$

The potentials $V_{\varkappa}(\rho)$ will be assumed to be of the form shown in the figure: repulsion (Coulomb and centrifugal) at $\rho\geqslant a_{\varkappa}$ and more or less arbitrary behavior in the interval $0\leqslant\rho<a_{\varkappa}$. It is presently assumed[2] that at $\rho<a_{\varkappa}$ the potential between the nuclei can have one of the forms marked by the numbers 1 and 2 in the figure. Then $V_{\varkappa}(\infty)=0$. The quantities in (1b) are equal to

$$\varepsilon_{\varkappa}=E-E_{\varkappa}, \tag{4}$$

where $E_{\varkappa}$ are certain constants. This channel is open if $\varepsilon_{\varkappa}>0$, and closed if $\varepsilon_{\varkappa}<0$. The functions $D_{\varkappa}(\rho)$ and $V_{\varkappa\varkappa'}(\rho)$ are assumed to be real, while the second of them is assumed also to be symmetrical with respect to the indices $\varkappa$ and $\varkappa'$. To make the exposition that follows as lucid as possible, we assume that

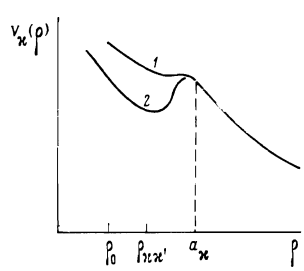
$$D_{\varkappa}(\rho)=\eta_{\varkappa}\delta(\rho-\rho_0),\quad V_{\varkappa\varkappa'}(\rho)=v_{\varkappa\varkappa'}\delta(\rho-\rho_{\varkappa\varkappa'}), \tag{5}$$

where ρ_0 and $\rho_{\varkappa\varkappa'}$ are constants.

We now formulate the assumptions that will help us later on to introduce the concept of diffusion in channel space:

a) the channel indices $\varkappa$ can be renumbered in such a way that all the quantities that depend on $\varkappa$, namely $E_{\varkappa}$, $V_{\varkappa}(\rho)$, $\eta_{\varkappa}$, $a_{\varkappa}$, $v_{\varkappa\varkappa'}$, $m_{\varkappa}$ vary little on going vary little on going from $\varkappa$ to the neighboring channel $\varkappa\pm1$;

b) the quantities $\eta_{\varkappa}$ and $v_{\varkappa\varkappa'}$ which enter in (5) tend rapidly enough to zero with increasing $\varkappa$ and $|\varkappa-\varkappa'|$, respectively.



The first of these assumptions enables us later on to regard the index $\varkappa$ not as a discrete quantity but as a continuous one. The second assumption ensures "locality" of the entire problem in the space of the continuous variable $\varkappa$.

2. EXPRESSION FOR THE S MATRIX

We shall assume that $E>0$ and that a large number of channels is open at this energy. We solve the corresponding problem of the continuous spectrum. We introduce for this purpose the auxiliary functions $G_0(\rho, \rho')$ and $G_\varkappa(\rho, \rho')$, which are the Green's functions of the homogeneous equations (1a) and (1b):

$$(\hat{h}_0-E)G_0(\rho,\rho')=\delta(\rho-\rho'),\quad G_0(\rho,\rho')=\sum_\lambda\frac{\psi_\lambda(\rho)\psi_\lambda(\rho')}{E_\lambda-E}; \tag{6}$$

$$(\hat{h}_\varkappa-\varepsilon_\varkappa)G_\varkappa(\rho,\rho')=\delta(\rho-\rho'),$$

$$G_\varkappa(\rho,\rho')=\frac{2m_\varkappa}{\hbar^2k_\varkappa}\chi_\varkappa(\rho_<)\varphi_\varkappa^{(+)}(\rho_>). \tag{7}$$

Here $\chi_\varkappa(\rho)$ and $\varphi_\varkappa^{(+)}(\rho)$ are the regular and irregular solutions of the equation

$$(\hat{h}_\varkappa-\varepsilon_\varkappa)\begin{pmatrix}\chi_\varkappa(\rho)\\ \varphi_\varkappa^{(+)}(\rho)\end{pmatrix}=0 \tag{8}$$

with asymptotic behavior

$$\chi_\varkappa(\rho)\sim\sin(k_\varkappa\rho+\delta_\varkappa),\quad \varphi_\varkappa^{(+)}(\rho)\sim\exp[i(k_\varkappa\rho+\delta_\varkappa)];$$

$$k_\varkappa=(2m_\varkappa\varepsilon_\varkappa)^{1/2}/\hbar. \tag{9}$$

Assuming that the entrance channel is $\bar{\varkappa}$ and using Eqs. (5)–(9) we obtain the solution of the system of equations (1):

$$\psi^{(\bar{\varkappa})}(\rho)=G_0(\rho,\rho_0)\sum_\varkappa\eta_\varkappa\varphi_\varkappa^{(\bar{\varkappa})}(\rho_0), \tag{10}$$

$$\varphi_\varkappa^{(\bar{\varkappa})}(\rho)=\delta_{\varkappa\bar{\varkappa}}\chi_{\bar{\varkappa}}(\rho)+\sum_{\varkappa'}\{G_\varkappa(\rho,\rho_{\varkappa'})v_{\varkappa\varkappa'}\varphi_{\varkappa'}^{(\bar{\varkappa})}(\rho_{\varkappa'})+G_\varkappa(\rho,\rho_0)\xi_{\varkappa\varkappa'}\varphi_{\varkappa'}^{(\bar{\varkappa})}(\rho_0)\}, \tag{11}$$

where $\xi_{\varkappa\varkappa'}\equiv G_0(\rho_0,\rho_0)\eta_\varkappa\eta_{\varkappa'}$.

