Theory of nonlinear oscillations of an isolated cylindrical magnetic domain

Yu. P. Mukhortov

All-Union Scientific Research Institute for Physicotechnical and Radiotechnical Measurements
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A study is made of nonlinear radial oscillations of an isolated magnetic cylindrical domain. The macroscopic Lagrangian of the system is found, and from it the nonlinear equation of motion is derived. The equation is investigated in the simplest case, in which all dissipative processes can be neglected. Three types of nolinearity are introduced: kinetic, potential, and mixed. In kinetic nonlinearity, an increase of the period of the oscillations occurs with increase of the energy of excitation. The maximum value of the period is 18% larger than the period of harmonic oscillations. In potential nonlinearity, the period increases without limit when the excitation energy approaches a certain limiting value. In mixed nonlinearity, the period decreases with increase of energy. The increase of the period does not exceed 10%. In all cases a limiting energy is found, which determines the characteristic energy scale and is an important parameter of the nonlinear theory. It is noted that in nonlinear oscillations of a cylindrical magnetic domain, the mean radius of the domain decreases.

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A cylindrical magnetic domain (CMD) is a closed region with reversed magnetization in a ferromagnetic film or plate with an axis of easy magnetization perpendicular to the surface, and subject to an external magnetic field that is also perpendicular to the surface of the film. In the plane of the film the ferromagnetic medium is homogeneous and isotropic. As was shown by Thiele[1,2], a motionless isolated cylindrical domain has a circular cross section; the domain radius is determined by the condition for a minimum of the energy.

Allowance for the inertia of a domain boundary in motion leads to the possibility of realizing harmonic oscillators. The problem of harmonic radial oscillations was first considered by Henry[3]. Later these oscillations were studied in considerable detail[4,5]. An attempt was made in[4] to construct a nonlinear theory of oscillations; a study of radial oscillations in a superlattice formed by CMD was made in[5]. In the latter case, oscillations of the domain boundaries are the analog of the optical branches of an ordinary crystal lattice[6]. In[7], a theory was developed for a more complicated type of oscillations, connected with departure of the domain shape from circularity.

Although the problem of nonlinear oscillations was also posed in[4], it remained virtually unsolved. Furthermore, the basic equation presented there does not agree with those derived below from first principles.

The aim of the present paper is to give a systematic derivation of the nontrivial part of the nonlinear equation of motion and to investigate it in cases in which it yields to analysis.

1. NONLINEAR EQUATION OF DYNAMICS

From the beginning we limit ourselves to the case of radial oscillations, leaving aside the possibility of excitation of a more complicated type of oscillations connected with change not only of the radius but also of the shape of the domain[7].

The starting point for the derivation of the equation will be the Hamiltonian function of the system. Then we shall obtain the Lagrangian function, from which follows the nonlinear equation that describes the dynamics of the radial oscillations of the CMD. The Lagrangian formalism enables us comparatively simply to allow for dissipative processes in the system, by introduction of a phenomenological Rayleigh function.

The role of Hamiltonian function is played by the total energy of a medium with an isolated CMD,

$$\mathcal{H}=E_W+E_H+E_M, \tag{1}$$

which contains three terms: E_W, the energy of the domain wall; E_H, the energy of interaction of the magnetization with the external constant magnetic field; and E_M, the magnetostatic energy of interaction between the magnetization of the domain and the magnetization of the ferromagnetic medium surrounding it.

In a state of rest of the domain boundary, the equation $\partial\mathcal{H}/\partial r = 0$ determines the equilibrium domain radius r_0. A detailed analysis of the equilibrium configuration was carried out in[1,2].

If the domain boundary moves, its energy E_W increases and becomes dependent on the velocity. At small velocities, the addition to the energy is proportional to the square of the velocity and can be treated as kinetic energy[3,8,9]. At high velocities, this approximation ceases to be valid. For the energy of a moving domain boundary, Enz[10] gives an expression identical in structure with the expression for the energy of a relativistic particle. Schlömann[11] obtained a more general expression, a result of which we shall also use. According to[11], the wall moves with the phase velocity of a spin wave that has an imaginary wave vector. Then the dispersion equation for spin waves determines the dependence of the energy density $\mathcal{E}$ of the domain boundary on its velocity of motion:

$$\mathcal{E}=\mathcal{E}_0 a(v), \tag{2}$$

$$\frac{v^2}{\gamma^2 H_a D}=-\frac{1+\sigma}{a^2}+2+\sigma-a^2. \tag{3}$$

Equation (3) is valid in a sufficiently weak magnetic field H, when $H \ll H_a$ or $H \ll Dv^2$, where v is the velocity of motion of the domain boundary, γ is the gyromagnetic ratio, D is the exchange constant, H_a is the anisotropy field, $\sigma = 4\pi M/H_a$, M is the saturation magnetization, and $\mathcal{E}_0 = 2M(H_a D)^{1/2}$ is the energy density of the wall at rest[1)]. According to[3], for the individual

terms in the expression (1) the following relations are valid:

$$E_M=\frac{8\pi}{3}M^2d^3\left\{1-\frac{1}{m}\left[\frac{h^2}{d^2}\mathrm{K}(m)-\left(1-\frac{h^2}{d^2}\right)\mathrm{E}(m)\right]\right\}, \quad (4)$$

