Some properties of the type IX cosmological model with moving matter

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The singular points and their separatrices are investigated (in the sense of the qualitative theory of differential equations) for the Einstein equations in the homogeneous type IX cosmological model with moving matter. A new power asymptotic behavior of the metric in the presence of moving matter, which generalizes the well-known Taub asymptotic behavior, is found. The typical states of the metric during the early stage of expansion are studied; it is shown that the typical states depend on the value of an integral M which is related to the velocity of the matter.

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INTRODUCTION

The homogeneous cosmological model of type IX in the Bianchi classification with moving matter was first considered by Belinskiĭ, Lifshitz, and Khalatnikov,[1] who showed that the oscillatory mode which they had previously discovered[2] is accompanied in the presence of moving matter by rotations of the "Kasner axes." Subsequently, Grishchuk, Doroshkevich, and Lukash[3] investigated the oscillatory mode in a coordinate system in which the metric is diagonal and showed that mixing does not occur.

The present paper is a continuation of the joint work of S. P. Novikov and the author[4] and uses the methods of the qualitative theory of differential equations employed in[4]. These methods enable one to obtain a complete picture of all the modes of the metric in this model in the neighborhood of the cosmological singularity.

When the space contracts, almost any metric (i.e., all metrics except those with a power asymptotic behavior) enters the Belinskiĭ–Lifshitz–Khalatnikov (BLKh) oscillatory mode. Therefore, this oscillatory mode is a typical state of the metric when space contracts. When the space expands from the cosmological singularity, the BLKh oscillatory mode terminates at a certain time. In[4], the problem was posed of the typical states of the metric during the early stage in the expansion of space that are realized near the cosmological singularity and follow directly after the BLKh oscillatory mode. These typical states for the diagonal type IX model are three power regimes of the metric—the quasi-isotropic regime which generalizes the Friedmann solutions, the regime found by S. P. Novikov, and the Taub regime (in this connection, see also[5]). In this paper, we study the typical states during the early stage of expansion in the type IX model with moving matter and we investigate the dependence of the typical states on the velocity of the matter.

1. EINSTEIN'S EQUATIONS IN THE PHASE SPACE

We write the metric g_{ij} of the homogeneous type IX cosmological model in the standard form (see[1,2]):

$$ds^2=g_{00}(\tau)\,d\tau^2-g_{ab}(\tau)\,e_\alpha{}^a e_\beta{}^b dx^\alpha dx^\beta, \tag{1.1}$$

where $g_{00}(\tau)$ will be determined below by the choice of the time scale. We suppose that the space is filled with matter with hydrodynamic energy-momentum tensor:

$$T_{ik}=(p+\varepsilon)\,u_i u_k-p g_{ik}, \tag{1.2}$$

and that the components of the four-velocity u_i in the standard basis e^a_α depend only on the time, $\varepsilon(\tau)$ is the energy density, $p(\tau)$ is the pressure, and the equation of state is $p=k\varepsilon$, $0\leqslant k<1$.

In studying the Einstein equations

$$R_{ij}-{}^1/{}_2 g_{ij}R=T_{ij} \tag{1.3}$$

for the type IX model we use the well-known variational principle (see[6]):

$$\delta\int R\sqrt{-g}\,dt=-\int T^{ik}\delta g_{ik}\sqrt{-g}\,dt. \tag{1.4}$$

The expression $T^{ik}\delta g_{ik}$ for the metric (1.1) has the form

$$T^{ik}\delta g_{ik}=\varepsilon\delta\ln\frac{g_{00}}{|g|^k}+(1+k)\,\varepsilon\,(g_{ab}u^a u^b g_{00}\delta g_{00}-u^i u^k\delta g_{ik}). \tag{1.5}$$

Here and below $|g|=\det\|g_{ab}\|$.

In order to simplify the expression (1.5), we choose the time τ such that

$$g_{00}(\tau)=|g|^k, \tag{1.6}$$

and then under the condition (1.6) the first term in (1.5) is zero. The time τ is related to the synchronous time $t(g_{00}(t)\equiv 1)$ by $dt=|g|^{k/2}d\tau$.

In accordance with the variational principle (1.4) and (1.5), the "tensor components" ($i, j=1, 2, 3$) of the Einstein equations are equivalent to the equations

$$\frac{\partial L}{\partial g_{ij}}-\frac{d}{d\tau}\frac{\partial L}{\partial \dot g_{ij}}=\frac{1+k}{2}\,\varepsilon\,(u^i u^j-k g_{ab}u^a u^b g^{ij})\,|g|^{(1+k)/2}, \tag{1.7}$$

where the function $L(g_{ij},\ \dot g_{ij})$ is obtained from $\frac{1}{2}R\sqrt{-g}$ by omitting the total derivative with respect to the time and has the form

$$L={}^1/{}_8|g|^{(1-k)/2}(\varkappa_\alpha{}^\alpha\varkappa_\beta{}^\beta-\varkappa_\alpha{}^\beta\varkappa_\beta{}^\alpha)-{}^1/{}_4|g|^{-(1-k)/2}(2|g|g^{\alpha\alpha}-g_{\alpha\beta}g_{\alpha\beta}), \tag{1.8}$$

where $\varkappa^\beta_\alpha=\dot g_{\alpha\gamma}g^{\gamma\beta}$.

The Einstein equation $R_0^0 - \frac{1}{2}R = T_0^0$ has the form

$$H=\varepsilon\left(\left(1+k\right)u_0^2 g^{00}-k\right)|g|^{(1+k)/2},$$
$$H={}^1/_8|g|^{(1-k)/2}(\varkappa_\alpha{}^\alpha\varkappa_\beta{}^\beta-\varkappa_\alpha{}^\beta\varkappa_\beta{}^\alpha)+{}^1/_4|g|^{-(1-k)/2}(2|g|g^{\alpha\alpha}-g_{\alpha\beta}g_{\alpha\beta}). \tag{1.9}$$

The Einstein equations $R_{0\alpha} = T_{0\alpha}$ have the form

$$-{}^1/_2\varkappa_\beta{}^\gamma C_{\alpha\gamma}{}^\beta=(1+k)\varepsilon u_0 u_\alpha, \tag{1.10}$$

where $C^\beta_{\alpha\gamma}$ are the structure constants of the type IX group ($SO(3)$) in the standard form.