To simplify the formulas further, we assume that, by virtue of the rapid damping of $v_{\varkappa\varkappa'}$ with increasing $|\varkappa-\varkappa'|$, the constants $\rho_{\varkappa\varkappa'}$ in the first term in the curly brackets can be set equal to

$$\rho_{\varkappa\varkappa'}=\rho_\varkappa=\rho_{\varkappa'}$$

and this term is written in the form[1)] $G_\varkappa(\rho,\rho_\varkappa)v_{\varkappa\varkappa'}\varphi_{\varkappa'}^{(\bar{\varkappa})}\times(\rho_{\varkappa'})$. Since $\xi_{\varkappa\varkappa'}$ also attentuates rapidly with increasing $\varkappa$ and $\varkappa'$, it follows that the second term can be expressed as $G_\varkappa(\rho,\rho_\varkappa)\xi_{\varkappa\varkappa'}\varphi_{\varkappa'}^{(\bar{\varkappa})}(\rho_{\varkappa'})$. Taking these simplifications into account, we can rewrite Eq. (10) in the form

$$\varphi_\varkappa^{(\bar{\varkappa})}(\rho)=\delta_{\varkappa\bar{\varkappa}}\chi_{\bar{\varkappa}}(\rho)+G_\varkappa(\rho,\rho_\varkappa)\sum_{\varkappa'}\omega_{\varkappa\varkappa'}\varphi_{\varkappa'}^{(\bar{\varkappa})}(\rho_{\varkappa'}), \tag{10'}$$

where the matrix

$$\omega_{\varkappa\varkappa'}\equiv v_{\varkappa\varkappa'}+\xi_{\varkappa\varkappa'} \tag{12}$$

is real and symmetrical in the indices $\varkappa$ and $\varkappa'$.

Thus, the solution $\varphi_\varkappa^{(\bar{\varkappa})}(\rho)$ is determined in all of space if we know the values of the functions $\varphi_{\varkappa'}^{(\bar{\varkappa})}$ at the points $\rho=\rho_{\varkappa'}$. To determine these values, we put $\rho=\rho_\varkappa$ in (10'). We then obtain for the determination of these functions the system of linear equations

$$\varphi_\varkappa^{(\bar{\varkappa})}=\delta_{\varkappa\bar{\varkappa}}\chi_{\bar{\varkappa}}+G\sum_{\varkappa'}\omega_{\varkappa\varkappa'}\varphi_{\varkappa'}^{(\bar{\varkappa})}. \tag{10''}$$

Here and below

$$\varphi_\varkappa^{(\bar{\varkappa})}\equiv\varphi_\varkappa^{(\bar{\varkappa})}(\rho_\varkappa),\quad \chi_{\bar{\varkappa}}\equiv\chi_{\bar{\varkappa}}(\rho_{\bar{\varkappa}}),\quad G_\varkappa\equiv G_\varkappa(\rho_\varkappa,\rho_\varkappa). \tag{13}$$

It follows from (10'') that the solution should be of the form

$$\varphi_\varkappa^{(\bar{\varkappa})}=(\delta_{\varkappa\bar{\varkappa}}+G_\varkappa A_{\varkappa\bar{\varkappa}})\chi_{\bar{\varkappa}}, \tag{14}$$

where the symmetrical matrix $A_{\varkappa\bar{\varkappa}}$ is a solution of the equation

$$A_{\varkappa\bar{\varkappa}}=\omega_{\varkappa\bar{\varkappa}}+\sum_{\varkappa'}\omega_{\varkappa\varkappa'}G_{\varkappa'}A_{\varkappa'\bar{\varkappa}}. \tag{15}$$

Formulas (7)–(10) determine completely the asymptotic behavior of the functions $\varphi_\varkappa^{(\bar{\varkappa})}(\rho)$. Calculating the coefficients of the diverging waves in the different channels, we easily obtain an expression for the S matrix:

$$S_{\varkappa\bar{\varkappa}}=\exp[i(\delta_\varkappa+\delta_{\bar{\varkappa}})]\{\delta_{\varkappa\bar{\varkappa}}+2i(G_2(\varkappa))^{1/2}A_{\varkappa\bar{\varkappa}}(G_2(\bar{\varkappa}))^{1/2}\}. \tag{16}$$

The phases $\delta_\varkappa$ are determined here by Eqs. (8) and (9), while $G_2(\varkappa)=\mathrm{Im}G_\varkappa$. The S matrix defined by (16) is symmetrical and unitary, so long as the matrix $\omega_{\varkappa\varkappa'}$ is symmetric and real.

The structure of the formula for the S matrix is quite simple. The phase factors in front of the curly brackets and the Kronecker symbol $\delta_{\varkappa\bar{\varkappa}}$ inside the curly brackets need no explanation. As to the second term in the curly brackets the factors $(G_2(\varkappa))^{1/2}$ and $(G_2(\bar{\varkappa}))^{1/2}$ are of pure kinematic origin, and since

$$G_2(\varkappa)=\frac{2}{\hbar v_\varkappa}\begin{cases}\chi_\varkappa^2 & \text{if}\quad \varepsilon_\varkappa>0\\ 0 & \text{if}\quad \varepsilon_\varkappa<0\end{cases};\qquad v_\varkappa=\left(\frac{2}{m_\varkappa}\varepsilon_\varkappa\right)^{1/2},$$

it follows that $G_2(\varkappa)$ characterizes the penetrability of the potential $V_\varkappa(\rho)$, and consequently describes the process of motion of the particles of the $\varkappa$-th channel from the point $\rho=\rho_\varkappa$ to the outside (or from infinity to the point $\rho=\rho_\varkappa$).

The entire complicated dynamics of the processes that occur at short distances, i.e., the transitions from channel to channel, the possible capture of one of the bound states (i.e., the coupling of Eq. (1b) with Eq. (1a)), etc., is described by the matrix $A_{\varkappa\bar{\varkappa}}$. It is precisely this matrix which describes the "wandering" in the channel space, and it is just for this matrix that we shall show below that, under the conditions formulated in Sec. 1, it describes diffusion in channel space and can be obtained as a solution of the corresponding diffusion equation.

3. TRANSITION TO CONTINUOUS CHANNEL SPACE

Using the assumption (a) made in Sec. 1, we shall now regard the index $\varkappa$ of the channel as a continuous quantity. All the formulas of the preceding section remain in force in this case. It is necessary only to replace all the sums over $\varkappa$ by integrals $\int d\varkappa$, and the

Kronecker symbol $\delta_{\varkappa\bar{\varkappa}}$ by the δ function $\delta(\varkappa-\bar{\varkappa})$. The indices $\varkappa$ and $\varkappa'$ are now arguments of functions and, as is customary, we shall write them in parentheses, for example $A(\varkappa, \bar{\varkappa})$ instead of $A_{\varkappa\bar{\varkappa}}$.