$$E_H={}^1/_2\pi d^2hMH, \quad (5)$$

$$E_w=\pi\mathcal{C}hd,$$

where h is the thickness of the ferromagnetic film, d = 2r is the domain diameter, K and E are elliptic integrals of the first and second kinds, and m = $(1+h^2/d^2)^{-1}$. Solution of equation (3) closes the system:

$$a(v)=\left\{1+\frac{1}{2}\left[\sigma-\frac{v^2}{c^2}-\left(\left(\sigma-\frac{v^2}{c^2}\right)^2-4\frac{v^2}{c^2}\right)^{1/2}\right]\right\}^{1/2}, \quad (7)$$

where $c=\gamma(H_aD)^{1/2}$. Analysis of equation (7) shows that the domain wall velocity cannot exceed its limiting value $v_m=c[(1+\sigma)^{1/2}-1]$. For $v=v_m$, the energy density of the domain wall reaches its highest value $\mathcal{C}_{max}=\mathcal{C}_0(1+\sigma)^{1/4}$ [11]. For $\sigma\gg1$, (7) yields Enz's[10] result

$$a(v)=(1-v^2/v_M^2)^{-1/2}, \quad v_M=\gamma(H_aD\sigma)^{1/2}. \quad (8)$$

For materials in which CMD are observed, the contrary condition $\sigma<1$ is realized. For $\sigma\ll1$, one obtains from (7) the approximate formula, which we shall use hereafter,

$$a(v)=1+\frac{\sigma}{4}\left[1-\left(1-\frac{v^2}{v_m^2}\right)^{1/2}\right], \quad v_m=\frac{\gamma\sigma}{2}(H_aD)^{1/2}. \quad (9)$$

Finally, formula (4) can be approximated with sufficient accuracy by a simple expression, containing only elementary functions. As was noted by Callen and Josephs, the derivative $\partial E_M/\partial x$ on the interval $0<x<10$ practically coincides with the function $-x(1+{}^3/_2x)^{-1}$ [12]. By using this relation and integrating it, after simple transformations we obtain the final expression for the Hamiltonian function:

$$\tilde{\mathcal{H}}=\frac{\lambda x}{2}[a(\dot{x})-1]+\frac{\lambda x}{2}+\frac{bx^2}{2}-\frac{2}{3}x+\frac{4}{9}\ln\left(1+\frac{3}{2}x\right), \quad (10)$$

where $x=r/h$, $\lambda=\mathcal{C}_0/4\pi M^2h$, $b=H/4\pi M$, and

$$\tilde{\mathcal{H}}=\mathcal{H}/16\pi^2M^2h^3.$$

We have taken as reference value the energy of a uniform plate without CMD; therefore $\mathcal{H}=0$ for $x=0$.

To obtain the Lagrangian function, we use the well-known relation[13]

$$\tilde{\mathcal{H}}=\dot{x}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}}-\tilde{\mathcal{L}}, \quad (11)$$

where we shall regard as a differential equation for $\tilde{\mathcal{L}}$. On solving this equation, we get

$$\tilde{\mathcal{L}}=\dot{x}\int_0^{\dot{x}}\frac{\tilde{\mathcal{H}}(x,\dot{x})}{\dot{x}^2}d\dot{x}+\dot{x}f(x), \quad (12)$$

where f(x) is an arbitrary function of the coordinates. It is well known that the Lagrangian function is determined only to within a total time derivative of an arbitrary function of the coordinates; therefore we may without loss of generality set f(x) = 0. Substitution in (12) of the limiting expressions (8) and (9) leads to the following Lagrangians:

$$\tilde{\mathcal{L}}=\frac{\lambda x}{2}[1-\sqrt{1-(\dot{x}\tau)^2/\sigma}]-\mathcal{E}(x), \quad \sigma\gg1, \quad (13)$$

$$\tilde{\mathcal{L}}=\frac{\lambda x\sigma}{8}[\sqrt{1-(\dot{x}\tau)^2}-1+\dot{x}\tau\arcsin\dot{x}\tau]-\mathcal{E}(x), \quad \sigma\ll1, \quad (14)$$

where $\tau=h/v_m$ and $\mathcal{E}(x)=\tilde{\mathcal{H}}(x,\dot{x})$ for $\dot{x}=0$—a quantity that plays the role of the potential energy in the dynamic problem.

The equation of motion is now obtained in the usual form:

$$\frac{d}{dt}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}}-\frac{\partial\tilde{\mathcal{L}}}{\partial x}=-\frac{\partial\tilde{Q}}{\partial\dot{x}}, \quad \frac{\partial\tilde{\mathcal{H}}}{\partial t}=\dot{x}\frac{\partial\tilde{Q}}{\partial\dot{x}}, \quad (15)$$

where $\tilde{Q}=\pi\tilde{\beta}h^4x\dot{x}^2$ is Rayleigh's dissipation function, and where $\tilde{\beta}$ is a phenomenological coefficient of viscosity (friction)[6]. For comparison with the equation of motion from reference[4], we write out equation (15) for small velocities, when

$$a(v)=1+\frac{\sigma}{8}\frac{v^2}{c^2}\equiv1+\frac{1}{\mathcal{C}_0}\frac{mv^2}{2}. \quad (16)$$

Then $\tilde{\mathcal{L}}={}^1/_4\lambda xmv^2-\mathcal{E}(x)$, and (15) gives

$$\frac{\lambda x}{2}mh^2\ddot{x}-\frac{\lambda}{2}\frac{mh^2\dot{x}^2}{2}+\frac{\partial\mathcal{E}}{\partial x}=-\frac{\partial\tilde{Q}}{\partial\dot{x}}. \quad (17)$$

Equation (17) differs from that of[4] by the presence of the term $\lambda mh^2\dot{x}^2/4$, where m is the density of effective mass of the domain boundary. The equations agree only for harmonic oscillations, where one must neglect all nonlinear terms. This is the case that was analyzed in[3,4].