From Eqs. (1.9) and (1.10) and the condition

$$u_0^2 g^{00}-u_a u_b g^{ab}=1 \tag{1.11}$$

we can obtain the expressions

$$\frac{\varepsilon u_0^2|g|^{(1+k)/2}}{g_{00}}=\frac{1}{2}\left(H+\left(H^2-\frac{16k}{(1+k)^2}X_a X_b g^{ab}|g|^k\right)^{1/2}\right), \tag{1.12}$$

$$\frac{1+k}{2}\varepsilon u^i u^j|g|^{(1+k)/2}=\frac{4X^iX^j|g|^k}{(1+k)\left(H+(H^2-16k(1+k)^{-2}X_aX_bg^{ab}|g|^k)^{1/2}\right)} \tag{1.13}$$

where $X_a = -\frac{1}{4}\varkappa^\gamma_\beta C^\beta_{\alpha\gamma}|g|^{(1-k)/2}$.

Substituting (1.13) into Eq. (1.7), we obtain a closed system of second-order differential equations for the components of the matrix $g_{ij}(\tau)$:

$$\frac{\partial L}{\partial g_{ij}}-\frac{d}{d\tau}\frac{\partial L}{\partial \dot g_{ij}}=h^{ij},$$
$$h^{ij}=\frac{4(X^iX^j-kg_{ab}X^aX^bg^{ij})|g|^k}{(1+k)\left(H+(H^2-16k(1+k)^{-2}X_aX_bg^{ab}|g|^k)^{1/2}\right)}. \tag{1.14}$$

We transform the system (1.14) into a system of first-order equations defined in the phase space of p^{ij} and g_{ij}. The momenta p^{ij} are defined by

$$p^{ij}=\frac{\partial L}{\partial \dot g_{ij}}=\frac{|g|^{(1-k)/2}}{4}\left(g^{ij}(\ln|g|)^{\boldsymbol\cdot}-\dot g_{kl}g^{ik}g^{jl}\right). \tag{1.15}$$

In the phase space, the system (1.14) goes over into the system

$$\dot p^{ij}=-\frac{\partial H}{\partial g_{ij}}-h^{ij},\quad \dot g_{ij}=\frac{\partial H}{\partial p^{ij}}. \tag{1.16}$$

The function H (see (1.9)) in the coordinates p^{ij} and g_{ij} has the form

$$H=\frac{1}{|g|^{(1-k)/2}}\left[(\mathrm{Sp}(p\circ g))^2-2\,\mathrm{Sp}(p\circ g\circ p\circ g)+\frac{1}{4}\left(2|g|g^{\alpha\alpha}-\mathrm{Sp}(g^2)\right)\right]. \tag{1.17}$$

Here, $\mathrm{Sp}(Y)$ is the spur of the matrix Y, $p\circ g$ is the product of the matrices $p = \| p^{ij}\|$ and $g = \|g_{jk}\|$.

The dynamical system (1.16) is, by virtue of the derivation, equivalent to the complete system of Einstein equations. The time dependence of the velocities u_α and the energy density ε can be determined from Eqs. (1.9)–(1.11).

2. TRANSFORMATION OF THE DYNAMICAL SYSTEM

To study the system (1.16), we transform it by means of three changes of coordinates to a more convenient form.

1. We introduce the coordinates $S^j_k = g_{ki}p^{ij}$. In accordance with (1.15),

$$S_k^j=\frac{|g|^{(1-k)/2}}{4}\left(\varkappa_\alpha{}^\alpha\delta_k{}^j-\varkappa_k{}^j\right), \tag{2.1}$$

where $\varkappa^j_k = \mathring{g}_{ki}g^{ij}$. The principal directions of the matrix S^j_k coincide with those of the matrix $\varkappa^j_k$, which in[1] are called the Kasner axes. The matrix $S = \|S^j_k\|$ for nonzero velocities u_α is not symmetric (in contrast to the matrices $g = \|g_{ij}\|$ and $p = \|p^{ij}\|$). To see this, we use (1.10) and (1.15), obtaining

$$X_\alpha=-{}^1/_4\varkappa_\beta{}^\gamma C_{\alpha\gamma}{}^\beta|g|^{(1-k)/2}={}^1/_2(1+k)\varepsilon u_0 u_\alpha|g|^{(1-k)/2}=S_\beta{}^\gamma C_{\alpha\gamma}{}^\beta. \tag{2.2}$$

Obviously, the matrices S and g satisfy the identity

$$g\circ S^t=S\circ g. \tag{2.3}$$

Here, S^t is the transposed matrix.

The system (1.16) in the coordinates S^j_k and g_{ij} takes the form

$$S_k^j=-\frac{1}{2|g|^{(1-k)/2}}\left[|g|(\delta_k{}^jg^{\alpha\alpha}-g^{kj})-g_{ki}g_{ij}\right]+\delta_k{}^j\left(\frac{1-k}{2}\right)H-g_{ki}h^{ij},$$
$$\dot g_{ij}=\frac{2}{|g|^{(1-k)/2}}\left(g_{ij}S_h{}^h-2g_{ik}S_j{}^k\right). \tag{2.4}$$

The system (2.4) has the two first integrals L and K:

$$L=X_1^2+X_2^2+X_3^2, \tag{2.5}$$

$$K=\left[\left(H^2-\frac{16k}{(1+k)^2}Z\right)^{1/2}+H\right]\left[\left(H^2-\frac{16k}{(1+k)^2}Z\right)^{1/2}-\frac{1-k}{1+k}H\right]^{(1-k)/(1+k)}; \tag{2.6}$$

where $Z = X_a X_b g^{ab}|g|^k$. Using Eqs. (2.2), (1.9), and (1.12), we obtain an expression for these integrals in terms of the velocities u_α and ε:

$$L=\left(\frac{1+k}{2}\right)^2\varepsilon^2\bar u_0^2|g|(u_1^2+u_2^2+u_3^2),$$
$$K=2\left(\frac{2k}{1+k}\right)^{(1-k)/(1+k)}\varepsilon^{2/(1+k)}\bar u_0^2|g|, \tag{2.7}$$

where $\bar u_0$ is the velocity component in the synchronous frame. As $k\to 0$, the integral K (2.6) must be replaced by dK/dk [formally, (2.6) gives $K = 0$ for $k = 0$]. The integrals (2.7) for the case $k = 1/3$ were given by Grishchuk *et al* in[3]. Here, using the expressions (2.5) and (2.6), we point out important applications of these integrals to the problem of the typical states of the metric during the early stage of expansion of space.