The entire exposition that follows is devoted to the investigation of an S matrix that is a function of two points $\varkappa$ and $\bar{\varkappa}$ in the space of the channels:

$$S(\varkappa, \bar{\varkappa})=\exp[i(\delta(\varkappa)+\delta(\bar{\varkappa}))]\{\delta(\varkappa-\bar{\varkappa}) +2i(G_2(\varkappa))^{1/2}A(\varkappa, \bar{\varkappa})(G_2(\bar{\varkappa}))^{1/2}\}, \tag{17}$$

where $A(\varkappa, \bar{\varkappa})$ satisfies the equation

$$A(\varkappa, \bar{\varkappa})=\omega(\varkappa, \bar{\varkappa})+\int d\varkappa'\,\omega(\varkappa, \varkappa')G(\varkappa')A(\varkappa', \bar{\varkappa}), \tag{18}$$

with

$$\omega(\varkappa, \varkappa')=v(\varkappa, \varkappa')+G_0\eta(\varkappa)\eta(\varkappa'), \tag{19}$$

$$G_0=\sum_\lambda \frac{\psi_\lambda^2(\rho_0)}{E_\lambda-E}. \tag{20}$$

It must be remembered that in (17) and (18) the phases $\delta(\varkappa)$ and $\delta(\bar{\varkappa})$ and the Green's function $G(\varkappa)$ depend on the energy (see (7)–(9)).

Equation (18) is transformed with the aid of the substitution

$$Q(\varkappa, \bar{\varkappa})=g(\varkappa)A(\varkappa, \bar{\varkappa})g(\bar{\varkappa}),\ K(\varkappa, \varkappa')=g(\varkappa)\omega(\varkappa, \varkappa')g(\varkappa'), \quad g(\varkappa)\equiv(G(\varkappa))^{1/2}\equiv g_1(\varkappa)+ig_2(\varkappa),\ g_1,\ g_2>0, \tag{21}$$

into an equation with a symmetrical complex kernel:

$$Q(\varkappa, \bar{\varkappa})=K(\varkappa, \bar{\varkappa})+\int d\varkappa'\,K(\varkappa, \varkappa')Q(\varkappa', \bar{\varkappa}). \tag{22}$$

The S matrix is expressed in terms of Q in the obvious manner:

$$S(\varkappa, \bar{\varkappa})=\exp[i(\delta(\varkappa)+\delta(\bar{\varkappa}))]\left\{\delta(\varkappa-\bar{\varkappa}) +2i\left(\frac{G_2(\varkappa)}{G(\varkappa)}\right)^{1/2}Q(\varkappa, \bar{\varkappa})\left(\frac{G_2(\bar{\varkappa})}{G(\bar{\varkappa})}\right)^{1/2}\right\}. \tag{17'}$$

Wherever it does not lead to misunderstanding, we shall use henceforth an abbreviated operator notation, in which Eq. (22), for example, takes the form

$$Q=K+KQ. \tag{22'}$$

The kernel K consists of two parts (see (12) and (21)):

$$K(\varkappa, \varkappa')=\bar{K}(\varkappa, \varkappa')+v(\varkappa)v(\varkappa'),\ \bar{K}\equiv g(\varkappa)v(\varkappa, \varkappa')g(\varkappa'); \quad v(\varkappa)\equiv g(\varkappa)\eta(\varkappa)G_0^{1/2}, \tag{23}$$

the first of which describes the direct coupling of the channels, and the second the coupling via the linking of Eqs. (1b) with Eq. (1a), which describes the "internal" states $\psi_\lambda(\rho)$ which are bound if $\eta_\varkappa\equiv 0$.

We separate in explicit form the dependence of the function Q on the properties of the "internal" states ψ_λ. We describe the separation method, since it will be systematically used below. We rewrite Eq. (22) in the form of a system of equations

$$Q(\varkappa, \bar{\varkappa})-\int d\varkappa'\,\bar{K}(\varkappa, \varkappa')Q(\varkappa', \bar{\varkappa})=\tau(\varkappa, \bar{\varkappa}), \tag{24a}$$

$$K(\varkappa, \bar{\varkappa})+v(\varkappa)\int d\varkappa'\,v(\varkappa')Q(\varkappa', \bar{\varkappa})=\tau(\varkappa, \bar{\varkappa}). \tag{24b}$$

We introduce further the resolvent $\bar{R}$ of the kernel $\bar{K}$:

$$\bar{R}=\bar{K}+\bar{K}\bar{R}=\bar{K}+\bar{R}\bar{K}. \tag{25}$$

We shall remember henceforth that the kernels K and $\bar{K}$, just as Q and $\bar{R}$, are symmetrical functions of their arguments.

It follows from (24a) that

$$Q(\varkappa, \bar{\varkappa})=\tau(\varkappa, \bar{\varkappa})+\int d\varkappa'\,\tau(\varkappa', \bar{\varkappa})\bar{R}(\varkappa, \varkappa'). \tag{26}$$

Substituting this expression for Q in (24b) and changing there the order of integration, we obtain for τ an equation with a degenerate kernel:

$$\tau(\varkappa, \bar{\varkappa})=K(\varkappa, \bar{\varkappa})+v(\varkappa)\int d\varkappa'\,v^*(\varkappa')\tau(\varkappa', \bar{\varkappa}), \quad v^*(\varkappa)\equiv v(\varkappa)+\int d\varkappa'\,\bar{R}(\varkappa, \varkappa')v(\varkappa'). \tag{27}$$

The solution of this equation is, obviously,

$$\tau(\varkappa, \bar{\varkappa})=K(\varkappa, \bar{\varkappa})+\frac{v(\varkappa)}{1-\langle vv^*\rangle}\int d\varkappa'\,v^*(\varkappa')K(\varkappa', \bar{\varkappa}), \tag{28}$$

where

$$\langle vv^*\rangle\equiv\int d\varkappa\,v(\varkappa)v^*(\varkappa).$$

Returning now to (26), we obtain after simple transformations, in which we use (25),

$$Q(\varkappa, \bar{\varkappa})=\bar{R}(\varkappa, \bar{\varkappa})+\frac{v^*(\varkappa)v^*(\bar{\varkappa})}{1-\langle vv^*\rangle}. \tag{29}$$

The first term describes here effects connected with the direct connection between the channels, and the second describes the effects connected with the connection via the internal states. All the functions that enter in (29) depend on the energy, since $G(\varkappa)$ and $g(\varkappa)$ are energy-dependent. At certain complex values of the energy the function Q, and consequently also the S matrix, has poles corresponding to resonances in the cross sections of all the processes $\bar{\varkappa}\to\varkappa$. The causes of the resonances may be different. They may occur when the energy E approaches the energy E_λ of one of the internal states ψ_λ; cases are also possible when the resonances are the result of direct interaction between channels, or else the result of the properties of only one preferred channel. All these cases will now be considered.

