Phase transitions in systems with two order parameters

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Phase transitions in systems with two order parameters φ_1 and φ_2 are considered by Wilson's method. A model with distinct transition points for φ_1 and φ_2 is proposed, and describes the phase transitions in the system He^3-He^4. It is shown that a Landau critical point of second-order transitions exists in this model. In the case when the transition temperatures coincide it is shown that the character of the fixed points depends essentially on the dimensionalities of the order parameters.

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1. In this paper a phase transition in a two-component system is considered by Wilson's method[1]. The Hamiltonian of such systems are functional of two order parameters: φ_1 and φ_2. Examples of such systesm are liquid solutions, He^3–He^4 mixtures, mixtures of boson liquids, and also various phase transitions in solids. The dimensionalities of the order parameters φ_1 and φ_2 can be unequal, generally speaking. The case when the critical temperatures with respect to φ_1 and φ_2 are different was considered earlier[2]. Here, therefore, we shall discuss (Sec. 2) only the one special model with distinct transition points that describes, in our opinion, the phase transition in a He^3–He^4 mixture, and show that, under certain conditions in this model, a Landau critical point of second-order transitions arises[3].

In Sec. 3 we consider a model of a two-component system with the same transition points for φ_1 and φ_2. The interaction between the components is assumed to be of the Baxter type. The possible fixed-point values of the parameters of the Hamiltonian of the binary system are found. It is shown that, for $n_1 \neq n_2$ (n_1 and n_2 are the dimensionalities of the parameters φ_1 and φ_2), besides fixed points of a symmetric type (the fixed-point values of the coupling constants of the interactions that are quartic in φ_1 and φ_2 are equal) there exist asymmetric fixed points.

In Sec. 4 the stability of the solutions obtained is investigated. It is shown that in the case when the total dimensionality of the variables φ_1 and φ_2 is $n_1 + n_2 < 4$ symmetric solutions are stable even when $n_1 \neq n_2$. In the case when $n_1 + n_2 > 4$ solutions of the asymmetric type are found to be stable. In this case the system is characterized by two susceptibilities with unequal critical exponents. In all the cases considered there is no isomorphism of the phase transitions in the two-component systems with transitions in systems with one order parameter. The violation of isomorphism here is connected both with an effective change in the dimensionality of the order parameter and with the increase in the number of thermodynamic variables.

2. We write the Hamiltonian $H\{\varphi_1\varphi_2\}$ of the two-component system in the form

$$H\{\varphi_1\varphi_2\}=H_1\{\varphi_1\}+H_2\{\varphi_2\}+H_{int}\{\varphi_1\varphi_2\}, \tag{1}$$

$$H_i\{\varphi_i\}=H_{i0}+H_{i1}=\int d^dr[\tau_i|\varphi_i|^2+|\nabla\varphi_i|^2]+u_i\int d^dr|\varphi_i|^4; \tag{2}$$

here $\tau_i = (T - T_i)/T_i$ are the deviations from the critical temperatures T_i in a system of noninteracting components. The integration is performed over all space, of dimensionality d.

The concrete form of the operator $H_{int}\{\varphi_1\varphi_2\}$ depends on the system being considered. In this Section we shall assume that the variables φ_1 and φ_2 have dimensionalities $n_1 = 1$ and $n_2 = 2$, respectively. Precisely this situation obtains in a He^3–He^4 mixture. The variable φ_2 in this case describes the λ-transition in helium, and φ_1 specifies the deviation of the He^3 concentration from its mean value. H_{int} has the form

$$H_{int}=g\int d^dr\varphi_1|\varphi_2|^2. \tag{3}$$

At a given pressure P in a He^3–He^4 system there exists a line of λ-transitions $T_2(\mu)$ (μ is the chemical potential of the He^3) and, in addition, for g = 0 there is an isolated critical phase-separation point $T_1(P)$.