The system (2.4) has the monotonic function

$$F=3\frac{d}{dt}(|g|^{1/6})=(S_h{}^h)|g|^{-1/3},\qquad \frac{dF}{d\tau}\leqslant 0. \tag{2.8}$$

When the direction of time is chosen in the direction of decreasing volume $|g|$, we have $F<0$ and $S^k_k<0$. It follows from the monotonicity of the function F that the trajectories of the system (2.4) do not leave the region $S^k_k \leqslant 0$. If the direction of time is chosen in the direction of increasing volume $|g|$, then $F>0$ and the function F decreases monotonically.

2. We introduce the coordinates

$$\bar s_k{}^j=\frac{S_k{}^j}{G},\quad y_{ij}=\frac{g_{ij}}{G},\quad G, \tag{2.9}$$

where

$$G=\left(\sum_{\alpha,\beta=1}^3 g_{\alpha\beta}{}^2\right)^{1/2}.$$

The coordinates y_{ij} satisfy the identity

$$\sum_{i,j=1}^3 y_{ij}{}^2=1.$$

The coordinates (2.9) are convenient for studying the behavior of the system (2.4) in the neighborhood of the state of maximal expansion, where det $\| g_{ij} \|$ reaches a maximum and $\bar{s}_k^k = 0$.

3. To study the behavior of the system (2.4) near the cosmological singularity (where det $\| g_{ij} \| = 0$) we introduce the coordinates

$$s_k{}^j = \frac{\bar{s}_k{}^j}{\bar{\mathfrak{s}}} = \frac{S_k{}^j}{\mathfrak{S}}, \quad y_{ij}, \quad w = \frac{G^2}{\mathfrak{S}^2}, \quad G, \tag{2.10}$$

where

$$\bar{\mathfrak{s}} = \Big(\sum_{\alpha,\beta=1}^{3} (\bar{s}_\alpha{}^\beta)^2 \Big)^{1/2}, \quad \mathfrak{S} = \Big(\sum_{\alpha,\beta=1}^{3} (S_\alpha{}^\beta)^2 \Big)^{1/2}.$$

The coordinates s_k^j satisfy the identity

$$\sum_{k,j=1}^{3} (s_k{}^j)^2 = 1.$$

The system (2.4) in the coordinates (2.10) after the change of the time

$$\frac{d\tau_1}{d\tau} = \frac{\mathfrak{S}^2}{2|g|^{(1-k)/2}} \tag{2.11}$$

takes the form

$$\begin{gathered}
\dot{s}_k{}^j = w[-|y|(\delta_k{}^j y^{\alpha\alpha} - y^{jk}) + y_{ki} y_{ij} + s_k{}^j (|y| (s_\alpha{}^\alpha y^{\beta\beta} - s_\alpha{}^\beta y^{\alpha\beta}) - y_{\alpha\beta} s_\gamma{}^\beta y_{\gamma\alpha})] \\
+ (\delta_k{}^j - s_k{}^j s_\alpha{}^\alpha) H_1 (1-k) - \frac{8|y|w}{(1+k)[H_1 + (H_1{}^2 - 16k(1+k)^{-2} w x_a x_b |y| y^{ab})^{1/2}]} \\
\times [x_k x_\alpha y^{\alpha j} - k x_a x_b y^{ab} \delta_k{}^j - s_k{}^j (x_\alpha x_\gamma y^{\beta\gamma} s_\alpha{}^\beta - k x_a x_b y^{ab} s_\gamma{}^\gamma)], \\
\dot{y}_{ij} = 8(-y_{ik} s_j{}^k + y_{ij} y_{\alpha\beta} s_\gamma{}^\beta y_{\gamma\alpha}), \qquad (2.12) \\
\dot{w} = 2w \Big[s_\alpha{}^\alpha (4 - (1-k) H_1) - 8 y_{\alpha\beta} s_\gamma{}^\beta y_{\gamma\alpha} + w \Big(|y| (s_\alpha{}^\alpha y^{\beta\beta} - s_\alpha{}^\beta y^{\alpha\beta}) - y_{\alpha\beta} s_\gamma{}^\beta y_{\gamma\alpha} \\
+ \frac{8}{(1+k)} \frac{(x_\alpha x_\gamma |y| y^{\gamma\beta} s_\alpha{}^\beta - k x_a x_b |y| y^{ab} s_\gamma{}^\gamma)}{(H_1 + (H_1{}^2 - 16k(1+k)^{-2} w x_a x_b |y| y^{ab})^{1/2})} \Big) \Big] \\
\dot{G} = 4G(s_\alpha{}^\alpha - 2 y_{\alpha\beta} s_\gamma{}^\beta y_{\gamma\alpha}).
\end{gathered}$$

Here

$$H_1 = H/\mathfrak{S}^2 = (s_\alpha{}^\alpha)^2 - 2 s_\alpha{}^\beta s_\beta{}^\alpha + {}^1/_4 w (2|y| y^{\alpha\alpha} - 1), \quad x_\alpha = s_\beta{}^\gamma C_{\alpha\gamma}{}^\beta.$$