Resonances are as a rule particularly clearly pronounced when the energies $\varepsilon_\varkappa$ in all the open channels are not too large, so that all channels lie below the barrier. If $G_1(\varkappa)>0$, we have

$$G_1\gg G_2,\ g_1\gg g_2,\ \zeta(\varkappa)\equiv g_2(\varkappa)/g_1(\varkappa)\ll 1. \tag{30}$$

This is precisely the case of greatest interest, and will be investigated below.[2)] In expressions for K and $\bar{K}$ we can then neglect the terms of second order in $\zeta(\varkappa)$ and write

$$\bar{K}(\varkappa, \varkappa')=\bar{K}_1(\varkappa, \varkappa')+i\bar{K}_2(\varkappa, \varkappa'), \tag{31}$$

where

$$\bar{K}_1(\varkappa, \varkappa')=g_1(\varkappa)v(\varkappa, \varkappa')g_1(\varkappa'), \quad \bar{K}_2(\varkappa, \varkappa')=\zeta(\varkappa)\bar{K}_1(\varkappa, \varkappa')+\bar{K}_1(\varkappa, \varkappa')\zeta(\varkappa'). \tag{31'}$$

4. RESONANCES CONNECTED WITH THE INTERNAL INTERNAL STATES (RESONANCES OF THE FIRST TYPE)

Let the energy E be close to one of the values E_λ (see (3)). Then only one term predominates in G_0, and taking only this term into account we obtain (see (6))

$$G_0=\psi_\lambda^2/\varepsilon,\quad \varepsilon=E_\lambda-E,$$
$$v(\varkappa)v(\varkappa')=\mu(\varkappa)\mu(\varkappa')/\varepsilon,\ \mu(\varkappa)\equiv g(\varkappa)\eta(\varkappa)\psi_\lambda; \tag{32}$$
$$Q(\varkappa,\bar\varkappa)=\bar R(\varkappa,\bar\varkappa)+\frac{\mu^*(\varkappa)\mu^*(\bar\varkappa)}{\varepsilon-\langle\mu\mu^*\rangle}. \tag{29'}$$

The function μ^* is connected here with μ in the same manner as ν^* with ν (see (27)).

To calculate the denominator in (29'), we note that if we are not near some singularity of the resolvent R, then it follows from (25), (31), and (31') that in the first order in ζ we have

$$\bar R=\bar R_1+i\bar R_2,\ \bar R_1=\bar R_1+\bar K_1\bar R_1, \tag{33}$$

where $\bar R_1$ is the real (and symmetrical) resolvent of the kernel $\bar K_1$, and the real and symmetrical function $\bar R_2$ is equal to

$$\bar R_2(\varkappa,\varkappa')=\zeta(\varkappa)\bar R_1(\varkappa,\varkappa')+\bar R_1(\varkappa,\varkappa')\zeta(\varkappa')$$
$$+2\int d\varkappa''\,\bar R_1(\varkappa,\varkappa'')\zeta(\varkappa'')\bar R_1(\varkappa'',\varkappa'). \tag{33'}$$

With the aid of these formulas we can easily obtain

$$\varepsilon-\langle\mu\mu^*\rangle=\varepsilon-\Delta_\lambda-{}^1/_2 i\Gamma_\lambda, \tag{34}$$

where the shift Δ_λ and the width Γ_λ are given by

$$\Delta_\lambda=\int d\varkappa\,\mu_1^2(\varkappa)+\int d\varkappa\,d\varkappa'\,\mu_1(\varkappa)\bar R_1(\varkappa,\varkappa')\mu_1(\varkappa'),$$
$$\Gamma_\lambda=\int d\varkappa\,\theta_\lambda^2(\varkappa),\quad \theta_\lambda(\varkappa)=2\zeta^{1/2}(\varkappa)\mu_1^*(\varkappa), \tag{35}$$
$$\mu_1^*(\varkappa)\equiv \mathrm{Re}\,\mu^*(\varkappa)=\mu_1(\varkappa)+\int d\varkappa'\,\bar R_1(\varkappa,\varkappa')\mu_1(\varkappa'),$$
$$\mu_1(\varkappa)\equiv g_1(\varkappa)\eta(\varkappa)\psi_\lambda.$$

Taking (17), (21), (29), and (4) into account, we obtain ultimately for the S matrix an expression that is valid near the resonance

$$S(\varkappa,\bar\varkappa)=\exp[i(\delta(\varkappa)+\delta(\bar\varkappa))]\left\{\delta(\varkappa-\bar\varkappa)\right.$$
$$\left.+i\left[4\zeta^{1/2}(\varkappa)\bar R_1(\varkappa,\bar\varkappa)\zeta^{1/2}(\bar\varkappa)-\frac{\theta_\lambda(\varkappa)\theta_\lambda(\bar\varkappa)}{(E-E_\lambda+\Delta_\lambda)+{}^1/_2 i\Gamma_\lambda}\right]\right\}. \tag{36}$$

The first term in the square brackets describes the direct transitions: $2\zeta^{1/2}(\bar\varkappa)$ is the amplitude at which the particles of the initial channel $\bar\varkappa$ reach the point $\rho=\rho_{\bar\varkappa}$ at which transitions to other channels become possible. The transition from the point $\bar\varkappa$ in channel space to the point $\varkappa$ is described by the amplitude $\bar R_1(\varkappa,\bar\varkappa)$, and finally, $2\zeta^{1/2}(\varkappa)$ is the amplitude for the escape of particles of the $\varkappa$-th channel to the outside.