From the form of the Hamiltonian function (10) it follows directly that there are three types of nonlinearity: 1) nonlinearity due to the nonquadratic dependence of $a(\dot{x})$ on $\dot{x}$; we shall call it kinetic; 2) nonlinearity due to the nonquadratic dependence of $\mathcal{E}(x)$ on x; we shall call it potential; 3) nonlinearity due to the presence of the cross term $x\dot{x}^2$ even in the lowest-order approximation in the expansion of the kinetic energy with respect to $\dot{x}$; we shall call it mixed.

Before passing on to the analysis of these nonlinearities, we shall make a number of approximations that enable us to obtain results in analytic form. First, we shall neglect dissipation. Then the Hamiltonian function (10) will be a first integral of the equation (15), and the latter need no longer be solved. Second, we shall restrict ourselves to the approximation (9) for a(v), since it corresponds more nearly to the actual situation, although from the mathematical point of view the investigation of the "relativistic" oscillator (8) is of considerable interest. In view of the foregoing remarks, we can write down directly the first intergral of (15):

$$\varepsilon_0=\frac{\lambda\sigma x}{8}(1-\sqrt{1-(\dot{x}\tau)^2})+\mathcal{E}(x),$$

$$\mathcal{E}(x)=\frac{\lambda x}{2}+\frac{bx^2}{2}-\frac{2}{3}x+\frac{4}{9}\ln\left(1+\frac{3}{2}x\right). \quad (18)$$

A graph of $\mathcal{E}(x)$ is shown in Fig. 1.

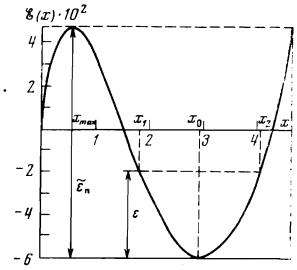

FIG. 1. Dependence of the potential energy (18) on domain radius x = r/h. The curve was drawn for $\lambda=0.5$, $b=0.1$. The energy of excitation ϵ is measured from the bottom of the potential well; x_1 and x_2 are the stopping points; $\tilde{\epsilon}_p$ is the maximum value of the excitation energy possible in finite motion.

2. KINETIC NONLINEARITY

If $|\dot{x}_T| \sim 1$, then kinetic nonlinearity becomes important. We are considering oscillations near equilibrium, where $\partial\mathscr{E}/\partial x = 0$; therefore the expansion of the potential energy begins with the quadratic term:

$$\mathscr{E}(x)=\mathscr{E}(x_0)+\frac{1}{2!}\frac{\partial^2\mathscr{E}}{\partial x^2}(x-x_0)^2+\frac{1}{3!}\frac{\partial^3\mathscr{E}}{\partial x^3}(x-x_0)^3+\ldots \quad (19)$$

The equilibrium domain radius x_0 is obtained by setting the first derivative of the potential energy equal to zero:

$$\lambda/2+bx_0-x_0(1+{}^3/_2x_0)^{-1}=0. \quad (20)$$

The larger of the two solutions of (20) corresponds to the minimum of the potential energy and determines the radius sought:

$$x_0=[(1-{}^3/_4\lambda-b)/3b]+(3b)^{-1}[(1-{}^3/_4\lambda-b)^2-3\lambda b]^{1/2}. \quad (21)$$

For the higher derivatives we get

$$\frac{\partial^2\mathscr{E}}{\partial x^2}=b-\left(1+\frac{3}{2}x_0\right)^{-2},\quad \frac{\partial^3\mathscr{E}}{\partial x^3}=3\left(1+\frac{3}{2}x_0\right)^{-3}. \quad (22)$$

In order to neglect the cubic term in the expansion of the potential energy in comparison with the remaining ones, we must require smallness of the displacements:

$$b-(1+{}^3/_2x_0)^{-2}\gg|x-x_0|_{max}(1+{}^3/_2x_0)^{-3}. \quad (23)$$

In addition, we shall suppose that

$$|x-x_0|_{max}\ll x_0. \quad (24)$$

If we use the parameter values for which the graph in Fig. 1 was constructed, then the criteria (23) and (24) lead respectively to the bounds $|x - x_0|_{max} \ll 1$ and $|x - x_0|_{max} \ll 3$. Such a relation between the criteria is typical in the region of stability far from collapse. The collapse field is found from the condition $\partial^2\mathscr{E}/\partial x^2 = 0$[12]:

$$b_{col}=1+{}^3/_4\lambda-(3\lambda)^{1/2}. \quad (25)$$

If $b > b_{col}$, existence of a CMD is impossible, and the medium goes over to the single-domain state. Contrariwise, at small fields there is formed a striped structure of serpentine domains. In the intermediate range of fields, there are isolated CMD[1,2].