We shall show that, with this choice of H_{int}, as the critical points come together the λ-transition line goes over at a certain point (the Landau critical point) to a line of first-order transitions. The partition function of the system has the form

$$Z=\mathrm{Sp}\{\exp[-H\{\varphi_1\varphi_2\}/T]\}. \tag{4}$$

Integrating (4) over the noncritical variable φ_1 near the λ-line, we reduce the problem of calculating the partition function (4) of a two-component system to that of calculating the partition function of a one-component system with an effective Hamiltonian $H_{eff}\{\varphi_2\}$ calculable from the formula

$$H_{eff}\{\varphi_2\}=-T\ln\int d\varphi_1\exp[-H\{\varphi_1\varphi_2\}/T]. \tag{5}$$

Expanding (5) in H_{int}, it is not difficult to find

$$H_{eff}\{\varphi_2\}=H_2\{\varphi_2\}-T\sum_n\frac{g^{2n}}{T^{2n}(2n)!}\int\Gamma_{2n}(r_1\dots r_{2n})\prod_{i=1}^{2n}\varphi_2^2(r_i)d^dr_i. \tag{6}$$

Here,

$$\Gamma_{2n}(r_1\dots r_{2n})=\langle\langle\varphi_1(r_1)\dots\varphi_1(r_{2n})\rangle\rangle \tag{7}$$

denotes Γ_{2n}—the irreducible correlator of order 2n.

The averaging in the calculation of Γ_{2n} is performed over the distribution function $\exp\{-H_1\{\varphi_1\}/T\}$. The principal contribution to the integrals (6) is given by distances of the order of $r_{1c} \ll r_{2c}$ (r_{ic} are the correlation lengths of the fluctuations of φ_1 and φ_2 respectively). Therefore, in the principal approximation in r_{1c}/r_{2c}, we can replace

$$\prod_{i=1}^{2n}\varphi_2^2(r_i)$$

by $\varphi_2^{4n}(r)$. Taking into account also, that in the renormalization-group transformations all operators φ_2^m with m > 4 "die out"[4], we can confine ourselves to the term with n = 1 in the expansion (6). Thus, we obtain

$$H_{\text{eff}}\{\varphi_2\}=H_2\{\varphi_2\}-\frac{g^2}{2T}G_1(\tau_1)\int\varphi_2^4(r)\,d^dr. \tag{8}$$

Here we have introduced the Green function

$$G_1(\tau_1)=\int d^dr\,\Gamma_2(r).$$

Since the Green function $G_1(\tau_1)$ increases on decrease of τ_1, it is obvious that for any bare values u_i and g there always exists a temperature T_0 at which the effective quartic-interaction constant is equal to zero:

$$u_2-\frac{g^2}{2T}G_1(\tau_{10})=0 \qquad \left(\tau_{10}=\frac{T_0-T_1}{T_1}\right). \tag{9}$$

This means that at the transition point at which the temperature and concentration are determined by Eq. (9) and the additional equality

$$T_2(\mu)=T_0, \tag{10}$$

the curve $T_2(\mu)$ of second-order phase transitions goes over into a curve of first-order transitions[3]. The point T_0 is the Landau critical point.

We emphasize that the susceptibility $\chi_1 = G_1(\tau_1)$ increases as this point is approached but nevertheless remains finite at the point itself.

It can be seen from (9) that it is possible, by choosing the temperature and chemical potential μ as the variables, to write the condition for the Landau critical point in a two-component system in a form isomorphous with this condition in a one-component system. The vanishing of the effective quartic-interaction constant leads to features of the thermodynamic quantities that differ from the features near the second-order transition points (in particular it leads to the result that all the fluctuations become Gaussian even in the case of three-dimensional space[5]).

3. As a second example we shall consider a model of a two-component system in which the interaction operator H_{int} has the form

$$H_{int}=u_3\int\varphi_1^2(r)\varphi_2^2(r)\,d^dr. \tag{11}$$

Such a case will certainly be realized when the dimensionalities of the variables φ_1 and φ_2 are not equal and are greater than unity (e.g., in a mixture of Bose liquids[6]). In cases when $n_1 = n_2$, besides an interaction of the type (11) other types of interaction are also possible.