The resonant term in (36) can be treated in the following manner: first, by a gradual transition from channel to channel, we ultimately reach the internal state ($\bar\varkappa\to\psi_\lambda$), which then decays in the same manner ($\psi_\lambda\to\varkappa$). This is seen particularly clearly if the channels $\bar\varkappa$ and $\varkappa$ are located "far" from ψ_λ, so that $\eta(\bar\varkappa)=\eta(\varkappa)=0$ (there is no direct linking of the corresponding equations (1b) with Eq. (1a); the functions $\eta(\varkappa)$ differ from zero only at small $\varkappa$). In this case we have for $\theta_\lambda(\varkappa)$, for example (see (35)),

$$\theta_\lambda(\bar\varkappa)=2\zeta^{1/2}(\bar\varkappa)\int d\varkappa'\,\mu_1(\varkappa')\bar R_1(\varkappa',\bar\varkappa)\approx 2\zeta^{1/2}(\bar\varkappa)\beta\bar R_1(0,\bar\varkappa). \tag{37}$$
$$\beta\equiv\int d\varkappa'\,\mu_1(\varkappa').$$

The partial amplitude $\theta_\lambda(\bar\varkappa)$ factorizes into the amplitude $2\zeta^{1/2}(\bar\varkappa)$ at which the channel particles reach the point $\rho=\rho_\varkappa$, and the factor $\bar R_1(0,\bar\varkappa)$ that describes the motion in channel space from the point $\bar\varkappa$ to the point $\varkappa=0$ near which direct transitions, described by the factor β, to the internal state ψ_λ become possible.

5. RESONANCES DUE TO COUPLING BETWEEN CHANNELS (RESONANCES OF THE SECOND TYPE)

If all the channels were closed ($g_2(\varkappa)\equiv 0$), then the system of equations (1a) and (1b) could have, besides bound states close to ψ_λ, also bound states of a different type, close to the eigenfunctions of the system (1b), in which we put $D_\varkappa(\rho)\equiv 0$. If at least some of the channels become open, then the bound states are transformed into quasi-stationary states. In this section we consider S-matrix resonances that are precisely of this origin. The cluster (quasi-molecular) states are particular cases of such resonances.

The kernel K of Eq. (22') is now written in the following manner, with the real and imaginary parts separated:

$$K=K_1+iK_2,$$
$$K_2(\varkappa,\varkappa')=\zeta(\varkappa)K_1(\varkappa,\varkappa')+K_1(\varkappa,\varkappa')\zeta(\varkappa'). \tag{38}$$

The real symmetrical kernel K_1 can be, as is well known, expanded in an eigenfunction series[3]:

$$K_1(\varkappa,\varkappa')=\sum_n\frac{1}{\lambda_n}u_n(\varkappa)u_n(\varkappa');$$
$$u_n(\varkappa)=\lambda_n\int d\varkappa'\,K_1(\varkappa,\varkappa')u_n(\varkappa'), \tag{39}$$
$$\int d\varkappa\,u_n(\varkappa)u_m(\varkappa)=\delta_{nm}.$$

The quantities λ_n and $u_n(\varkappa)$ are functions of the energy E, since the kernel K_1 depends on the energy. Therefore we shall sometimes write, for example, $\lambda_n(E)$. We shall soon see that resonance is the result of that term of the expansion in (39), for which $\lambda_p\approx 1$. We separate this term explicitly:

$$K_1(\varkappa,\varkappa')=\frac{1}{\lambda_p}u_p(\varkappa)u_p(\varkappa')+\sum_{n\neq p}\equiv\frac{1}{\lambda_p}u_p(\varkappa)u_p(\varkappa')+\bar K_1(\varkappa,\varkappa'),$$
$$K(\varkappa,\varkappa')=\frac{1}{\lambda_p}u_p(\varkappa)u_p(\varkappa')+\bar K(\varkappa,\varkappa'), \tag{40}$$
$$\bar K(\varkappa,\varkappa')=\bar K_1(\varkappa,\varkappa')+iK_2(\varkappa,\varkappa').$$

Equation (22') for Q is solved by the same method as in Secs. 3 and 4, and we obtain

$$Q(\varkappa,\bar\varkappa)=\bar R(\varkappa,\bar\varkappa)+\frac{u_p^*(\varkappa)u_p^*(\bar\varkappa)}{\lambda_p-\langle u_p u_p^*\rangle}, \tag{41}$$

where $\tilde R=\tilde R_1+i\tilde R_2$ is the resolvent of the kernel $\tilde K$ ($\tilde R=\tilde K+\tilde K\tilde R$), and

$$u_p^*(\varkappa)=u_p(\varkappa)+\int d\varkappa'\,\tilde{R}(\varkappa,\varkappa')u_p(\varkappa'). \tag{42}$$

Taking into account the smallness of K_2 and the orthogonality of u_p to $\tilde{K}_1$ and $\tilde{R}_1$, we can easily express the imaginary part of the resolvent $\tilde{R}_2$ in terms of the real part (again confining ourselves to the first order in ζ):

$$\begin{aligned}\tilde{R}_2(\varkappa,\varkappa')&=\zeta(\varkappa)\tilde{R}_1(\varkappa,\varkappa')+\tilde{R}_1(\varkappa,\varkappa')\zeta(\varkappa')\\&+2\int d\varkappa''\,\tilde{R}_1(\varkappa,\varkappa'')\zeta(\varkappa'')\tilde{R}_1(\varkappa'',\varkappa')+\frac{1}{\lambda_p}\Big\{\zeta(\varkappa)u_p(\varkappa)u_p(\varkappa')\\&+u_p(\varkappa)u_p(\varkappa')\zeta(\varkappa')+\int d\varkappa''[\tilde{R}_1(\varkappa,\varkappa'')\zeta(\varkappa'')u_p(\varkappa'')u_p(\varkappa')\\&+u_p(\varkappa)u_p(\varkappa'')\zeta(\varkappa'')\tilde{R}_1(\varkappa'',\varkappa')]\Big\}.\end{aligned}$$

Here $\tilde{R}_1$ is the resolvent of the kernel $\tilde{K}_1$. Calculating next the denominator in (41) and retaining in the numerator only the terms that do not contain ζ, we obtain

$$Q(\varkappa,\bar{\varkappa})=\tilde{R}_1(\varkappa,\bar{\varkappa})+u_p(\varkappa)u_p(\bar{\varkappa})\Big/\Big[\lambda_p(E)-1+\frac{2i}{\lambda_p}\int d\varkappa\,\zeta(\varkappa)u_p^2(\varkappa)\Big]. \tag{43}$$