The system (2.12) contains a closed subsystem in the coordinates s_k^j, y_{ij}, w. This system can be considered in the region S, which has dimension 11, which is determined by the natural conditions

$$\begin{gathered}
y \circ s^t = s \circ y, \quad y_{ij} = y_{ji}, \quad \sum_{i,j=1}^{3} y_{ij}{}^2 = 1, \quad \sum_{k,j=1}^{3} (s_k{}^j)^2 = 1, \\
|y| = \det \|y_{ij}\| > 0, \quad 0 < w < \infty, \quad K \geqslant 0.
\end{gathered} \tag{2.13}$$

It follows from the condition $K \geqslant 0$ in particular that

$$H_1 \geqslant 0, \quad H_1{}^2 \geqslant 16k(1+k)^{-2} w x_a x_b |y| y^{ab}.$$

The boundary Γ of S [see (2.13)] consists of three components: Γ_0, Γ_1, and Γ_w, which are determined by the following conditions: Γ_0: det $\| y_{ij} \| = 0$; Γ_1: $K = 0$; Γ_w: $w = 0$. It is obvious that the system (2.12) can be extended continuously to the components Γ_0 and Γ_w and to the component Γ_1 for $H_1 \neq 0$. As $H_1 \to 0$ the expressions containing H_1 in the denominator [see (2.12)] are, by virtue of the condition

$$H_1{}^2 \geqslant 16\, k(1+k)^{-2} w x_a x_b |y| y^{ab},$$

bounded above by

$$c H_1 \frac{|x_k x_j |y| y^{ij}|}{x_a x_b |y| y^{ab}}, \qquad c|x_k| (|y| y^{ii} w)^{1/2}.$$

Therefore, these expressions tend to zero as $H_1 \to 0$, except for points at which the matrix y_{ij} is simply degenerate, and $x_a x_b | y | y^{ab} = 0$, $w \neq 0$, $(x_1, x_2, x_3) \neq 0$. Thus, everywhere except these exceptional points, which are distinguished by the two conditions det $\|y_{ij}\| = 0$ and $x_a x_b | y | y^{ab} = 0$, the system (2.12) in the limit $H_1 \to 0$ can be extended continuously to the boundary Γ.

The components Γ_0, Γ_1, Γ_w of the boundary are invariant manifolds of the system (2.12), i.e., a trajectory that begins on the boundary remains on it for all time.

3. POWER-LAW ASYMPTOTIC BEHAVIORS. TYPICAL STATES DURING THE EARLY STAGE IN THE EXPANSION OF SPACE

We find the power-law (with respect to t) asymptotic behaviors of the metric of the type IX model with moving matter as space contracts. A metric having this form can be represented in the coordinates (2.10) by a trajectory of the system (2.12) entering one of the singular points of this system. By virtue of the existence of the monotonic function $F = s_k^k / w^{1/2} | y |^{1/3}$ all the singular points of the system (2.12) lie on the boundary components Γ_0 ($| y | = 0$) or Γ_w($w = 0$). The singular points form six sets: Φ_{LKh}, N, T, A, B, and K.

1. The set Φ_{LKh} has dimension 5 and is determined by the conditions $s_k^j = -3^{-1/2} \delta_k^j$, $w = 0$, and y_{ij} arbitrary; $H_1(\Phi_{\mathrm{LKh}}) = 1$. A six-dimensional separatrix enters the set Φ_{LKh} from the physical region S, this separatrix corresponding to diagonalizable metrics (no moving matter) with the quasi-isotropic asymptotic behavior found by Lifshitz and Khalatnikov[7]:

$$g_{ij}(t) \approx t^{4/3(1+k)} g_{ij}{}^0 \tag{3.1}$$

(here, the synchronous time tends to zero, $t \to 0$).

2. The set N, which has dimension 2, is determined by the conditions

$$\|y_{ij}\| = Q_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_1{}^t, \quad \|s_k{}^j\| = Q_1 \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_2 \end{pmatrix} Q_1{}^t.$$

Here and below, Q_1 is an arbitrary orthogonal matrix;

$$\begin{gathered}
s_1 = -2(3+k)(43+2k+3k^2)^{-1/2}, \quad s_2 = -(5-k)(2(43+2k+3k^2))^{-1/2}, \\
w = \frac{8(1+3k)(1-k)}{43+2k+3k^2}, \qquad H_1(N) = \frac{8(5-k)}{43+2k+3k^2}.
\end{gathered}$$

A five-dimensional separatrix enters the singular points of N from the physical region S, this separatrix representing diagonalizable metrics with the asymptotic behavior

$$g_{ij}(t) \approx Q_1 \begin{pmatrix} C_1 t^{(1-k)/(1+k)} & 0 & 0 \\ 0 & C_2 t^{(3+k)/2(1+k)} & 0 \\ 0 & 0 & C_3 t^{(3+k)/2(1+k)} \end{pmatrix} Q_1{}^t. \tag{3.2}$$

The singular points of Φ_{LKh} and N are nondegenerate and unstable.

3. The set T, which has dimension 5, is determined by the conditions

$$\|y_{ij}\| = Q_1 \begin{pmatrix} 2^{-1/2} & 0 & 0 \\ 0 & 2^{-1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_1{}^t, \quad \|s_k{}^j\| = Q_1 \begin{pmatrix} s & 0 & 0 \\ 0 & s & x \\ 0 & 0 & 0 \end{pmatrix} Q_1{}^t.$$

Here, the coordinates s and x satisfy the conditions $s \leqslant 0$, $2s^2 + x^2 = 1$, $w \geqslant 0$ an arbitrary number, $H_1(T) = 0$.

A seven-dimensional separatrix enters the set T from the physical region S; it represents metrics (with rotation of the axes for $x \neq 0$) with asymptotic behavior generalizing that found by Taub:

$$g_{ij}(t) \approx Q_1 \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_1 + C_2 x^2 t^2 & -C_2 xst^2 \\ 0 & -C_2 xst^2 & C_2 s^2 t^2 \end{pmatrix} Q_1'. \qquad (3.3)$$

The singular points of T for $s \neq 0$ are nondegenerate and unstable. The boundary of the set T for $s = 0$ is the set of degenerate singular points T^0.