In the case when the transition points with respect to φ_1 and φ_2 are different, essentially the whole analysis of Sec. 2 can be applied to this case also. However, there is an important difference, in two respects: 1) unlike in the preceding results, here there is a transition-temperature shift, determined by the quantity $u_3\langle\varphi_1^2(r)\rangle$; 2) the effective Hamiltonian $H_{\text{eff}}\{\varphi_2\}$ in this case is determined by the irreducible correlators of the energy $\varphi_1^2(r_i)$. In particular, the renormalization of the bare quartic interaction constant u_2 will be determined by the quantity

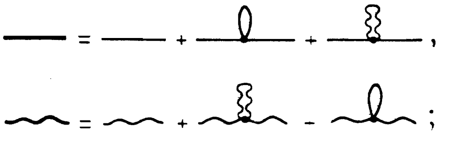
$$\Gamma_2'=\int\langle\langle\varphi_1^2(0)\varphi_1^2(r)\rangle\rangle\,d^dr,$$

which is proportional to the corresponding specific heat. Since this quantity depends weakly on the temperature, the equation analogous to Eq. (9) can hold only when there is a certain relationship between the bare constants u_2 and u_3. Otherwise, the phase transition in a two-component mixture with separated transition points is isomorphous with a phase transition in a one-component system[2].

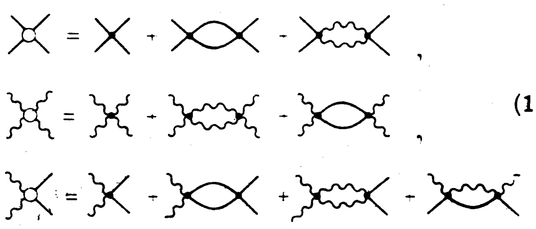
We shall consider the case when the transition points with respect to the variables φ_1 and φ_2 coincide ($\tau_1 = \tau_2 = \tau$). The analogous problem for the case of an anisotropic antiferromagnet was solved in the paper by Nelson et al.[8] Changing in (1) to the Fourier representation

$$\varphi_i(r)=\int\varphi_{ik}e^{ikr}\frac{d^dk}{(2\pi)^d}$$

and making use of Wilson's ϵ-expansion[1], we write down equations for the Green functions and vertex parts in leading order in $\epsilon = 4 - d$:



(12a)



(12b)

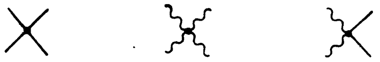
Here a straight line denotes

$$G_{10}(k)=\int d\varphi_{1k}|\varphi_{1k}|^2\exp\left[-\frac{H_{10}\{\varphi_1\}}{T}\right]\Big/\int d\varphi_{1k}\exp\left[-\frac{H_{10}\{\varphi_1\}}{T}\right]=\frac{1}{\tau+k^2}$$

a wavy line denotes

$$G_{20}(k)=\int d\varphi_{2k}|\varphi_{2k}|^2\exp\left[-\frac{H_{20}\{\varphi_2\}}{T}\right]\Big/\int d\varphi_{2k}\exp\left[-\frac{H_{20}\{\varphi_2\}}{T}\right]=\frac{1}{\tau+k^2}$$

and the vertices u_1, u_2 and u_3 respectively have the form



The momentum conservation law is fulfilled at each vertex. Integration is performed over all independent momenta.

Estimating the left-hand sides of Eqs. (12b) by scaling theory and making use of the condition for matching scaling theory and perturbation theory, we can obtain equations for the fixed-point values of u_i and the critical indices γ_1 and γ_2 determining the behavior of the corresponding susceptibilities $G_1(k = 0) = \tau^{-\gamma_1}$ and $G_2(k = 0) = \tau^{-\gamma_2}$:

$$\gamma_1-1=\pi^2[4u_1(n_1+2)+2n_2u_3],$$
$$\gamma_2-1=\pi^2[4u_2(n_2+2)+2n_1u_3]; \tag{13a}$$

$$\tfrac{1}{2}u_1\varepsilon=\pi^2[4u_1^2(n_1+8)+n_2u_3^2],$$
$$\tfrac{1}{2}u_2\varepsilon=\pi^2[4u_2^2(n_2+8)+n_1u_3^2],$$
$$\tfrac{1}{2}u_3\varepsilon=\pi^2u_3[4u_1(n_1+2)+4u_2(n_2+2)+8u_3]. \tag{13b}$$

In the case when any one of the vertices u_i is equal to zero, the corresponding equation in (12b) becomes an identity and drops out of the system (13b).