This formula shows that the position E_p of the resonance is determined by the condition $\lambda_p(E_p)=1$. Expanding $\lambda_p(E)$ in a series about this point

$$\lambda_p(E)-1=-\alpha(E-E_p)$$

and using (17) and (21), we obtain a final expression for the S matrix, which is valid near resonance:

$$\begin{aligned}S(\varkappa,\bar{\varkappa})&=\exp[i(\delta(\varkappa)+\delta(\bar{\varkappa}))]\Big\{\delta(\varkappa-\bar{\varkappa})\\&+i\Big[4\zeta^{1/2}(\varkappa)\tilde{R}_1(\varkappa,\bar{\varkappa})\zeta^{1/2}(\bar{\varkappa})-\frac{\theta_p(\varkappa)\theta_p(\bar{\varkappa})}{E-E_p+{}^1/_2i\Gamma_p}\Big]\Big\},\end{aligned} \tag{44}$$

$$\theta_p(\varkappa)\equiv\frac{2}{\alpha^{1/2}}\zeta^{1/2}(\varkappa)u_p(\varkappa),\quad \Gamma_p=\int d\varkappa\,\theta_p^2(\varkappa). \tag{44'}$$

6. RESONANCES CONNECTED WITH SOME SINGLE CHANNEL (RESONANCES OF THE THIRD TYPE)

The quantities $G_\varkappa$ in (15), regarded as functions of the energy E, can have poles corresponding to bound or quasistationary states of the homogeneous equation of the $\varkappa$-th channel:

$$(\bar{h}_\varkappa-\varepsilon_\varkappa)\chi_\varkappa(\rho)=0. \tag{45}$$

Let, for example, the energy E be such that we are located near a quasistationary state of the channel $\varkappa_1$ (its energy is $\varepsilon_{\varkappa_1 0}$, its width is $\Gamma_{\varkappa_1}$):

$$\varepsilon_{\varkappa_1}=E-E_{\varkappa_1}\approx\varepsilon_{\varkappa_1 0}-i\Gamma_{\varkappa_1}/2. \tag{46}$$

Then the Green's function $G_{\varkappa_1}$ admits, in the vicinity of this point, of the parametrization

$$G_{\varkappa_1}(E)=\frac{\Phi(\rho_{\varkappa_1})\Phi(\rho_{\varkappa_1})}{-E+E_{\varkappa_1 0}-{}^1/_2i\Gamma_{\varkappa_1}},\quad E_{\varkappa_1 0}\equiv\varepsilon_{\varkappa_1 0}+E_{\varkappa_1}. \tag{47}$$

Here $\Phi(\rho)$ is a regular solution of Eq. (45), normalized by the condition

$$\int d\rho\,\Phi^2(\rho)=1, \tag{48}$$

where the integration is carried out over the entire internal region from $\rho=0$ to the point where the particles of the $\varkappa_1$ channel go out from under the barrier. If we are dealing with a bound state, then we must put $\Gamma_{\varkappa_1}=0$ in (46) and (47).

In the transition from the system of algebraic equations (15) to the integral equation (18) it was assumed that the properties of all the quantities, including $G_\varkappa$, vary smoothly on going from one channel to the other. This condition is violated in the case of resonance in the channel $\varkappa_1$, and before we change over to a continuous distribution with respect to $\varkappa$ we must separate the channel $\varkappa_1$ in explicit form:

$$A_{\varkappa\bar{\varkappa}}=\omega_{\varkappa\bar{\varkappa}}+\omega_{\varkappa\varkappa_1}G_{\varkappa_1}A_{\varkappa_1\bar{\varkappa}}+\sum_{\varkappa'\neq\varkappa_1}\omega_{\varkappa\varkappa'}G_{\varkappa'}A_{\varkappa'\bar{\varkappa}}. \tag{49}$$

In the last term $G_{\varkappa'}$ can be regarded already as a smooth function of $\varkappa'$, and we can change over to the integral

$$\sum_{\varkappa'\neq\varkappa_1}\to\int d\varkappa'\,\omega(\varkappa,\varkappa')G(\varkappa')A(\varkappa',\bar{\varkappa}).$$

Here $G(\varkappa')$ must already be regarded as a smooth function that contains no singularities at all at $\varkappa'=\varkappa_1$. Carrying this program through to conclusion, we obtain for $A(\varkappa,\bar{\varkappa})$ the equation

$$\begin{aligned}A(\varkappa,\bar{\varkappa})&=w(\varkappa,\bar{\varkappa})+\int d\varkappa'\,w(\varkappa,\varkappa')G(\varkappa')A(\varkappa',\bar{\varkappa}),\\w(\varkappa,\varkappa')&\equiv\omega(\varkappa,\varkappa')+\frac{\omega(\varkappa,\varkappa_1)G_{\varkappa_1}\omega(\varkappa_1,\varkappa')}{1-\omega_{\varkappa_1\varkappa_1}G_{\varkappa_1}}\end{aligned} \tag{50}$$

when (50) and (47) are taken into account, the kernel K in Eq. (22) for Q now turns out to be

$$K(\varkappa,\varkappa')=g(\varkappa)\omega(\varkappa,\varkappa')g(\varkappa')+\frac{\sigma(\varkappa)\sigma(\varkappa')}{-E+E_{\varkappa_1 0}-{}^1/_2i\Gamma_{\varkappa_1}}, \tag{51}$$

$$\sigma(\varkappa)\equiv g(\varkappa)\omega(\varkappa,\varkappa_1)\Phi(\rho_{\varkappa_1}). \tag{52}$$

Using further the same procedure as in the preceding sections, we obtain

$$Q(\varkappa,\bar{\varkappa})=Z(\varkappa,\bar{\varkappa})-\frac{\sigma^*(\varkappa)\sigma^*(\bar{\varkappa})}{E-E_{\varkappa_1 0}+\langle\sigma\sigma^*\rangle+{}^1/_2i\Gamma_{\varkappa_1}} \tag{53}$$

where Z is the resolvent of the kernel $g(\varkappa)\omega(\varkappa,\varkappa')g(\varkappa')$, and

$$\sigma^*(\varkappa)=\sigma(\varkappa)+\int d\varkappa'\,Z(\varkappa,\varkappa')\sigma(\varkappa'). \tag{54}$$

For the S matrix we obtain in the lowest order in ζ:

$$\begin{aligned}S(\varkappa,\bar{\varkappa})&=e^{i(\delta(\varkappa)+\delta(\bar{\varkappa}))}\Big\{\delta(\varkappa-\bar{\varkappa})\\&+i\Big[4\zeta^{1/2}(\varkappa)Z_1(\varkappa,\bar{\varkappa})\zeta^{1/2}(\bar{\varkappa})-\frac{\vartheta(\varkappa)\vartheta(\bar{\varkappa})}{(E-E_{\varkappa_1 0}+\Delta)+i(\Gamma+\Gamma_{\varkappa_1})/2}\Big]\Big\},\\\vartheta(\varkappa)&\equiv 2\zeta^{1/2}(\varkappa)\sigma_1^*(\varkappa),\quad \Gamma=\int d\varkappa\,\vartheta^2(\varkappa),\\\Delta&=\int d\varkappa\,\sigma_1^2(\varkappa)+\int d\varkappa\,d\varkappa'\,\sigma_1(\varkappa)Z_1(\varkappa,\varkappa')\sigma_1(\varkappa').\end{aligned} \tag{55}$$

The subscript 1 denotes here, as before, the real part of the corresponding quantity, for example, $\sigma_1=\mathrm{Re}\,\sigma$. The phase shifts $\delta(\varkappa)$ in (55) are smooth functions of and do not contain the resonant term at $\varkappa=\varkappa_1$.