On supposing that the inequalities (23) and (24) are satisfied, we obtain from (18)

$$\varepsilon={}^1/_8\lambda x_0\sigma(1-\sqrt{1-(\dot{x}\tau)^2})+{}^1/_2k(x-x_0)^2, \quad (26)$$

where $k = (\partial^2\mathscr{E}/\partial x^2)_{x=x_0}$ is the elasticity modulus, and where $\epsilon = \epsilon_0 - \mathscr{E}(x_0)$ is the energy of the excitation measured from the bottom of the potential well.

Before passing on to the solution of (26), we note that in general the total kinetic energy cannot exceed a certain greatest value. We shall determine this value. We rewrite (10) in the form

$$\varepsilon_0=T(x,\dot{x})+\mathscr{E}(x). \quad (27)$$

By solving (27) for $\dot{x}$ we determine the $\dot{x}(x)$ dependence. On setting the derivative dT/dx equal to zero, taking account of the $\dot{x}(x)$ dependence just found, we find the value of x for which the extremum is attained. This procedure can be simplified by means of (27):

$$\frac{dT}{dx}=\frac{d}{dx}[\varepsilon_0-\mathscr{E}(x)]=-\frac{\partial\mathscr{E}}{\partial x}=0. \quad (28)$$

The kinetic energy is maximum at the equilibrium value of the domain radius; this is entirely reasonable physically, since it is at this value that the potential energy is minimum. It has already been mentioned above that the energy density of the domain boundary has a greatest value, attained at the limiting velocity of motion. On the basis of the previous discussion, the greatest value of the kinetic energy determines the maximum value of the energy of excitation

$$\varepsilon_k=(\lambda x_0/2)[(1+\sigma)^{1/2}-1]. \quad (29)$$

For $\sigma \ll 1$,

$$\varepsilon_k=\lambda x_0\sigma/8. \quad (30)$$

On substituting $\lambda = 0.5$, $x_0 = 3$, and $\sigma = 0.2$, we get $\epsilon_k = 3.75 \times 10^{-2}$.

We shall now find the period of the oscillations of the CMD. For this purpose we solve equation (26) for $\dot{x}$:

$$\frac{dx}{dt}=\frac{1}{\tau}\left[\left(\frac{\varepsilon}{\varepsilon_k}-\frac{\mathscr{E}(x)}{\varepsilon_k}\right)\left(2-\frac{\varepsilon}{\varepsilon_k}+\frac{\mathscr{E}(x)}{\varepsilon_k}\right)\right]^{1/2}. \quad (31)$$

We integrate (31) with respect to the coordinate x between stopping points[2)], determined by the equation $\mathscr{E}(x) = \epsilon_0$, and with respect to t from zero to $\frac{1}{2}T$:

$$\int_0^{T/2}\frac{dt}{\tau}=\int_{-z_0}^{z_0}\left[\left(\frac{\varepsilon_0}{\varepsilon_k}-\frac{\mathscr{E}(z)}{\varepsilon_k}\right)\left(2-\frac{\varepsilon_0}{\varepsilon_k}+\frac{\mathscr{E}(z)}{\varepsilon_k}\right)\right]^{-1/2}dz, \quad (32)$$

where $z = x - x_0$, $z_0 = (2\epsilon/k)^{1/2}$. The integral on the right side can be reduced to a complete elliptic integral of the first kind by the change of variable of integration $z = z_0 \sin\theta$:

$$T=2\tau\left(\frac{\varepsilon_k}{k}\right)^{1/2}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{[1-(\varepsilon/2\varepsilon_k)\cos^2\theta]^{1/2}}, \quad (33)$$

whence we finally obtain the expression for the period

$$T=T_0\frac{2}{\pi}K(\sqrt{\varepsilon/2\varepsilon_k}), \quad (34)$$

where

$$T_0=2\pi\tau(\varepsilon_k/k)^{1/2} \quad (35)$$

is the period of the harmonic oscillations, which were investigated in detail earlier[3-5]; therefore we shall restrict ourselves to analysis of the dependence of the period of the oscillations on the energy. For small values of the argument, the elliptic integrals can be expanded as series ([14], 8.113),

$$K(m)=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2m^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2m^4+\ldots\left[\frac{(2n-1)!!}{2^n n!}\right]^2m^{2n}+\ldots\right\} \quad (36)$$

and truncated after the first nonvanishing term of the expansion that depends on the energy:

$$(T-T_0)/T_0=\varepsilon/8\varepsilon_k. \quad (37)$$

The largest value of the period is attained for $\epsilon = \epsilon_k$ and is

$$T=2T_0\pi^{-1}K(\sqrt{1/2})=1.18\ T_0. \quad (38)$$

A graph of the dependence of the period (34) on the energy of excitation is given in Fig. 2.

To understand the reason for the increase of period is quite simple if we calculate the generalized momentum and its time derivative:

$$\tilde{P}=\frac{\partial\tilde{\mathscr{L}}}{\partial\dot{x}}=\frac{\lambda x_0\sigma\tau}{8}\arcsin(\dot{x}\tau),\quad \frac{d\tilde{P}}{dt}=\frac{\lambda x_0\sigma\tau^2\ddot{x}}{8[1-(x\tau)^2]^{1/2}}. \quad (39)$$

It is evident that for $|\dot{x}_T| \to 1$ ($|v| \to v_m$) the derivative $d\tilde{P}/dt$ becomes infinite, whereas the potential force

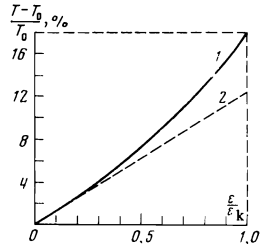

FIG. 2. Relative change of period $(T - T_0)/T_0$, in percent, for kinetic nonlinearity. Curve 1 is drawn according to formula (34), the straight line 2 according to (37).

remains always finite. This means that the time that the domain spends near an equilibrium position (when the velocity of the motion of the boundary is large) increases with increase of the energy of excitation. In the end this leads to increase of the period of the oscillations. In contrast to the relativistic case, the energy itself remains finite for $|v| \to v_m$.

3. POTENTIAL NONLINEARITY

As has already been mentioned in Sec. 1, potential nonlinearity is due to a nonquadratic dependence of the potential energy $\mathscr{E}(z)$ on z. It occurs even when the kinetic energy is quadratic in the velocity ($|\dot{z}_\tau|^2 \ll 1$) and contains no cubic terms ($|z| \ll x_0$). The Hamiltonian of the problem,

$$\varepsilon_0 = \varepsilon_k (\dot{z}_\tau)^2/2 + \mathscr{E}(z) \tag{40}$$

is obtained from the original (18) if in it we retain in the kinetic energy only the principal term of the expansion. The period of the oscillation is found by solution of the differential equation (40) similarly to what was done in Sec. 2:

$$T = 2\tau \varepsilon_k^{1/2} \int_{z_1}^{z_2} \frac{dz}{[2(\varepsilon_0 - \mathscr{E}(z))]^{1/2}}. \tag{41}$$

We were not successful in calculating the integral (41) with the function (18). We shall use the expansion (19), truncating after the cubic term. The following term of the expansion, that of fourth degree, may be neglected if $|z| \ll (1 + \frac{3}{2}x_0)$. For calculation of the integral it is convenient to introduce the new variable $u = z(1 + \frac{3}{2}x_0)^{-1}$; then after simple transformations we obtain from (41), with the aid of (19) and (22),

$$T = 2\tau \left(1 + \frac{3}{2}x_0\right) \varepsilon_k^{1/2} \int_{u_1}^{u_3} \frac{du}{(2\varepsilon - pu^2 - u^3)^{1/2}}, \tag{42}$$

where $p = k(1 + \frac{3}{2}x_0)^2$. The limits of integration in (41) and (42) are none other than the coordinates of the stopping points, which are obtained by setting the radicand equal to zero. Of the three roots of the equation, it is necessary to choose the two that are smallest in absolute value and correspond to the range of finite motion. The third root is located in the region of instability (collapse). On solving the cubic equation, we can reduce the integral to an elliptic intergral ([14], 3.131):

$$\int_{u_1}^{u_2} \frac{du}{[(u_2-u)(u-u_1)(u-u_3)]^{1/2}} = \frac{2}{(u_2-u_3)^{1/2}} K\left[\left(\frac{u_2-u_1}{u_2-u_3}\right)^{1/2}\right]. \tag{43}$$

where

$$u_1 = \frac{p}{3}\left[2\cos\left(\frac{\alpha}{3} + \frac{\pi}{3}\right) - 1\right], \quad u_2 = \frac{p}{3}\left[2\cos\left(\frac{\alpha}{3} - \frac{\pi}{3}\right) - 1\right], \tag{44}$$

$$u_3 = -\frac{p}{3}\left[2\cos\frac{\alpha}{3} + 1\right], \quad \cos\alpha = 1 - 2\frac{\varepsilon}{\varepsilon_p}, \quad 0 \leqslant \alpha \leqslant \pi,$$

$$\varepsilon_p = 2(p/3)^3. \tag{45}$$

The quantity (45) plays the role of limiting energy in potential nonlinearity. Its physical meaning is very simple: for $\epsilon = \epsilon_p$ the roots u_1 and u_3 of the equation coincide. This means that the line of constant energy is tangent to the $\mathscr{E}(z)$ curve at its maximum. For energies $\epsilon > \epsilon_p$, finite motion is impossible.