4. The sets A and B, which have dimension 6, are determined by the conditions

$$\|y_{ij}\| = Q_1 \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_1', \quad \|s_k^j\| = Q_1 \begin{pmatrix} s_1 & 0 & s_1^3 \\ 0 & s_1 & s_2^3 \\ 0 & 0 & s_2 \end{pmatrix} Q_1', \quad w=0.$$

On A, $s_1 = (1/4) s_2$ and on B, $s_2 = 0$; $H_1(A) = H_1(B) = 0$. The singular points of A and B do not have separatrices reaching them from the physical region S, so that no asymptotic behaviors correspond to them.

5. The set K has dimension 7 and is determined by the conditions

$$\|y_{ij}\| = Q_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_1', \quad \|s_k^j\| = Q_1 \begin{pmatrix} s_1^1 & s_1^2 & s_1^3 \\ 0 & s_2^2 & s_2^3 \\ 0 & 0 & s_3^3 \end{pmatrix} Q_1', \qquad (3.4)$$
$$w=0, \; H_1(K)=0.$$

The set K lies in the intersection $\Gamma_0 \cap \Gamma_1 \cap \Gamma_w$ of the boundary components. These singular points are nondegenerate (for $s_1^1 \neq s_2^2 \neq s_3^3$) and unstable. The separatrices of the singular points of K lie on the boundary Γ and move (for time directed in the direction of contraction of space) from one singular point of K to another. The singular points of K do not correspond to any power asymptotic behaviors. These singular points together with their separatrices are an approximation (see the Appendix) of the most general mode of the metric for contracting space—the BLKh oscillatory mode.[1,2]

Note that all these sets of singular points and power asymptotic behaviors (3.1), (3.2), and (3.3) were present in the diagonal type IX model (see[4]).

We shall now show that in the presence of moving matter the power behaviors (3.1) and (3.2) are not realized. From the integrals L (2.5) and K (2.6) one can form an integral M that is invariant under the transformations

$$g_{ij} \to \lambda^2 g_{ij}, \; S_k^j \to \lambda^2 S_k^j, \; t \to \lambda t. \qquad (3.5)$$

The integral M in the coordinates s_k^j, y_{ij}, w has the form

$$M = L \cdot K^{-2(1+k)/(1+3k)} = (x_1^2 + x_2^2 + x_3^2)\,|y|^{2(1-k)/(1+3k)} w^{3(1-k)/(1+3k)}$$
$$\times \left(Z_1^{1/2} - \frac{1-k}{1+k} H_1 \right)^{-2(1-k)/(1+3k)} (Z_1^{1/2} + H_1)^{-2(1+k)/(1+3k)}$$
$$= C(k)\, \varepsilon^{2(3k-1)/(1+3k)} \bar{u}_0^{2(k-1)/(1+3k)} |g|^{(k-1)/(1+3k)} (u_1^2 + u_2^2 + u_3^2), \qquad (3.6)$$

where

$$Z_1 = H_1^2 - 16k(1+k)^{-2} w x_\alpha x_\beta |y| y^{\alpha\beta}.$$

If the metric right up to the singularity has the behavior (3.1) or (3.2), then the corresponding trajectory of the system (2.12) enters the singular points of Φ_{LKh} or N. The integral M (3.6) is equal to zero at the singular points of Φ_{LKh} and N. Therefore, a trajectory entering these singular points corresponds to a metric without moving matter, i.e., the behaviors (3.1) and (3.2) are not realized when moving matter is present and space contracts right down to the singularity.

Note that the parameter value $k = 1/3$ is distinguished by the fact that the integral M for $k = 1/3$ does not depend on the energy density ε:

$$M\left(\frac{1}{3}\right) = \frac{2^{4/3}}{9} \bar{u}_0^{-2/3} |g|^{-1/3} (u_1^2 + u_2^2 + u_3^2).$$

It is natural to say that the matter moves fast if $M \gg 1$ and slowly if $M \lesssim 1$.

If the direction of time coincides with contraction of space, all the trajectories of the system (2.12) approach the boundary Γ because of the presence of the monotonic function F (2.8). At the same time, $F \to -\infty$ along each trajectory since $F = -\infty$ on the boundary. Having reached a small neighborhood of the boundary Γ defined by the condition $|F| \gg 1$, a trajectory of the system (2.12) begins to move along the trajectories of this system that lie on Γ. All the trajectories of the system (2.12) on Γ are separatrices of singular points and lead from one singular point to another (we do not give the separatrix diagram here since it is essentially the same as that given earlier in[4] for the diagonal type IX model). After a finite number of transitions (there is never more than three) along the separatrices of the singular points of Φ_{LKh}, N, T, A, and B, the trajectory arrives in the neighborhood of the singular points of K and begins to move along their separatrix. During this motion of the trajectory, the metric is in the BLKh oscillatory mode (see the Appendix). Thus, all metrics of the type IX model with moving matter except for the metrics that have the Taub asymptotic behavior (3.3) reach the BLKh oscillatory mode, which is therefore a typical state of the metric when space contracts.

If the direction of time coincides with expansion of space, the BLKh oscillatory mode ends at a certain time. Let us consider the question of the typical states of the metric that follow the BLKh oscillatory mode (see[4,8]). The precise formulation of this problem is based on the following important property of the monotonic function $F = d(|g|^{1/6})/dt$: If the direction of time coincides with expansion of space, the function F along each solution decreases from $+\infty$ to 0, and $F = 0$ at the time of maximal expansion. It is natural to say that the states of the metric for which $F \gg 1$ are the early stage of the expansion. Note that the function F is invariant under the scale transformations (3.5) and has a simple physical meaning: It is the rate of change of the mean radius of the universe defined as $|g|^{1/6}$ ($|g|^{1/6}$ has the dimensions of a length); for the Friedmann solution, $F = \dot{a}$, where a is the radius of the three-dimensional sphere (the function $F = \partial(|g^{1/6}|)/\partial t$ is monotonic along any (for example, inhomogeneous) solution of the Einstein equations in a synchronous frame).