7. "WANDERING" IN CHANNEL SPACE. DIFFUSION APPROXIMATION

We proceed now to the main question. We examine first the nonresonant situation. It follows from (17') and (29) that in this case

$$S(\varkappa,\bar{\varkappa})=e^{i(\delta(\varkappa)+\delta(\bar{\varkappa}))}\left\{\delta(\varkappa-\bar{\varkappa})+4i\zeta^{1/2}(\varkappa)\left[\bar{R}(\varkappa,\bar{\varkappa})+\frac{\nu^*(\varkappa)\nu^*(\bar{\varkappa})}{1-\langle\nu\nu^*\rangle}\right]\zeta^{1/2}(\bar{\varkappa})\right\} \tag{56}$$

where $\bar{R}$ is the resolvent of the kernel $\bar{K}$ (see (23)), and ν^* is defined by (27). The first term in the square brackets describes direct transitions from the input channel $\bar{\varkappa}$ to the channel $\varkappa$, while the second describes transitions in which at first the system goes from channel $\bar{\varkappa}$ to the internal states ψ_λ, forming something like a compound nucleus, and then goes over to the output channel $\varkappa$. Such an interpretation becomes particularly obvious if the channels $\varkappa$ and $\bar{\varkappa}$ are far enough from the internal states (there is no direct linkage with Eq. (1a): $\eta(\varkappa)=\eta(\bar{\varkappa})=0$). It then follows from (27) that

$$\nu_1^*(\varkappa)=\int d\varkappa'\,\bar{R}_1(\varkappa,\varkappa')\nu_1(\varkappa')\approx\gamma\bar{R}_1(\varkappa,0),\quad \gamma=\int d\varkappa'\,\nu_1(\varkappa'); \tag{57}$$

$$\nu_1^*(\varkappa)\nu_1^*(\bar{\varkappa})\approx\gamma^2\bar{R}_1(\varkappa,0)\bar{R}_1(0,\bar{\varkappa}). \tag{58}$$

Thus, there is every reason for calling the resolvent R the propagation function in the channel space.

In the case of small "penetrabilities" considered by us, $\zeta\ll 1$, the resolvent $\bar{R}=\bar{R}_1+i\bar{R}_2$ can be regarded as real, since its imaginary part R_2 contains the small parameter ζ. The equation for R_1 is of the form (see (25) and (31))

$$\bar{R}_1=\bar{K}_1+\bar{K}_1\bar{R}_1, \tag{59}$$

where the kernel $\bar{K}_1(\varkappa,\varkappa')$, according to the assumption b) made in Sec. 1, tends sufficiently rapidly to zero with increasing $|\varkappa-\varkappa'|$.

Equations of this type can frequently be solved in the diffusion approximation, assuming that $\bar{R}_1$ is a smoother function of its arguments than $\bar{K}_1$. Let, for example $\bar{K}_1(\varkappa,\varkappa')$ depend only on the difference $|\varkappa-\varkappa'|$ between its arguments. Thus, following the usual procedure, we obtain

$$\int d\varkappa'\,\bar{K}_1(|\varkappa-\varkappa'|)\bar{R}_1(\varkappa',\bar{\varkappa})\approx A\bar{R}_1(\varkappa,\bar{\varkappa})+B\frac{d^2}{d\varkappa^2}\bar{R}_1(\varkappa,\bar{\varkappa}),$$
$$A\equiv\int d\varkappa'\,K_1(|\varkappa-\varkappa'|),\quad B\equiv\frac{1}{2}\int d\varkappa'\,(\varkappa'-\varkappa)^2\bar{K}_1(|\varkappa-\varkappa'|) \tag{60}$$

and in place of the integral equation (59) we obtain the differential equation

$$\frac{d^2}{d\varkappa^2}\bar{R}_1(\varkappa,\bar{\varkappa})-t^2\bar{R}_1(\varkappa,\bar{\varkappa})=-\frac{1}{B}\bar{K}_1(|\varkappa-\bar{\varkappa}'|),\quad t^2\equiv\frac{1-A}{B}. \tag{61}$$

It must be borne in mind that the channel index $\varkappa$ has generally speaking several components: $\varkappa=\{\varkappa_{(1)},\varkappa_{(2)},\ldots,\varkappa_{(n)}\}$ and therefore it can be regarded as a vector $\varkappa$ in n-dimensional space.

This does not change the gist of the matter, but $d^2/d\varkappa^2$ in (61) should be replaced by the Laplacian

$$\frac{d^2}{d\varkappa^2}\to\sum_{i=1}^{n}\frac{\partial^2}{\partial\varkappa_{(i)}^2}\equiv\Delta_n.$$

That is to say, in this approximation the wandering of our system in multidimensional space of channels is described by an equation of the type of the stationary-diffusion equation with absorption or multiplication (depending on the sign of t^2) and a source. In the general case the diffusion coefficient depends, of course, on $\varkappa$.

We consider now the case when the system energy is close to resonant.

Resonances of the first type (Sec. 4). The nonresonant part of the S matrix (36) contains the propagation function $\bar{R}_1$ considered above. The partial widths $\theta_\lambda(\varkappa)$ which enter in the resonant part of the S matrix were already discussed in Sec. 4 (see (37)). The arguments presented there confirm the possibility of interpreting $\bar{R}_1$ as a propagation function in channel space.

Resonances of the third type (Sec. 6). This case differs from the preceding one only in that the resonance is connected not with the internal state ψ_λ, but with the quasistationary (or stationary) state in the channel $\varkappa_1$.