By means of (43) and (44), we obtain from (42) the final expression for the period of the oscillations:

$$\frac{T}{T_0} = \frac{2}{\pi} \frac{3^{1/4}}{[2\sin(1/3\alpha + 1/3\pi)]^{1/2}} K\left[\left(\frac{\sin 1/3\alpha}{\sin(1/3\alpha + 1/3\pi)}\right)^{1/2}\right]. \tag{46}$$

For small values of the energy, $\epsilon/\epsilon_p \ll 1$, the expansion (36) can be used:

$$(T - T_0)/T_0 = 5/36\, \varepsilon/\varepsilon_p. \tag{47}$$

The behavior of the period (46) for $\epsilon \to \epsilon_p$ is interesting. With the aid of the well-known formula ([14], 8.113)

$$K(m) = 1/2 \ln[16(1-m^2)^{-1}], \quad m^2 \to 1 \tag{48}$$

(46) yields, with sufficient accuracy,

$$\frac{T - T_0}{T_0} = \frac{1}{2\pi} \ln\left(\frac{532}{1 - \varepsilon/\varepsilon_p}\right) - 1 \approx \frac{1}{2\pi} \ln(1 - \varepsilon/\varepsilon_p)^{-1}, \quad \varepsilon \to \varepsilon_p. \tag{49}$$

Thus for $\epsilon \to \epsilon_p$ the period becomes infinite. The reason for this phenomenon is as follows. The derivative $\partial\mathscr{E}/\partial z$ determines the generalized force that acts on the domain during the oscillation process and tends to return it to the equilibrium position. For negative z, with increase of the energy of excitation the magnitude of the restoring force diminishes, until it becomes zero. At the point $\partial\mathscr{E}/\partial z = 0$, the energy $\mathscr{E}(z)$ has a maximum value $\mathscr{E}(x_0) + \epsilon_p$. If ϵ is exactly equal to ϵ_p, the oscillator, having begun its motion at some point and going to the stopping point, will remain there, since the restoring force at the extremum is zero and the oscillator arrives at this point with zero velocity. In formulas (46) and (49) this situation corresponds to an infinite period of the oscillations. In contrast to the kinetic-nonlinearity case analyzed above, the increase of the period in the present case is due to increase of the time that the domain spends near the stopping point.

For large energies, formulas (46) and (49) for the period are not valid, since the expansion (19) of the potential energy holds only for small departures from equilibrium. Therefore for large energies they give only a qualitative indication of the trend in the dependence of the period on the energy. The asymptotic behavior at large energies can be obtained by expanding $\mathscr{E}(x)$ near the maximum, which is reached at the point

$$x_{max} = [(1 - 3/4\lambda - b)/3b] - (3b)^{-1}[(1 - 3/4\lambda - b)^2 - 3\lambda b]^{1/2}. \tag{21a}$$

Thus we should obtain the correct energy dependence of the period; for as was explained above, for large energies the principal contribution to the period comes from the neighborhood of the stopping point. Starting with formula (41) and repeating calculations analogous to those made in the derivation of (49), we arrive at the expression

$$T = A \ln[B(1 - \varepsilon/\tilde{\varepsilon}_p)^{-1}], \tag{49a}$$

$$\tilde{\varepsilon}_p = \mathscr{E}(x_{max}) - \mathscr{E}(x_0), \tag{45a}$$

A and B are constants independent of the energy.

Thus for large energies of excitation, the period actually diverges logarithmically. The characteristic energy $\tilde{\epsilon}_p$ differs from the energy ϵ_p of (45) in that it is determined from the exact potential curve (18) and

not from the expansion (19), which is valid only at small departures from equilibrium. The energy ϵ_p is a characteristic parameter of the theory for small energies of excitation. The energy $\tilde{\epsilon}_p$, which has the direct physical meaning of limiting energy for the actual potential curve, is a characteristic parameter of the theory for large energies of excitation.

The energy dependence (46) of the period is shown in Fig. 3, where (dotted) the dependence according to formula (47) is also shown.

In closing this section, we note still another effect that accompanies potential nonlinearity in CMD oscillations. This is a change of the mean radius of the domain.

The mean radius of the domain, $\bar{x} = (x_1 + x_2)/2$, for small oscillations coincides with the equilibrium value: $\bar{x} = x_0$. If the $\mathscr{E}(x)$ curve is asymmetric with respect to the point $x = x_0$, then the maximum excursions of the domain boundary are not equal to each other: $|x_1 - x_0| \neq |x_2 - x_0|$, as is quite obvious from Fig. 1. Therefore $\bar{x} \neq x_0$ for anharmonic oscillations.

We shall calculate the change of mean radius for small $x - x_0$, when the expansion of the potential energy can be truncated after the cubic term. In the lowest nonvanishing order with respect to the energy, we get with the aid of (44)

$$\bar{x}-x_0=\left(1+\frac{3}{2}x_0\right)\frac{u_1+u_2}{2}-x_0$$
$$\approx-\frac{2}{27}p\frac{\varepsilon}{\varepsilon_p}\left(1+\frac{3}{2}x_0\right)=-\varepsilon k^{-2}\left(1+\frac{3}{2}x_0\right)^{-3}. \quad (50)$$

For $b = 0.1$, $x_0 = 3$, and $\epsilon = 3.7 \cdot 10^{-2}$ we get $x_0 - \bar{x} = 7.2 \cdot 10^{-2}$, which amounts to 2.4% of the equilibrium radius. From Fig. 1 it follows directly that the smallest value of the mean radius that can in principle be attained in nonlinear oscillations with the illustrative dependence $\mathscr{E}(x)$ chosen is ~2.6.