The determination of typical states of the metric during the early stage of expansion consists of the following. Suppose that for $F=F_1 \gg 1$ in the phase space of the coordinates g_{ij} and the momenta p^{ij} initial conditions are given in some manner (for example, the distribution can be taken uniform on the surface $F=F_1$). By virtue of the Einstein equations, these initial conditions are displaced in the phase space and for some $F=F_2 \gg 1$ ($F_2 < F_1$) can be concentrated in a small neighborhood of certain special points of the phase space. At the same time, the metric will be approximated by certain special regimes, which we shall call typical states of the metric during the early stage of the expansion.

In the model considered, the condition $F=F_1 \gg 1$ means that the initial data on the manifold S are in a small neighborhood of the boundary Γ. Therefore, trajectories emanating from these initial data move along the separatrices of the singular points of the system (2.12) lying on the boundary Γ until they arrive in the neighborhood of singular points that have separatrices going into the physical region S. These singular points are the points of the sets Φ_{LKh}, N, and T. Thus, the original distribution of initial data is transformed into a distribution concentrated in the neighborhood of the sets Φ_{LKh}, N, and T of singular points. At the same time, the trajectories remain near the boundary Γ, and therefore $F \gg 1$. As a trajectory moves along the separatrices of the singular points of Φ_{LKh}, N, and T that go into the physical region S, the metric can be approximated by the power laws (3.1), (3.2), and (3.3), and these are the typical states of the metric of the type IX model during the early stage of expansion (at the same time, the function F reduces to values $F \sim 1$).

Note that the time t_0 required for the metric to reach one of these power regimes (t_0 is longer than the duration of the BLKh oscillatory mode) may be arbitrarily short and it depends strongly on the solution itself [this is already obvious from the presence of the scale transformations (3.5)]. For all solutions, $t_0 \ll t_m$, where t_m is the time that elapses from the singularity (at $t=0$) to the time of maximal expansion.

In the presence of moving matter, the typical states of the metric during the early stage of the expansion defined above by the condition $F=d(|g|^{1/6})/dt \gg 1$ depend strongly on the value of the integral M. If $M \gg 1$ for some solution (i.e., the matter moves fast), the corresponding trajectory of the system (2.12) can never be in a neighborhood of the sets Φ_{LKh} or N since $M=0$ on these sets. Therefore, for $M \gg 1$ the only typical state of the metric during the early stage of expansion is the power regime (3.3), which generalizes the Taub regime. But if $M \lesssim 1$ (slowly moving matter), the power regimes (3.1), (3.2), and (3.3) are typical states of the metric during the early stage of expansion, as for the diagonal metric.

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APPENDIX: COMBINATORIAL MODEL OF THE OSCILLATORY MODE

The Appendix contains a complete integration of the separatrices of the singular points of K and derivation of a combinatorial model of the oscillatory mode by means of an approximation of a trajectory of the system (2.12) by a sequence of separatrices of the singular points of K past which this trajectory moves (as space contracts).

We give an invariant description of the set K. The point P of this set is determined by two matrices: $P=(y_{ij}, s_j^k)$ (at the same time $w=0$, $H_1(P)=(\mathrm{Sp}(s))^2 = 2\mathrm{Sp}(s^2)=0$, $s=\| s_j^k \|$). In accordance with (3.4), the matrix y_{ij} has rank 1; let e_y be the eigenvector of y_{ij} corresponding to its unit eigenvalue. In accordance with (3.4), the vector e_y is also an eigenvector of the matrix s_j^k; let s be the eigenvalue of this matrix corresponding to it. (An equivalent formulation is this: the matrix y_{ij} is a projector onto some principal direction of the matrix s_j^k.)

Suppose $s_1 \geqslant s_2 \geqslant s_3$ are the three eigenvalues of the matrix s_j^k. It is convenient to split the set K into three subsets K_1, K_2, K_3 defined by the following condition: $s=s_l$ on K_l.

The eigenvalues of the system (2.12) and their principal (eigen) directions at the singular points of K_l are

$$\begin{gathered}\lambda_1=2(1-k)(s_1+s_2+s_3), \text{ variables } s_k^j,\\ \lambda_2=8(s_n+s_m-s_l), \text{ variable } w,\\ \lambda_3=8(s_l-s_n), \quad \lambda_4=8(s_l-s_m), \text{ variables } y_{ij}.\end{gathered} \tag{A.1}$$

Here, $(s_l, s_m, s_n)=(s_1, s_2, s_3)$. The remaining seven eigenvalues $\lambda_5, \ldots, \lambda_{11}$ are equal to zero and correspond to directions that are tangent to the set K. By virtue of the conditions

$$H_1(K)=(s_1+s_2+s_3)^2-2(s_1^2+s_2^2+s_3^2)=0 \text{ and } s_1+s_2+s_3 \leqslant 0$$

we find that $s_i \leqslant 0$ and the signs of the eigenvalues λ_1, λ_2, λ_3, λ_4 are different, i.e., the singular points of K_l are nondegenerate and unstable.