The nonresonant part of the S matrix (55) has the same form as (56), since the resolvent Z practically coincides with Q in formula (29). Therefore all the statements made at the beginning of this section concerning the nonresonant case are applicable here. The motion of the system in channel space is described by the propagation function $Z_1\approx\bar{R}_1$. The partial widths $\vartheta(\varkappa)$ which enter in the resonant part of the S matrix (55) have properties analogous to the properties of the widths $\theta_\lambda(\varkappa)$. Thus, if the channel $\varkappa$ is far enough from $\varkappa_1$, then (see (52)–(55))

$$\vartheta(\varkappa)\approx 2\zeta^{1/2}(\varkappa)cZ_1(\varkappa,\varkappa_1),\quad c=\int d\varkappa'\,\sigma(\varkappa'),$$

and it is seen that the reduced amplitude $\vartheta/2\zeta^{1/2}$ is a product of factors Z_1c, the first of which describes motion in channel space from $\varkappa$ to $\varkappa_1$, and the second a transition to a state of channel $\varkappa_1$, and this state leads in turn to resonance. The only complication in comparison with the preceding case is that Z_1 contains two terms (see (56)), one of which describes a direct transition $\varkappa\to\varkappa_1$, and the second a transition via the internal states ψ_λ: $\varkappa\to 0\to\varkappa_1$.

Resonances of the second type. In this case, in contrast to all the cases considered above, one cannot introduce the concept of "wandering" or diffusion in channel space. Formally this is connected with the fact that when a term with $\lambda_p\approx 1$ is separated from the kernel K_1 (see (40)), then the remainder $\tilde{K}_1(\varkappa,\varkappa')$ is no longer "short-range," that is, it no longer tends to zero with increasing $|\varkappa-\varkappa'|$. Therefore the integral equation for the resolvent $\tilde{R}$ cannot be reduced to a differential one. Physically, on the other hand, the reason lies in the fact that in a case of resonance of the second type the system is in a definite quantum state u_p, which encompasses the entire aggregate of the channels.

The arguments presented above show that within the framework of the assumptions (a) and (b) made in Sec. 1 it is always possible to introduce the concept of a propagation function $\mathcal{R}$ in channel space. To find this function it is necessary to solve the integral equation

$\mathcal{R}=\mathcal{K}+\mathcal{K}\mathcal{R}$ with a short-range kernel $\mathcal{K}$; equations of this type can frequently be solved in the diffusion approximation, and in this case $\mathcal{R}$ is determined from an equation describing stationary diffusion (with absorption or multiplication and a source) in the channel space. The only exception is the case of resonances of the second type, when a long-range interaction of sorts enters into the channel space, due to the proximity to the stationary (or quasistationary) state of the aggregate of channels.

8. CONCLUSION

The analysis presented in this paper is valid only for a system with a very large number of channels, open or closed. The approach indicated above is in essence statistical, although not in the usual sense employed, for example, in nuclear physics. The assumptions (a) and (b) formulated in Sec. 1 are sufficient to make the formulas derived in the preceding sections correct. A necessary condition here is (b), while the first condition can apparently be made much less stringent.

The classification introduced by us for the resonances is not formal but physical. For example, at a given input channel $\bar{\varkappa}$, the cross sections $\sigma_{\varkappa\bar{\varkappa}}$ of different reactions, which are proportional to $|S(\varkappa,\bar{\varkappa})|^2$, will have entirely different dependences on $\varkappa$. This can be easily seen by comparing formulas (36), (44), and (55). We shall not discuss this in detail at present.

One final remark. In all real cases the channel index has several (n) components and can be regarded as vector in n-dimensional space. In particular, two components of the index $\varkappa$ describe the excitation energies of two particles of the channel. As the total excitation energy is gradually increased, we ultimately fall into the region of closed channels. On the other hand, the diffusion laws, as is well known, depend essentially on the dimensionality of the space in which the diffusion takes place. It would therefore be quite incorrect to confine oneself in the approach developed above, only to allowance for the open channels. All the channels, open and closed, must be taken into account in the scheme.

To apply the procedure described in the preceding section to the calculation of concrete systems it is necessary, first, to renumber the channels in correspondence with the requirements of Sec. 1 and, second, to introduce an explicit expression for the kernel K or for the diffusion coefficient. The channels can be numbered by using physical considerations. The form of the kernel K or of the diffusion coefficient should, as a rule, be chosen by starting from the experimental data. Both questions are the subject of a separate investigation.

[1)]This can be done accurately by expanding all the corresponding quantities in powers of $\varkappa-\varkappa'$, but the small corrections that result from this procedure are of no interest.

[2)]The case $G_1<0$ when $\xi\gg1$ is considered in a perfectly analogous manner, but the expansion must now be in powers of ξ^{-1} and not ξ. The formulas of Secs. 4–7 are then slightly modified, but all the physical results remain the same.

[1]A. I. Baz' and M. V. Zhukov, Yad. Fiz. **16**, 60, 958 (1972) [Sov. J. Nucl. Phys. **16**, (1973)].

[2]C. Ngo, B. Tamain, J. Galin, M. Beiner, and R. Lombard, Nucl. Phys. **A240**, 353 (1975).

[3]V. I. Smirnov, Kurs vysshei matematiki (Course of Higher Mathematics), Vol. IV, Gostekhizdat, 1951, Chap. 1, p. 22.

Translated by J. G. Adashko

Bound states of electron-positron pairs in a strong electric field

A. B. Migdal

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted August 12, 1975)
Zh. Eksp. Teor. Fiz. **70**, 411–422 (February 1976)

It is shown that taking account of the interaction leads to a splitting of electron-positron levels in the field of the nucleus. The levels differ by the number of pairs, each pair consisting of an electron in the K shell and a positron in a quasistationary state. A bound state arises in a small range of Z. The energy spectrum of the positrons, which are emitted upon critical approach of heavy nuclei, should contain several maxima which differ in energy by 10 to 30 keV.

PACS numbers: 36.10.Dr

As is well known, the Dirac equation in the field of a point charge loses meaning at $Z>Z_c=137$. In actual fact the ground state energy is of the form ($\hbar=m=c=1$)

$$\varepsilon_0=[1-(Ze^2)^2]^{1/2}$$

and becomes imaginary for $Z>137$. Allowance for the finite size of the nucleus[1–3] removes this difficulty. However, at a value $Z\approx170$ the energy of the lowest state reaches the value $\varepsilon_0=-1$ and the total energy of a pair vanishes, that is, the vacuum becomes unstable with respect to the creation of electron-positron pairs. Thus, at $Z=Z_c$ the Dirac equation loses the meaning of