4. MIXED NONLINEARITY

We shall suppose that $|\dot{x}\tau| \ll 1$ and that the condition (23) is satisfied. Then in the expansion of the kinetic energy we need to retain only the lowest nonvanishing order in the velocity, and in the expansion of the potential energy only the lowest nonvanishing order in the coordinate. Mixed nonlinearity arises because of the presence of the cross term $z\dot{z}^2$ in the kinetic energy:

$$\varepsilon=\varepsilon_k\left(1+\frac{z}{x_0}\right)\frac{(\dot{z}\tau)^2}{2}+\frac{kz^2}{2}. \quad (51)$$

In order that this type of nonlinearity may dominate over the potential, still another inequality must be satisfied:

$$\varepsilon_k\frac{z}{x_0}(\dot{z}\tau)^2\gg z^3\left(1+\frac{3}{2}x_0\right)^{-3}. \quad (52)$$

On using for an estimate the value (35) of the period of harmonic oscillations, we get from (52)

$$k(1+{}^3/_2x_0)^3\gg 4\pi^2x_0. \quad (53)$$

By solution of equation (51) for $\dot{z}$ and integration,

$$\tau\left(\frac{\varepsilon_k}{kx_0}\right)^{1/2}\int_{-z_0}^{z_0}\frac{dz(x_0+z)^{1/2}}{(z_0^2-z^2)^{1/2}}=\int_0^{T/2}dt, \quad z_0=(2\varepsilon/k)^{1/2}, \quad (54)$$

the change of variables $z = z_0 \cos\theta$ reduces the integral to a tabulated integral:

$$T=T_0\frac{1}{\pi}\int_0^{\pi}\sqrt{1+\frac{z_0}{x_0}\cos\theta}\,d\theta, \quad (55)$$

hence, by ([14], 2.576), we finally get the following expression for the period:

$$\frac{T}{T_0}=\frac{2}{\pi}(1+\sqrt{\varepsilon/\varepsilon_m})^{1/2}\mathrm{E}\left[\left(\frac{2\varepsilon^{1/2}}{\varepsilon^{1/2}+\varepsilon_m^{1/2}}\right)^{1/2}\right], \quad (56)$$

where $\epsilon_m = kx_0^2/2$ is a characteristic energy, which the energy of excitation may not exceed in mixed nonlinearity. It corresponds to the potential energy at a displacement z equal to the domain radius x_0. Actually this value can never be attained, since the minimum domain radius is necessarily limited by the bounds of the region (21a) in which finite motion is possible.

The difference of mixed nonlinearity from those considered above lies in the fact that it leads to a diminution of the period. For small energies of excitation ($\epsilon \ll \epsilon_m$), one can use the expansion of the elliptic integral of the second kind in powers of the argument ([14], 8.114)

$$\mathrm{E}(m)=\frac{\pi}{2}\left[1-\frac{1}{4}m^2-\frac{3}{64}m^4-\ldots\right],$$

this leads to the following approximate formula:

$$(T-T_0)/T_0=-{}^1/_{16}\varepsilon/\varepsilon_m. \quad (57)$$

The smallest value of the right member of (56) is $(8/\pi^2)^{1/2} = 0.9$. Figure 4 shows the dependence of the period on the energy of excitation according to formulas (56) and (57).

The reason for the diminution of the period of the oscillations is the following. The presence of the factor $(1 + z/x_0)$ before the kinetic energy in (51) leads to the result that on the interval $(-z_0, 0)$ the mean rate of change of the kinetic energy is somewhat larger, and on the section $(0, z_0)$ somewhat smaller, than for a harmonic oscillator. This means that the time during which a change of kinetic energy from 0 to ϵ is occurring on the interval $(-z_0, 0)$ is always less, and on the interval $(0, z_0)$ always greater, than for a harmonic oscillator. For small ϵ, the total effect can be estimated by introducing a correction factor $(1 \pm z_0/x_0)$ to the time of stay of the harmonic oscillator on the different intervals during motion from one stopping point to the other:

$$T\sim\frac{T_0}{2}\left(1-\frac{z_0}{x_0}\right)+\frac{T_0}{2}\left(1+\frac{z_0}{x_0}\right).$$

Hence it follows that in the lowest order in z_0/x_0, the period is unchanged. In the next order such a change

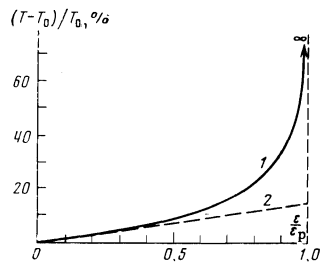

FIG. 3. Relative change of period $(T - T_0)/T_0$, in percent, for potential nonlinearity. Curve 1 is drawn according to formula (46), the straight line according to (47).

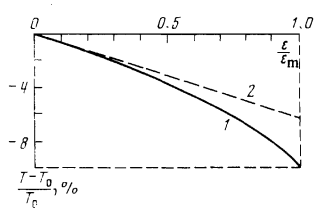

FIG. 4. Relative change of period $(T - T_0)/T_0$, in percent, for mixed nonlinearity. Curve 1 is drawn according to formula (56), the straight line 2 according to (57).

occurs, because the expansion (57) begins with the quadratic term $\epsilon = kz_0^2/2$, whereas the term linear in z_0 is absent.

In closing this section, we shall give an expression for the change of period when all three types of nonlinearity manifest themselves and the energy of excitation is small in comparison with all the characteristic energies (ϵ_k, ϵ_p, $\tilde{\epsilon}_p$, and ϵ_m). For this purpose it is obviously necessary to sum the expressions (37), (47), and (57):

$$\frac{T-T_0}{T_0}=\frac{1}{8}\frac{\varepsilon}{\varepsilon_k}+\frac{5}{36}\frac{\varepsilon}{\varepsilon_p}-\frac{1}{16}\frac{\varepsilon}{\varepsilon_m}, \tag{58}$$

where

$$\varepsilon_k=\frac{\lambda x_0\sigma}{8}, \quad \varepsilon_p=2\left[\frac{k(1+{}^3/_2x_0)^2}{3}\right]^3, \quad \varepsilon_m=\frac{kx_0^2}{2}. \tag{59}$$

Formula (58) enables us to determine which of the types of nonlinearity is the main one in each concrete case, for small energies of excitation. By way of example we shall give numerical values for $\lambda = 0.5$, $\sigma = 0.2$, $b = 0.1$. Then $k = 6.7 \times 10^{-2}$, $\epsilon_k = 3.7 \times 10^{-2}$, $\epsilon_p = 0.6$, $\epsilon_m = 0.3$. In this case the characteristic value of the energy for onset of kinetic nonlinearity is an order of magnitude smaller than the other values; therefore it will begin to show up sooner than the others. The potential and mixed nonlinearities almost exactly compensate each other, and in (58) only the first term is important (for the chosen values of the parameters).

It must be stipulated yet again that the calculated value (45) or (59) of the limiting energy for potential nonlinearity is correct only for comparatively small excitation energy, when the expansion (19) of the potential energy can be truncated after the cubic term. Directly from the graph of Fig. 1 and from (45a), one gets $\tilde{\epsilon}_p = 0.1$, which is six times smaller than the value calculated by formula (45). If the true value of $\tilde{\epsilon}_p$ is of the order of ϵ_k or smaller, then at large energies the principal increase of period (up to infinity) may occur precisely because of the presence of potential nonlinearity, although at small oscillations its contribution may be insignificant in comparison with the other types of nonlinearity. In Henry's paper[3] there is a reference to unpublished experimental data according to which the observed value of the frequency is less than that calculated by the linear theory by more than a factor three. Part of the discrepancy may be attributable to uncertainty in the values of the material parameters. It is quite possible, however, that it is a consequence of the anharmonic character of the oscillations[3)]. In every case, qualitatively, the decrease of resonance frequency can be connected with the influence of potential and kinetic nonlinearities.

We note finally that the elasticity modulus k, which plays an important role in the nonlinear theory (see (59)), can be directly determined by experiment[3]. In fact, if one makes a slight increment ΔH of the magnetic field, the domain radius will change by Δr. Supposing that the increments are small, we get from the equation $\partial \mathscr{E}/\partial x = 0$

$$\frac{\partial^2\mathscr{E}}{\partial x\,\partial b}\Delta b+\frac{\partial^2\mathscr{E}}{\partial x^2}\Delta x=0,$$

whence follows the desired relation

$$\frac{\Delta H}{4\pi M}+k\frac{\Delta r}{r_0}=0. \tag{60}$$

In the derivation of (60) we have calculated the derivative $\partial^2\mathscr{E}/\partial x\,\partial b$ by means of (20) and have used the definition of the elasticity modulus.

Another important quantity, ϵ_k, can be determined from the period T of oscillation of the domain for $\epsilon \to 0$ (35):

$$\varepsilon_k=k(\omega_0\tau)^{-2}, \quad \omega_0=2\pi/T_0, \tag{61}$$

if the value of the limiting velocity v_m is known.

CONCLUSION

At the basis of the theory constructed above lay two important assumptions. First, we treated the domain wall as infinitely thin, with an energy density independent of the radius of curvature. This assumption is fulfilled better, the larger the quality factor of the material $q = 1/\sigma$[1]. Second, the expression used for the energy density was that obtained for a uniformly moving plane boundary[11]. The range of applicability of this approximation has been discussed in[11] (see also[15]). Furthermore, the obtained Eq. (15) does not enable us to analyze the more complicated type of oscillations connected with change of shape of the CMD (profile of the domain wall)[7].

It should be mentioned also that allowance for viscosity in the problem of radial oscillations of a CMD leads to a fourth type of nonlinearity, which may be called dissipative. It is due to the presence of a cubic term in the Rayleigh function.

[1)] The notation and the normalization of constants correspond to those adopted in [11]. In the literature, as a rule, a different normalization is used: $D = 2A/M$, $H_a = 2K_u/M$, where A and K_u are the exchange and anisotropy constants, respectively. [1] The quantity $q = 1/\sigma$ is called the quality factor. The larger it is, the more justified becomes the approximation of the domain wall to zero thickness, which lies at the basis of the theory being developed. In the contrary case there arises, in particular, a dependence of the energy density of the domain boundary on the domain radius.

[2)] According to the general property of reversibility, the time of motion between stopping points does not depend on which of these points the motion begins at. Therefore the period of the oscillations is twice the time of a single passage between stopping points [13].

[3)] The excitation energy cannot be lowered to an arbitrarily small value. This is due first to the problem of detection of weak signals, and second to the presence of a coercive force.

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191