Taking what has been said into account, it is not difficult to find all the possible fixed points for a two-component system with the Hamiltonian (1):

1) $u_1=u_2=u_3=0$;

$$2)\ u_1=u_3=0,\qquad u_2=\frac{\varepsilon}{8\pi^2}\frac{1}{n_2+8};$$

$$3)\ u_2=u_3=0,\qquad u_1=\frac{\varepsilon}{8\pi^2}\frac{1}{n_1+8};$$

$$4)\ u_1=u_2=\frac{u_3}{2}=\frac{\varepsilon}{8\pi^2}\frac{1}{n_1+n_2+8}; \tag{14}$$

$$5)\ u_1=\frac{\varepsilon}{16\pi^2}\frac{n}{n^2+8}\left[1-s\frac{n-2}{2n(n-1)}\right],$$

$$u_2=u_1\left[1+s\frac{n-2}{n(n-1)}\right],\qquad u_3=u_1\frac{2(4-n)}{n}\left[1+s\frac{n-2}{2n(n-1)}\right];$$

$$6)\ u_3=0,\qquad u_1=\frac{\varepsilon}{8\pi^2}\frac{1}{n_1+8},\qquad u_2=\frac{\varepsilon}{8\pi^2}\frac{1}{n_2+8}.$$

In obtaining the fifth solution we have introduced the notation $n = \frac{1}{2}(n_1 + n_2)$, $s = \frac{1}{2}(n_1 - n_2)$ and have assumed that $|s|/n \ll 1$. Generally speaking, there exist two further fixed points, which are complex for $n_i > 1$.

4. To investigate the stability of the solutions obtained it is necessary to find the eigenvalues of the determinant of the linearized system of equations (13b):

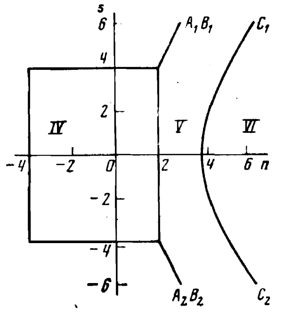
$$\begin{gathered}[\varepsilon-4(n_1+8)u_1]\delta u_1-n_2u_3\delta u_3=0,\\ [\varepsilon-4(n_2+8)u_2]\delta u_2-n_1u_3\delta u_3=0,\\ -2(n_1+2)u_3\delta u_1-2(n_2+2)u_3\delta u_2+[\varepsilon-2(n_1+2)u_1-2(n_2+2)u_2-8u_3]\delta u_3=0.\end{gathered} \tag{15}$$

The solution is stable if the three corresponding eigenvalues are all less than zero. Thus, it can be shown that the first three solutions (14) are unstable. The complex fixed points are also unstable, even in the region (with $n_i < 1$) where they become real.

Substituting the fourth solution into (15), we find that all three eigenvalues $\lambda_i < 0$, under the condition $\frac{1}{2}(n_1+n_2) \le 2$. The phase transition in this case is found to be isomorphous with a phase transition in a one-component system with an order parameter of dimensionality $m = n_1 + n_2$.

Under the condition $\frac{1}{2}(n_1 + n_2) > 2$ the fifth solution is found to be stable. It is important to emphasize that, for $n_1 \neq n_2$, two different susceptibilities, with $\gamma_1 \neq \gamma_2$, correspond to this solution (as can be seen from Eqs. (12a)).

On further increase of $n_1 + n_2$ the fifth solution also becomes unstable. For $\frac{1}{2}(n_1 + n_2) > 4$, the sixth solution, corresponding to two noninteracting components φ_1 and φ_2, is found to be stable. For $n_1 \neq n_2$, this solution is also asymmetric. The regions of stability of the fixed points in the (n, s)-plane are shown in the Figure. The line C_1C_2 is described by the expression $s^2 = (n - 4)(n + 8)$, and the lines A_1B_1 and A_2B_2 for $n - 2 \ll 1$ have the form $|s| = 2n$.



Regions of stability of the fixed points in the (n, s)-plane. The Roman numbers indicate the regions of stability of the corresponding solutions (14).

The situation considered can occur in the case of transitions in solids whose thermodynamic potential is described by means of an irreducible representation of sufficiently high dimensionality n. When some external field, not conjugate to the multi-dimensional order parameter, is applied, the corresponding irreducible representation of high dimensionality can decompose into two representations of dimensionalities n_1 and n_2. The transitions in the perovskites, e.g., in $SrTiO_3$,[7] can serve, apparently, as examples of such transitions.

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108