We now integrate the separatrices of the singular points of K. From every point $\overline{P}(\overline{y}_{ij}, \overline{s}_k^j)$ belonging to K_1 there emanates a two-dimensional separatrix that moves along the boundary component Γ_w ($w=0$) at the level $H_1=0$ [see (A.1)]. This separatrix has the form

$$w=0, \quad s_k^j(\tau_1)=\overline{s}_k^j, \quad y_{ij}(\tau_1)=\frac{(\exp(-8\tau_1\overline{s}_k^j))\circ g_0}{[\mathrm{Sp}((\exp(-8\tau_1\overline{s}_k^j))\circ g_0)^2]^{1/2}}, \tag{A.2}$$

where g_0 is a symmetric matrix such that $g_0\overline{s}^t=\overline{s}g_0$ [then for all τ_1: $y(\tau_1)\overline{s}^t=\overline{s}y(\tau_1)$, see (2.13)] and $y_{ij}(-\infty)=\overline{y}_{ij}$. We denote $y_{ij}^1=y_{ij}(+\infty)$. It is obvious that the matrix y_{ij}^1 has rank 1 and $y^1\overline{s}^t=\overline{s}y^1$. It is not difficult to verify that the equation $y^1\overline{s}^t=\overline{s}y^1$ means that the matrix y^1 is a projector onto a principal direction of the matrix $\overline{s}$, i.e., the point $P^1=(y_{ij}^1, \overline{s}_k^j)$ belongs to either the set K_2 or the set K_3. It follows from (A.1) that only a one-dimensional separatrix enters a point of K_2 along the manifold $w=0$, $H_1=0$; therefore, almost the whole of the two-dimensional separatrix that emanates from the point $\overline{P}$ enters the point $P^1=(y_{ij}^1, \overline{s}_k^j)$ belonging to K_3, and a one-dimensional separatrix is separated from it and enters the point $(y_{ij}^2,\ \bar{s}_k^j)$ of K_2. The matrices y_{ij}^1 and y_{ij}^2 are projectors onto the principal directions of the matrix $\bar{s}_k^j$ corresponding to the eigenvalues s_3 and s_2, respectively.

From each point $\bar{P}=(\bar{y}_{ij},\ \bar{s}_k^j)$ belonging to K_2 there emanates a one-dimensional spearatrix of the form (A. 2) which enters the point $P^1=(y_{ij}^1,\ \bar{s}_k^j)$ of K_3.

From each point $\bar{P}=(\bar{y}_{ij},\ \bar{s}_k^j)$ belonging to K_3 there emanates a one-dimensional separatrix having the form

$$y_{ij}(t)=\bar{y}_{ij},\qquad s_k^j(t)=\bar{s}_k^j\frac{\operatorname{ch}t_0}{\operatorname{ch}t}+\frac{\operatorname{sh}t-\operatorname{sh}t_0}{\operatorname{ch}t}\bar{y}_{ij},$$
$$w(t)=-4(\operatorname{sh}t-\operatorname{sh}t_0)[\operatorname{sh}t+\operatorname{sh}t_0-2(s_1+s_2)\operatorname{ch}t_0]\operatorname{ch}^{-2}t. \tag{A. 3}$$

Here, the time t is related to τ_1 by $dt=w(t)\,d\tau_1$ and the constant t_0 is determined by the condition $\tanh t_0=s_3$, where $0\geqslant s_1\geqslant s_2\geqslant s_3$ are the eigenvalues of the matrix $\bar{s}_k^j$. The separatrix (A. 3) is defined for $t_0\leqslant t\leqslant t_1<0$. For $t=t_0$, we obtain the initial point $\bar{P}=(\bar{y}_{ij},\ \bar{s}_k^j)$; for

$$t=t_1=\frac{2(s_1+s_2)-s_3}{(1-(s_3)^2)^{1/2}}$$

we obtain the final point $P^1=(\bar{y}_{ij},\ \bar{s}_k^j(t_1)),\ w(t_1)=0,$ $H_1(t)\equiv 0$. It follows from (A. 3) that the final matrix $s_k^j(t_1)$ is obtained as the first point of intersection of the shorter arc of the great circle that passes (on the sphere $\sum_{k,j=1}^3(s_k^j)^2=1$) through the two matrices, $\bar{s}_k^j$ and $\bar{y}_{ij}$, and the surface $H_1(s_k^j)=(\mathrm{Sp}(s))^2-2\mathrm{Sp}(s^2)=0$. The final point $P^1=(\bar{y}_{ij},\ s_k^j(t_1))$ belongs, as is readily verified, to the set K_1 or the set K_2.

The results of the integration of the separatrices are reflected in a separatrix diagram of the form

$$\begin{tikzcd} & K_1 \arrow[dl, shift left, "a_1^3"] \arrow[dr, "a_1^2"] & \\ K_3 \arrow[ur, shift left, "a_3^1"] \arrow[rr, shift right, "a_3^2"'] & & K_2 \arrow[ll, shift right, "a_2^3"'] \end{tikzcd} \tag{A. 4}$$

Here, the arrow and the symbol a_i^j denote a transition along a separatrix from the set K_i to the set K_j. Note that from each point of K_1 two transitions are possible: a_1^3 and a_1^2, but, as is readily verified, $a_2^3\circ a_1^2=a_1^3$, i.e., after these two transitions one and the same point on the set K_3 is obtained.

As we noted earlier, a general trajectory of the system (2.12) after a certain time begins to move along the separatrices of the singular points of K_l and can therefore be approximated in accordance with (A. 4) by an infinite sequence of these separatrices and singular points. When the trajectory is in the neighborhood of the singular points of K_l, one of the eigenvalues of the metric $g_{ij}(t)$ is much greater than the other two (since the matrix

$$y_{ij}=g_{ij}\Big/\Big(\sum_{\alpha,\beta=1}^3 g_{\alpha\beta}^2\Big)^{1/2}$$

has rank 1 on the set K). To this maximal eigenvalue there corresponds a principal direction of the matrix g_{ij} which is asymptotically close to the common eigenvector of the matrices y_{ij} and s_k^j. Thus, the infinite sequence of separatrices defined by the diagram (A. 4) is an approximation of the oscillatory mode of the metric $g_{ij}(t)$. This approximation is mapped as follows into the BLKh[1] piecewise approximation of the oscillatory mode by Kasner solutions.

Let $e_i^1,\ e_i^2,\ e_i^3$ be the eigenvectors of the matrix s_k^j (as we noted earlier, they coincide with those of the matrix $\varkappa_k^j$, which are called "Kasner axes" in[4]) and $s_1,\ s_2,\ s_3$ be the corresponding eigenvalues. In the case of motion of the trajectory along the separatrices of $a_1^3,\ a_1^2,\ a_2^3$ the metric $g_{ij}(t)$ can be approximated by the following Kasner solution:

$$g_{ij}(t)=C_1t^{2p_1}e_i^1e_j^1+C_2t^{2p_2}e_i^2e_j^2+C_3t^{2p_3}e_i^3e_j^3,$$

where the Kasner exponents p_1 are determined by

$$p_i=1-\frac{2s_i}{s_1+s_2+s_3}.$$

To motion of the trajectory along the separatrices of a_3^2 and a_3^1 in the BLKh model there corresponds a "change of the Kasner exponents and rotation of the Kasner axes." It is not difficult to show that the Kasner exponents p_i and eigenvectors e_i^a obtained after the transitions a_3^2 and a_3^1 are the same as after the "change of Kasner exponents and rotation of the Kasner axes" in the BLKh model. Thus, the separatrix approximation of the metric $g_{ij}(t)$ determined by the diagram (A. 4) is isomorphic to the BLKh approximation provided the basic transition along the two-dimensional separatrix of a_1^3 is chosen from the two possible transitions a_1^3 and a_1^2.

Let us describe briefly the combinatorial model obtained here for the oscillatory mode. In it, the trajectory of the system (2.12) is periodically in the neighborhood of the singular points of K, and these points are obtained from one another by successive application of some mapping T.

A point of the set K is a pair of matrices $(y_{ij},\ s_k^j)$ satisfying the conditions

$$s_k^k\leqslant 0,\ H_1(s)=(\mathrm{Sp}\,s)^2-2\,\mathrm{Sp}\,(s^2)=0,$$
$$\sum_{i,j=1}^3 y_{ij}^2=1,\qquad \sum_{i,j=1}^3(s_k^j)^2=1, \tag{A. 5}$$

where the matrix y_{ij} has rank 1 and is the projector onto a certain (real) principal direction of the matrix s_k^j. Let s_y be the eigenvalue of the matrix s_k^j corresponding to this direction [by virtue of (A. 5), $s_y\leqslant 0$].

On the set K there acts a mapping T defined as follows. If s_y is not the minimal eigenvalue of the matrix s_k^j, then the mapping T may be two-valued and carry the point $(y_{ij},\ s_k^j)$ to the point $(\bar{y}_{ij},\ s_k^j)$, where $\bar{y}_{ij}$ is the projector onto the other principal direction of the matrix s_k^j corresponding to the eigenvalue smaller than s_y. If s_y is the smallest eigenvalue of the matrix s_k^j, the mapping T is single valued and carries the point $(y_{ij},\ s_k^j)$ to the point $(y_{ij},\ \bar{s}_k^j)$, where the matrix $\bar{s}_k^j$ is the first point of intersection of the shorter arc of the great circle (on the sphere $\sum_{k,j=1}^3(s_k^j)^2=1$) passing through the points s_k^j and y_{ij} and the surface $H_1(s)=0$.

The mapping of this combinatorial model into the BLKh combinatorial model is determined by the fact that the principal directions of the matrix s^j_k coincide with the "Kasner axes," and the Kasner exponents p_i are determined by the equations

$$p_i=1-\frac{2s_i}{s_1+s_2+s_3},$$

where s_1, s_2, s_3 are the eigenvalues of the matrix s^j_k.

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Properties of second vacuum pole P' in the theory of the pomeron as a Goldstone particle

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It is proposed that the pomeron is a Goldstone particle that appears upon spontaneous symmetry breaking of the system of vacuum poles and P and P'. The properties of the pomeron are barely affected by the interaction and are determined by its bare (unrenormalized) characteristics. The properties of P' depend strongly on the interaction with the pomerons. The contribution of P' at low energies s contains terms that decrease in power-law fashion (and can, generally speaking, also oscillate as functions of ln s, depending on the choice of the model). At high energies this contribution goes over into an expression analogous to the usual negative contribution of non-enhanced reggeon branch cuts, but those containing a small cutoff radius and therefore strongly dependent on ln s. This can result in a rather rapid growth of the total cross section even in the experimental energy region. At a momentum transfer $t\neq 0$, a mixed state is produced in the system of two pomerons and its contribution to the angular distribution leads to the appearance of a second maximum at $t\neq 0$. The existence of such a state can therefore explain the known anomalies in the angular distributions of pp scattering at high energies.

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INTRODUCTION

In the theory of complex angular momenta, the $\omega=j=1$ Pomeranchuk pole P is the analog of a nonrelativistic massless excitation. An illustrative confirmation of this property is the fact that the positions $\omega=0$ of all the singularities corresponding to exchange of an arbitrary number of pomerons coincide (at reggeon momenta $\mathbf{k}=0$).[1,2]

For most nonrelativistic physical systems, the appearance of a massless Goldstone excitation is evidence of spontaneous breaking of the continuous symmetry existing in the system.[3] This phenomenon is well known in solid state physics,[4] namely, the onset of zero-gap excitations in a phase transition. It is therefore natural to assume that the existence of a pomeron is also due to an analogous cause, namely spontaneous breaking, at momentum transfers $t<0$, of the symmetry of a certain continuous group characterizing hadron interactions in a vacuum channel of positive signature ($t=-k^2$).

A phenomenological identification of the type of group that can be responsible for the appearance of a pomeron as a Goldstone particle is afforded by the character of the excitations with the aid of which it is customary to describe the vacuum channel. At $t<0$ this channel contains a second vacuum trajectory P' besides the pomeron. It was therefore proposed in[5], henceforth referred to as I, that the Pomeranchuk pole is produced as a Goldstone boson following spontaneous breaking of the symmetry of a system of two interacting reggeons P and P'. This hypothesis, as shown in I, leads to hindrances and constraints on the constants of the reggeon interactions, and makes it possible to find their possible forms when the interactions are expanded in powers of the reggeon momenta $\mathbf{k}_i$.

The traditional representations (see[6]) call for the contribution of P' to the cross sections of the processes to be small and to decrease in power-law fashion with increasing energy s:

