Stationary Langmuir and electromagnetic wave spectra when a relativistic electron beam heats a plasma

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We find the stationary turbulence spectra for the case when the plasma contains a given external source (relativistic electron beam) that excites Langmuir oscillations. The stationary state is maintained because non-linear interactions of the Langmuir oscillations with one another and with electromagnetic waves take them out of resonance with the beam and cause them later to be absorbed because of Coulomb collisions. The non-linear interaction mechanism is the induced scattering of the oscillations by the ions. We show that when we take into account the interaction between the Langmuir and the electromagnetic waves it is possible to secure collisional dissipation of the oscillations even well above threshold ($\gamma \gg \nu$, where γ is the growth rate of the instability and ν the collision frequency). This enables us to moderate considerably the conditions for the applicability of the weak-turbulence theory to the problem of plasma heating by a beam.

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1. INTRODUCTION

When studying the heating of a plasma by a relativistic electron beam or a powerful electromagnetic wave, the problem arises of the non-linear limitation of the level of the Langmuir oscillations which are excited by the external source in the plasma. If we do not go beyond the framework of the weak turbulence theory[1-3] the basic non-linear effect is usually the induced scattering of Langmuir waves by ions. Two scattering channels are known: the scattering of Langmuir into Langmuir waves (ll) and the scattering of Langmuir into electromagnetic waves (lt). The probabilities for these processes have the same order of magnitude, but due to a small optical depth of the plasma or inhomogeneity of its density the lt-scattering may be suppressed. The ll-scattering is then the main one. It leads to a transfer of the oscillations which are excited by the source to the long-wavelength part of the spectrum where there is no generation. If the change in the dispersive addition to the wave frequency in each elementary scattering process is small (differential transfer) the corresponding kinetic equation for the waves turns out to be relatively simple. The stationary Langmuir turbulence spectra were obtained for that case in[4] where it was shown that when the excitation is anisotropic the oscillation spectrum is concentrated on a few surfaces (jets) in wavevector space.[1)] When the long-wavelength oscillations are sufficiently strongly damped the jets are cut off in the long-wavelength region, while for small damping the solution here corresponds to a constant flux of Langmuir quanta along the spectrum. We note that the total energy of the Langmuir oscillations in this solution depends linearly on the instability growth rate while the power released in the plasma is proportional to the square of the growth rate.

We obtain in the present paper stationary turbulence spectra for the case when both ll- and lt-scattering are allowed. The different characteristics of these spectra consist in the fact that when the ratio of the growth rate γ of the instability of the Langmuir waves to the frequency ν of the electron collisions is large the stabilization of the instability is secured mainly because of the electromagnetic oscillations (their energy is much larger than that of the Langmuir oscillations). The power dissipated in the plasma is then directly proportional to γ. It is important that the spectra obtained do not require the inclusion of additional dissipation mechanisms of the long-wavelength oscillations as is the case when $\gamma \gg \nu$ when there is no lt-scattering when there is a constant flux of Langmuir quanta along the spectrum in the long-wavelength region.

We have chosen the following sequence of exposition in the present paper. We give in Sec. 2 the kinetic equations for the Langmuir and electromagnetic oscillations assuming that the spectral transfer is differential in character. We consider in Sec. 3 the problem of stationary turbulence spectra when the excitation of the Langmuir oscillations is isotropic. Already this simple model shows that the lt-scattering (when it is allowed) qualitatively changes the form of the stationary spectra. We obtain in Sec. 4 the spectrum corresponding to the excitation of Langmuir oscillations by two relativistic electron beams injected in opposite directions into the plasma. The last (fifth) section of the paper contains a discussion of the results obtained.

2. BASIC EQUATIONS

We write the electric field of the oscillations in the plasma as a superposition of the fields of the Langmuir (l) and electromagnetic (t) waves with slowly changing amplitudes:

$$\mathbf{E}(\mathbf{r};t)=\int E^l(\mathbf{k})\frac{\mathbf{k}}{k}\exp\{i\mathbf{kr}-i(\omega^l+\omega_0)t\}d^3\mathbf{k}$$
$$+\int \mathbf{E}^t(\mathbf{k})\exp\{i\mathbf{kr}-i(\omega^t+\omega_0)t\}d^3\mathbf{k}+\text{c.c.}, \tag{1}$$
$$\omega^l(\mathbf{k})\equiv\frac{3}{2}\omega_0k^2r_D^2,\quad \omega^t(\mathbf{k})\equiv\frac{1}{2}\omega_0\frac{k^2c^2}{\omega_0^2}.$$

As all frequencies of the oscillations which interest us are close to the electron plasma frequency ω_0 we have explicitly split off this quantity in Eq. (1) and introduced the dispersive additions ω^l and ω^t. In what follows it is implied that the phases of the waves are random so that

$$\langle E^l(\mathbf{k})E^{l*}(\mathbf{k}')\rangle=2\pi(\omega_0+\omega^l)N^l(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}'), \tag{2}$$
$$\langle E_\alpha^t(\mathbf{k})E_\beta^{t*}(\mathbf{k}')\rangle=2\pi(\omega_0+\omega^t)N_{\alpha\beta}^t(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}'). \tag{3}$$

The pointed brackets indicate averaging over the phases. The quantities $N^l(\mathbf{k})$ and $N_{\alpha\beta}^t(\mathbf{k})$ are the spectral functions of the l- and t-oscillations which are connected with the corresponding energy densities U^l and U^t through the following relations:

$$U^l=\int(\omega_0+\omega^l)N^l(\mathbf{k})d^3\mathbf{k}, \tag{4}$$

$$U^t=\int(\omega_0+\omega^t)N_{\alpha\alpha}{}^t(\mathbf{k})d^3\mathbf{k}. \tag{5}$$

We note that in Eq. (3) we have not averaged over the polarization of the electromagnetic oscillations. The fact is that because of the degeneracy of the dispersion law the difference in phase of t-waves with different polarizations can, in general, not be considered to be random. Because of this the spectral function $N^t_{\alpha\beta}$ turns out to be a tensor. The description of the t-waves by means of a single scalar quantity (the spectral density of the number of quanta) usually assumes that the waves are unpolarized. This is, of course, valid in the spherically symmetric case but there is no foundation for it when there is less symmetry (in particular, when there is axial symmetry).

The set of equations for the spectral functions which describes the ll- and lt-scattering by ions has the form

$$\frac{\partial}{\partial t}N^l(\mathbf{k})=N^l(\mathbf{k})\int\frac{(\mathbf{k}\mathbf{k}')^2}{k^2k'^2}N^l(\mathbf{k}')\,\mathrm{Im}\,G_{\mathbf{k}-\mathbf{k}';\,\omega^l-\omega'^l}\,d^3\mathbf{k}'$$
$$+N^l(\mathbf{k})\int\frac{k_\alpha k_\beta}{k^2}N_{\alpha\beta}{}^t(\mathbf{k}')\,\mathrm{Im}\,G_{\mathbf{k}-\mathbf{k}';\,\omega^l-\omega'^t}\,d^3\mathbf{k}, \tag{6}$$

$$\frac{\partial}{\partial t}N_{\alpha\beta}{}^t(\mathbf{k})=-\frac{i}{2}\Gamma_{\alpha\mu}(\mathbf{k})N_{\mu\beta}{}^t(\mathbf{k})+\frac{i}{2}\Gamma_{\beta\mu}{}^*(\mathbf{k})N_{\alpha\mu}{}^t(\mathbf{k}),$$

$$\Gamma_{\alpha\beta}(\mathbf{k})\equiv\int\frac{[k_\alpha'k^2-k_\alpha(\mathbf{k}\mathbf{k}')][k_\beta'k^2-k_\beta(\mathbf{k}\mathbf{k}')]}{k^4k'^2}N^l(\mathbf{k}')G_{\mathbf{k}-\mathbf{k}';\,\omega^t-\omega'^l}\,d^3\mathbf{k}'. \tag{7}$$

The function $G_{\mathbf{k};\omega}$ which occurs in Eqs. (6) and (7) is given by the following formula:

$$G_{\mathbf{k};\omega}=\frac{\omega_0^2}{2n}\int\frac{\mathbf{k}\,\partial f/\partial\mathbf{p}}{\mathbf{k}\mathbf{v}-\omega}d^3\mathbf{p}\left[1-T\int\frac{\mathbf{k}\,\partial f/\partial\mathbf{p}}{\mathbf{k}\mathbf{v}-\omega}d^3\mathbf{p}\right]^{-1}, \tag{8}$$

where f is the equilibrium ion distribution function, normalized to unity, $\mathbf{p}$ the ion momentum, n the plasma density, and T the electron temperature. When evaluating the integral in Eq. (8) we must use Landau's rule to go round the pole so that $G_{-\mathbf{k};-\omega}=G^*_{\mathbf{k};\omega}$.

The set of Eqs. (6) and (7) can be obtained by the general methods of the weak turbulence theory[1-3], but the calculation turns out to be more compact if we follow Zakharov[5] and right from the start separate the equations for the fast (electron) and the slow (ion) motions, and afterwards average in these equations over the random phases. Such a derivation of the expression for the probability for ll-scattering was given in[4]. It can easily be generalized to the lt-scattering case.[6] One can also easily in the initial equations take into account the scattering of electromagnetic into electromagnetic waves, but this addition is unimportant as it contains an additional small parameter T/mc^2 as compared to the contribution from the ll- and lt-processes (see, e.g.,[3], p. 313).

We shall be interested in what follows in spectra that have axial symmetry. This enables us to simplify the set of Eqs. (6) and (7). We introduce a unit vector $\mathbf{n}$ giving the preferred direction and two mutually perpendicular unit vectors $\mathbf{e}_1(\mathbf{k})$ and $\mathbf{e}_2(\mathbf{k})$ corresponding to the two directions of the electric field of an electromagnetic wave with wavevector $\mathbf{k}$:

$$\mathbf{e}_1(\mathbf{k})=\frac{\mathbf{n}k^2-\mathbf{k}(\mathbf{n}\mathbf{k})}{|\mathbf{n}k^2-\mathbf{k}(\mathbf{n}\mathbf{k})|},\quad \mathbf{e}_2(\mathbf{k})=\frac{[\mathbf{n}\mathbf{k}]}{|[\mathbf{n}\mathbf{k}]|}. \tag{9}$$

One checks easily that in the axially symmetric case the matrix $\Gamma_{\alpha\beta}(\mathbf{k})$ can be written in the following form:

$$\Gamma_{\alpha\beta}(\mathbf{k})=\Gamma_1e_{1\alpha}e_{1\beta}+\Gamma_2e_{2\alpha}e_{2\beta}.$$

The tensor $N^t_{\alpha\beta}(\mathbf{k})$ can also be expanded in terms of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$N_{\alpha\beta}{}^t(\mathbf{k})=N_1e_{1\alpha}e_{1\beta}+N_2e_{2\alpha}e_{2\beta}+N_{12}e_{1\alpha}e_{2\beta}+N_{12}{}^*e_{1\beta}e_{2\alpha}. \tag{10}$$

We note now that owing to the axial symmetry only the first two terms in Eq. (10) give a non-vanishing contribution to the right-hand side of Eq. (9). It is therefore more convenient to deal with three equations for the quantities $N^l(\mathbf{k})$ and $N_\lambda(\mathbf{k})$ ($\lambda=1,2$) instead of the initial set of equations:

$$\frac{\partial}{\partial t}N^l(\mathbf{k})=N^l(\mathbf{k})\int\frac{(\mathbf{k}\mathbf{k}')^2}{k^2k'^2}N^l(\mathbf{k}')\,\mathrm{Im}\,G_{\mathbf{k}-\mathbf{k}';\,\omega^l-\omega'^l}\,d^3\mathbf{k}'$$
$$+N^l(\mathbf{k})\sum_{\lambda=1,2}\int\frac{(\mathbf{k}\mathbf{e}_\lambda')^2}{k^2}N_\lambda(\mathbf{k}')\,\mathrm{Im}\,G_{\mathbf{k}-\mathbf{k}';\,\omega^l-\omega'^t}\,d^3\mathbf{k}', \tag{11}$$
$$\frac{\partial}{\partial t}N_\lambda(\mathbf{k})=N_\lambda(\mathbf{k})\int\frac{(\mathbf{k}'\mathbf{e}_\lambda)^2}{k'^2}N^l(\mathbf{k}')\,\mathrm{Im}\,G_{\mathbf{k}-\mathbf{k}';\,\omega^t-\omega'^l}\,d^3\mathbf{k}'.$$

We shall consider a not too narrow kind of spectrum of oscillations (such spectra that for them the characteristic value of the phase velocity of the beats $(\omega-\omega')/|\mathbf{k}-\mathbf{k}'|$ is much larger than the sound velocity). We can then change in Eqs. (11) to the differential approximation. Formally this reduces to writing the imaginary part of the function $G_{\mathbf{k};\omega}$ in the following form:

$$\mathrm{Im}\,G_{\mathbf{k};\omega}=\frac{\pi\omega_0^2}{2nM}\delta'\left(\frac{\omega}{k}\right). \tag{12}$$

Here M is the ion mass and the prime on the δ-function indicates differentiation with respect to its argument.

It follows from the fact that the frequencies of the interacting l- and t-wave are close to one another (see (12)) that the wavevector of the electromagnetic wave which takes part in the lt-scattering is small compared to the wavevector of the Langmuir wave ($k^t/k^l\sim(T/mc^2)^{1/2}\ll1$). We can therefore replace the difference $\mathbf{k}-\mathbf{k}'$ by the wavevector of the Langmuir wave in the argument of the function G which determines the interaction between the l- and t-waves.

To simplify further the way Eq. (11) are written we change in them to dimensionless variables ω, x, and τ, where ω is the dimensionless dispersive addition to the wave frequency (we choose for the unit of frequency the quantity $\frac{3}{2}\omega_0T/mc^2$), x is the cosine of the angle between the wavevector and the preferred direction, and τ the dimensionless time ($\tau\equiv\frac{3}{2}\omega_0Tt/mc^2$). The dimensionless spectral functions $N^l(\omega;x)$, $N_1(\omega;x)$, and $N_2(\omega;x)$ are defined as follows:

$$N^l(\omega;x)\,d\omega\,dx\equiv\frac{8\pi^2}{27}\frac{m}{M}\left(\frac{mc^2}{T}\right)^2\frac{\omega_0}{nT}N^l(\mathbf{k})k^2\,dk\,dx,$$
$$N_\lambda(\omega;x)\,d\omega\,dx\equiv\frac{8\pi^2}{27}\frac{m}{M}\left(\frac{mc^2}{T}\right)^2\frac{\omega_0}{nT}N_\lambda(\mathbf{k})k^2\,dk\,dx. \tag{13}$$

Using Eqs. (9) and (12), and averaging the kernels of the integrals in (11) over the azimuthal angle we get in the new notation the following set of equations:

$$\frac{\partial}{\partial\tau}N^l(\omega;x)=N^l(\omega;x)\left[\omega^{1/2}\frac{\partial}{\partial\omega}\omega^{1/2}\int_{-1}^{1}N^l(\omega;x')T(x;x')\,dx'\right.$$
$$\left.+\sum_{\lambda=1;2}\omega\frac{\partial}{\partial\omega}\int_{-1}^{1}N_\lambda(\omega;x')T_\lambda(x;x')\,dx'+2\gamma(\omega;x)-\nu\right], \tag{14}$$
$$\frac{\partial}{\partial\tau}N_\lambda(\omega;x)=N_\lambda(\omega;x)\left[\frac{\partial}{\partial\omega}\omega\int_{-1}^{1}N^l(\omega;x')T_\lambda(x';x)\,dx'-\nu\right],$$

where

$$T(x;x')\equiv1-x^2-x'^2-3xx'+3x^2x'^2+3xx'^3+3x^3x'-5x'^3x^3, \tag{15}$$
$$T_1(x;x')\equiv x^2+{}^1/_2x'^2-{}^3/_2x^2x'^2, \tag{16}$$
$$T_2(x;x')\equiv{}^1/_2(1-x^2). \tag{17}$$

We included in Eqs. (14) the growth rate $\gamma(\omega; x)$ of the build-up of the Langmuir oscillations by the external sources and the damping rate $\nu/2$ of the collisional damping of the oscillations. These quantities are made dimensionless in the same way as the dispersive addition to the frequency.

In concluding this section we give the original equations for the spherically symmetric case with which we shall start the consideration of the stationary spectra. These equations are obtained from the set (14) by putting $N_1(\omega; x) = N_2(\omega; x) = N^t(\omega)/2$, $N^l(\omega; x) = N^l(\omega)$ and integrating over x':

$$\frac{\partial}{\partial\tau}N^l=N^l\left[\frac{4}{3}\omega^{1/2}\frac{\partial}{\partial\omega}\omega^{1/2}N^l+\frac{2}{3}\omega\frac{\partial}{\partial\omega}N^t+2\gamma-\nu\right], \qquad (18)$$

$$\frac{\partial}{\partial\tau}N^t=N^t\left[\frac{2}{3}\frac{\partial}{\partial\omega}\omega N^l-\nu\right]. \qquad (19)$$

Apart from the notation this set is the same as the one given on p. 313 of the book by Tsytovich[3] (see also[7], p. 194).

3. ISOTROPIC SOLUTIONS

The aim of this section is to elucidate how the character of the stationary solution of Eqs. (18) and (19) changes as the growth rate of the instability increases.[2)] We shall assume to fix the ideas that the growth rate $\gamma(\omega)$ vanishes for small and for large ω, is positive in between, and has a single maximum. We assume to start with that there are only Langmuir oscillations in the spectrum and that $N^t = 0$ (we shall show below that such a situation corresponds to being just above the threshold for producing the instability). Putting the right-hand side of Eq. (18) equal to zero we get

$$N^l=\begin{cases}0, & \omega>\omega_+,\\ \dfrac{3}{4\omega^{1/2}}\displaystyle\int_\omega^{\omega_+}\frac{2\gamma-\nu}{\omega^{1/2}}d\omega, & \omega<\omega_+\end{cases} \qquad (20)$$

where ω_+ is the larger of the two roots of the equation $\gamma(\omega) = \nu/2$. When $\omega > \omega_+$ there are no oscillations, since the spectral transfer occurs with a diminishing of the frequency and the source itself does not excite oscillations with frequencies above ω_+. If we are just above the instability threshold, so that the condition

$$\int_0^{\omega_+}\frac{2\gamma-\nu}{\omega^{1/2}}d\omega<0 \qquad (21)$$

is satisfied, the spectrum is cut off at a point $\omega = \tilde{\omega} > 0$ where the quantity N^l given by Eq. (20) vanishes. We must then add to Eq. (20) the condition

$$N^l=0, \quad \omega<\tilde{\omega}, \qquad (22)$$

which means that the oscillations excited by the source manage to get absorbed due to the Coulomb collisions before they reach the point $\omega = 0$ as a result of the spectral transfer. If, however,

$$\int_0^{\omega_+}\frac{2\gamma-\nu}{\omega^{1/2}}d\omega>0, \qquad (23)$$

there is a constant sink of oscillations at the point $\omega = 0$ and the problem of the dissipation of these waves arises. We note, however, that a solution with a sink at the point $\omega = 0$ is unstable with regard to the excitation of electromagnetic oscillations. We can check this by using (19) to evaluate the growth rate of the excitation of t-waves:

$$\Gamma^t=\frac{1}{8\omega^{1/2}}\int_\omega^{\omega_+}\frac{2\gamma-\nu}{\omega^{1/2}}d\omega-\frac{\gamma}{2}-\frac{\nu}{4}.$$

If inequality (23) is satisfied, clearly $\Gamma^t > 0$ for low frequencies. The spectrum with a cut-off at $\omega = \tilde{\omega}$ is, on the other hand, stable. Indeed, in that case the quantity $\omega^{1/2}(\Gamma^t + \gamma/2)$ is negative (it is negative for $\omega = 0$ and decreases with increasing ω). The quantity Γ^t is therefore also negative. When we are just above threshold (when inequality (21) is satisfied) the stationary spectrum is thus given by Eqs. (20) and (22) and consists of Langmuir waves only, while when the growth rate increases there appear electromagnetic oscillations in the spectrum.

In that range of the spectrum where the quantity N^t is different from zero the stationary solution of Eqs. (18) and (19) has the form

$$N^l=\frac{3}{2}\nu+\frac{A}{\omega}, \quad N^t=-\frac{A}{\omega}+B+3\int_\omega^{\omega_+}\frac{\gamma\,d\omega}{\omega}; \qquad (24)$$

A and B are integration constants. In the point where the quantity N^t vanishes the solution (24) must be joined to the solution (20). Moreover, the functions N^l and N^t are, because of their meaning, positive for all values of ω. Determining the constants A and B from these two conditions we get finally the following formula for the spectrum referring to the case of being well above threshold (inequality (23) is satisfied):

$$N^l=\begin{cases}0, & \omega>\omega_+\\ \dfrac{3}{4\omega^{1/2}}\displaystyle\int_\omega^{\omega_+}\frac{2\gamma-\nu}{\omega^{1/2}}d\omega, & \omega^*<\omega<\omega_+,\\ {}^3/_2\nu, & \omega<\omega^*\end{cases} \qquad N^t=\begin{cases}0, & \omega>\omega^*.\\ 3\displaystyle\int_\omega^{\omega^*}\frac{\gamma}{\omega}d\omega, & \omega<\omega^*\end{cases} \qquad (25)$$

Here ω^* is a root of the equation

$$\int_{\omega^*}^{\omega_+}\frac{2\gamma-\nu}{\omega^{1/2}}d\omega=2\nu\omega^{*1/2}.$$

Using Eq. (25) we can easily calculate the total number N of quanta in the system:

$$N=\int_{-1}^{1}dx\int_0^\infty d\omega\,(N^l+N^t)=3\int_0^{\omega_+}(2\gamma-\nu)\,d\omega. \qquad (26)$$

Apart from small corrections of order $k^2r_D^2$ this quantity is proportional to the total energy density of the oscillations $U \equiv U^l + U^t$. To make things clear we give the result in terms of variables with dimensions:

$$U=\frac{27}{\pi}nT\frac{M}{m}\frac{T}{m\omega_0^2}\int_0^{k_+}[2\gamma(k)-\nu]k\,dk \qquad (27)$$

(k_+ is the largest root of the equations $\gamma(k) = \nu/2$).

We emphasize that for the spectrum given by Eq. (25) there is no sink at the point $\omega = 0$: the energy lost by the source is completely absorbed through the Coulomb collisions. The power released in the plasma is thus equal to νU (in variables with dimensions). It is clear that the power depends linearly on the source strength γ. We note also that if γ is much larger than ν the energy is mainly concentrated in the electromagnetic oscillations while the energy of the Langmuir waves is small. We shall show in the next section that the qualitative statements enumerated here refer equally also to the anisotropic spectrum.

4. SPECTRUM EXCITED BY AN ANISOTROPIC SOURCE

We now consider a concrete example for which it is possible to find the stationary spectrum analytically. We shall assume that the Langmuir oscillations are excited by two identical relativistic electron beams which are injected in opposite directions into the plasma. The solution of the problem for any other source does not entail any difficulties in principle but may require a numerical integration of the equations.

We use the result of the calculation of the growth rate of the instability of a relativistic electron beam with a small angular spread $\Delta\theta$ (see, e.g.,[8]). As to beam parameters, we assume that they satisfy the following inequalities:

$$\Delta\theta > \max\left[\left(\frac{n_b}{n}\frac{mc^2}{\mathscr{E}}\right)^{1/4};\ \left(\frac{n_b}{n}\right)^{1/4}\left(\frac{mc^2}{\mathscr{E}}\right)^{1/2}\right],\quad \Delta\theta > \frac{mc^2}{\mathscr{E}},$$

where n_b is the beam density and $\mathscr{E}$ the electron energy. The first of these inequalities means that the instability is a kinetic one and the second enables us to neglect the effect of the spread in energy on the spread in velocity in the beam. It is important for what follows that in the $k_\parallel$, $k_\perp$ plane ($k_\parallel$ and $k_\perp$ are the longitudinal and transverse components of the wavevector with respect to the beam axis) the growth rate is different from zero in a narrow region around the line $k_\parallel = \omega_0/c$. For a given value of $k_\perp$ the growth rate (as function of $k_\parallel$) has a steep maximum. The following estimate holds the maximum:

$$\gamma_m \approx \omega_0 \frac{n_b}{n}\frac{mc^2}{\mathscr{E}}\frac{1}{\Delta\theta^2}\frac{\omega_0^2}{\omega_0^2+k_\perp^2c^2}.$$

For long-wavelength oscillations ($k < \omega_0/c$) the growth rate vanishes as their phase velocity is larger than the speed of light.

Hence it follows that in terms of the dimensionless variables used by us the growth rate $\gamma(\omega; x)$ corresponding to two beams has for each value of $\omega > 1$ two narrow maxima in terms of x at the points $x = \pm\omega^{-1/2}$, while

$$\gamma(\omega;x)=\gamma_0/\omega \text{ when } x=\pm\omega^{-1/2}\ (\omega>1), \tag{28}$$

where

$$\gamma_0 = \frac{2}{3}\frac{n_b}{n}\frac{mc^2}{\mathscr{E}}\frac{mc}{T}\frac{1}{\Delta\theta^2}$$

Moreover,

$$\gamma(\omega; x)=0 \text{ when } \omega<1.$$

It will become clear in what follows that these facts about the growth rate are sufficient to determine the stationary spectrum of the oscillations.

We turn now to Eqs. (14). In the stationary case they are of the form

$$N^l(\omega; x)\Gamma^l(\omega; x)=0,\quad N_\lambda(\omega; x)\Gamma_\lambda(\omega; x)=0, \tag{29}$$

where we have denoted by Γ^l and Γ_λ ($\lambda = 1, 2$) the expressions in the square brackets. Apart from these relations we must also satisfy the requirement of stability:

$$\Gamma^l(\omega; x)\leqslant 0,\quad \Gamma_\lambda(\omega; x)\leqslant 0. \tag{30}$$

Owing to the symmetry of the source the growth rate $\gamma(\omega; x)$ is symmetric under the substitution $x \to -x$. It is natural to assume that the solution possesses the same symmetry. It is then sufficient to consider instead of the interval $-1 < x < 1$ the range $0 < x < 1$. Clearly only the part of the kernel $T(x, x')$ which is even in x will then contribute to the equations (see Eq. (15)), i.e., we can put

$$T(x; x')=1-x^2-x'^2+3x^2x'^2. \tag{31}$$

We notefurther that the quantity Γ_2 is independent of x and that $\Gamma_2(\omega) = \Gamma_1(\omega; 1)$. Moreover, the contribution from the function N_2 to the quantity Γ is exactly the same as from the function

$$N_1=\delta(x-1+0)\int_0^1 N_2\,dx.$$

This means that if the function N^l, N_1, and N_2 satisfy Eqs. (29) and (30) the functions

$$\overline{N}^l=N^l,\quad \overline{N}_1=N_1+\delta(x-1+0)\int_0^1 N_2\,dx,\quad \overline{N}_2=0$$

also satisfy there relations while the spectra (N^l, N_1, N_2) and ($\overline{N}^l$, $\overline{N}_1$, $\overline{N}_2$) correspond to the same total energy of the oscillations. In other words, without loss of generality we can put $N_2 = 0$ and thereby reduce the problem to solving only the first two equations of the set (29). The quantities Γ^l and Γ_1 in these equations have the following form:

$$\begin{aligned}\Gamma^l(\omega; x)&=2\gamma(\omega; x)-\nu+2\omega^{-1/2}\frac{\partial}{\partial\omega}\omega^{1/2}\int_0^1 N^l(\omega; x)T(x; x')\,dx'\\&\quad+2\omega\frac{\partial}{\partial\omega}\int_0^1 N_1(\omega; x')T_1(x; x')\,dx',\end{aligned} \tag{32}$$

$$\Gamma_1(\omega; x)=-\nu+2\frac{\partial}{\partial\omega}\omega\int_0^1 N^l(\omega; x')T_1(x'; x)\,dx',$$

where $T(x; x')$ is given by Eq. (31) and $T_1(x; x')$ by Eq. (16). We consider the function Γ^l. In the range $\omega > 1$ the terms occurring in it depend in essentially different ways on x ($\gamma(\omega; x)$ is a function with a steep maximum at $x = \omega^{-1/2}$ while the integrals are of the form $a(\omega)x^2 + b(\omega)$). The equation $\Gamma^l = 0$ can thus for each fixed value of ω be satisfied in separate points $x_i = x_i(\omega)$ ($i = 1, 2, \ldots$). It is clear from Eq. (29) that just in those points the spectral density N^l of the quanta must be concentrated, i.e., using the terminology of[4] the spectrum has a jet-like structure. As the zeroes of the function Γ^l at the same time are its maxima, there are in the case of interest to us altogether three possible positions for the jets: $x = \omega^{-1/2}$, $x = 0$, and $x = 1$, and the number of jets for each value of ω is at most two. Hence it follows that when $\omega > 1$ we must consider the following variants:

$$1)\ \Gamma^l(\omega; \omega^{-1/2})=\Gamma^l(\omega; 0)=0.$$

when

$$\begin{gathered}N^l=A_1(\omega)\delta(x-\omega^{-1/2})+B_1(\omega)\delta(x-0);\\
2)\ \Gamma^l(\omega; \omega^{-1/2})=\Gamma^l(\omega; 1)=0,\\
N^l=A_2(\omega)\delta(x-\omega^{-1/2})+B_2(\omega)\delta(x-1+0);\\
3)\ \Gamma^l(\omega; \omega^{-1/2})=0,\quad \Gamma^l(\omega; 1)<0,\quad \Gamma^l(\omega; 0)<0,\\
N^l=A_3(\omega)\delta(x-\omega^{-1/2});\\
4)\ \Gamma^l(\omega; 0)=0,\quad \Gamma^l(\omega; \omega^{-1/2})<0,\quad \Gamma^l(\omega; 1)<0,\\
N^l=A_4(\omega)\delta(x-0);\\
5)\ \Gamma^l(\omega; 1)=0,\quad \Gamma^l(\omega; \omega^{-1/2})<0,\quad \Gamma^l(\omega; 0)<0,\\
N^l=A_5(\omega)\delta(x-1+0);\\
6)\ \Gamma^l(\omega; x)<0,\quad 0\leqslant x\leqslant 1,\\
N^l=0.\end{gathered} \tag{33}$$

Similar considerations applied to the quantity Γ_1 yield the following list (in the entire range of frequencies):

$$1)\ \Gamma_1(\omega; 0)=0,\quad \Gamma_1(\omega; 1)<0,$$

when

$$N_1=C_1(\omega)\delta(x-0);$$
$$2)\ \Gamma_1(\omega; 1)=0,\quad \Gamma_1(\omega; 0)<0,$$

$$N_1=C_2(\omega)\delta(x-1+0);$$
$$3)\ \Gamma_l(\omega;0)=\Gamma_1(\omega;1)=0; \tag{34}$$

this means that $\Gamma_1(\omega; x) = 0$ for all values of x; the spectrum can in that case not have a jet-like structure (see below);

$$4)\ \Gamma_1(\omega;x)<0 \text{ when } 0\leqslant x\leqslant 1.$$
$$N_1=0.$$

If we replace here the index 1 by the index l we get the corresponding catalog for the Langmuir oscillations in the region $\omega < 1$. It is further necessary to consider in turn all combinations occurring when we combine one variant from the group (33) with one from the group (34). The conditions imposed on the quantities Γ^l and Γ_1 give a set of ordinary differential equations the solution of which can easily be written down. After that it is necessary to take into account that the functions N^l and N_1 are continuous in ω and that they are positive. This enables us to construct the required spectrum unambiguously from the solutions obtained. The whole procedure turns out to be uncomplicated but rather tedious. We give here only the result. For the sake of simplicity we restrict ourselves to the case when we are well above the instability threshold ($\gamma_0 \gg \nu$). One can also find the spectrum in the other cases, but when $\gamma_0 \gg \nu$ the role of the lt-scattering is the sharpest (see Sec. 3).

In the situation of interest to us the whole of the frequency range splits up into five regions with different functional behavior of $N^l(\omega; x)$ and $N_1(\omega; x)$. We describe each of the stretches separately, starting with large values of ω.

1) $\omega > 2\gamma_0/\nu$. The growth rate $\gamma(\omega; x)$ is here below the threshold for the beam instability, and therefore

$$N^l(\omega;x)=N_1(\omega;x)=0. \tag{35}$$

2) $\gamma_0/2\nu < \omega < 2\gamma_0/\nu$. In this range, as before, there are no electromagnetic oscillations, and

$$N^l(\omega;x)=\left[\nu^{1/2}-\left(\frac{2\gamma_0}{\omega}\right)^{1/2}\right]^2\delta(x-\omega^{-1/2}). \tag{36}$$

3) $3 < \omega < \gamma_0/2\nu$. In the point $\omega = \gamma_0/2\nu$ an electromagnetic oscillations jet appears at the position $x = 1$:

$$\begin{gathered}N_1(\omega;x)=\left[-4\nu+\frac{2\nu}{\omega-1}-\frac{2\nu}{(\omega-1)^2}+(3\nu-2\gamma_0)\ln\frac{\omega-1}{\omega}\right]\delta(x-1+0),\\ N^l(\omega;x)=\nu\frac{\omega}{\omega-1}\delta(x-\omega^{-1/2}).\end{gathered} \tag{37}$$

4) $1 < \omega < 3$. At the upper limit ($\omega = 3$) a second jet of Langmuir waves appears at the position $x = 0$:

$$N^l(\omega;x)=\frac{\nu\omega}{2}\delta(x-\omega^{-1/2})+\frac{\nu}{2}(3-\omega)\delta(x-0). \tag{38}$$

In this range the quantity $\Gamma_1(\omega; x)$ vanishes for all values of x, i.e., the spectrum of the electromagnetic oscillations has here, in general, not a jetlike structure. Conditions (29) and (30) give in such a situation only the values of two moments of the angular distribution of the oscillations:

$$\begin{gathered}\int_0^1 N_1x^2\,dx=-\frac{7}{2}\nu+(2\gamma_0-3\nu)\ln\frac{3}{2},\\ \int_0^1 N_1(1-x^2)\,dx=\gamma_0\ln\frac{3}{\omega}.\end{gathered} \tag{39}$$

In other words, in the case considered there is a whole set of stationary solutions that differ from one another in the values of the higher-order angular momenta. We note, however, that for all these solutions the reaction of the oscillations on the source turns out to be the same, since it is characterized merely by the spectrum of the Langmuir oscillations which interact with the beams and that spectrum is uniquely determined. In particular, all solutions correspond to the same magnitude of the energy lost by the beams in the plasma.

5) $0 < \omega < 1$. Here $\Gamma_1(\omega; x) = \Gamma^l(\omega; x) = 0$ for all values of x and accordingly only the moments of the spectral functions are given (see sub 4):

$$\begin{gathered}\int_0^1 N^lx^2\,dx=\frac{\nu}{2},\qquad \int_0^1 N^l(1-x^2)\,dx=\nu;\\ \int_0^1 N_1x^2\,dx=-\frac{7}{2}\nu+(2\gamma_0-3\nu)\ln\frac{3}{2},\qquad \int_0^1 N_1(1-x^2)\,dx=\gamma_0\ln 3.\end{gathered} \tag{40}$$

We recall that the spectrum is symmetric under the substitution of x by $-x$, while Eqs. (35) to (40) refer to the interval $0 < x < 1$. These formulae have been written down up to and including terms of first order in the parameter ν/γ_0. However, we have already noted that the condition $\nu/\gamma_0 \ll 1$ is not necessary for an analytical solution of the problem.

Evaluating the total energy density U of the oscillations in the spectrum (35) to (40) gives the following results (in variables with dimensions):

$$U=\frac{18}{\pi}\frac{M}{m}\frac{n_b}{n}\frac{nT}{\Delta\theta^2}\frac{T}{\mathscr{E}}\ln\left(\frac{\omega_0}{\nu}\frac{n_b}{n}\frac{mc^2}{\mathscr{E}}\frac{1}{\Delta\theta^2}-2\right). \tag{41}$$

We have used the explicit expression for γ_0 (see (28)). The energy is basically determined by the electromagnetic oscillations but, because the growth rate $\gamma(\omega; x)$ decreases rather slowly ($\propto\omega^{-1}$), the ratio U^t/U^l turns out to be proportional to the logarithm of γ_0/ν rather than proportional to γ_0/ν itself. As in the isotropic case (see section 4) there is no condensation of oscillations at the point $\omega = 0$ and a power, equal to νU (in variables with dimensions) is released per unit volume of the plasma.

5. DISCUSSION OF THE RESULTS

Let us list the conditions for the applicability of the solutions obtained in Secs. 3 and 4.

We neglected in the initial equations the loss of electromagnetic waves from the plasma. This can be done, if the time of escape, which is equal to L/v_g (L size of the system, v_g group velocity of the wave) is considerably longer than the time for collisional damping. The group velocity of the electromagnetic waves which appear as a result of lt-scattering when the plasma is heated by a relativistic beam is of the order of magnitude of the electron thermal velocity. The restriction on L thus takes the form $L > \lambda$, where λ is the mean free path of the electrons. This is a rather stringent requirement. The more realistic case is when the electromagnetic oscillations turn out to be trapped for another reason (because the plasma density is somewhat lower in the region where the beam passes through than outside it). For the suppression of the oscillations which interest us we need a drop in density $\delta n/n$ on the order of T/mc^2. If such a drop exists the limitation $L > \lambda$ is removed.

The next condition pertains to the tt-scattering process, which was also not included in the initial equations. We can neglect the effect of this process provided that

$$\tau^{tt} > \nu^{-1}, \tag{42}$$

where τ^{tt} is the characteristic time for tt-scattering. As we have already noted, the quantity τ^{tt} is connected with the time for lt-scattering: $\tau^{tt} \sim mc^2\tau^{tl}/T$ (see[3]). On the other hand, we showed in Secs. 3 and 4 that if we are well above threshold the time for lt-scattering is equal to the reciprocal of the growth rate of the beam instability. Hence it is clear that inequality (42) gives the following restriction on the beam and plasma parameters:

$$\gamma/\nu < mc^2/T. \tag{43}$$

In the present paper we have been interested right from the start only in stationary spectra and we have not considered at all the problem of the establishment of the stationary state. This process is as yet unexplained and must be considered separately. It is of particular interest in the case when we are well above threshold when lt-scattering changes the form of the stationary spectrum qualitatively. We must here bear in mind the following. If lt-scattering is forbidden the estimate of the energy density of the Langmuir oscillations interacting with the beam is not very sensitive to whether the spectrum is truly stationary (see[8]) while in the case where we are well above threshold both solutions (stationary as well as non-stationary) correspond to an accumulation of oscillations in the long-wavelength part of the spectrum ($k < r_D^{-1}(m/M)^{1/2}$). When lt-scattering is taken into account the difference between the solutions can be much more significant. Kaplan and Tsytovich[9] have shown, by solving numerically the probes with the initial conditions, that in the transient regime allowance for lt-scattering leads only to the appearance, besides the Langmuir oscillations, of electromagnetic waves with an energy density of the same order of magnitude as that of the Langmuir oscillations. Otherwise the situation remains qualitatively the same as in the case when there are only ll interactions. As to the stationary spectrum, lt-scattering affects it to a much larger extent, as we already noted. Unfortunately, it is impossible to use the results of[9] to reach any conclusions about the establishment of a stationary state as in all variants of the calculations time intervals were considered which were small compared to the reciprocal of the collision frequency. In accordance with what we have said it would be interesting to perform calculations analogous to those performed in[9], increasing the time range at least to a few times the inverse collision frequency. As the result may depend significantly on the initial conditions it is desirable to consider not only natural conditions (thermal noise) but also other possibilities (in particular, the case when the initial spectrum does not differ strongly from the stationary spectrum).

In conclusion we show that the spectrum found in Sec. 4 corresponds (see, e.g.,[8]) to the following estimate for the stopping length of the beam:

$$l \sim \frac{1}{10}\frac{c}{\nu}\cdot\frac{m}{M}\left(\frac{\mathscr{E}}{T}\right)^2\left(\ln\frac{\omega_0}{\nu}\frac{n_b}{n}\frac{mc^2}{\mathscr{E}}\right)^{-1}. \tag{44}$$

In the case of a sufficiently dense plasma this estimate gives (from an experimental point of view) a completely acceptable value of l (for $n \sim 10^{18}\ \mathrm{cm}^{-3}$, $n_b \sim 10^{13}\ \mathrm{cm}^{-3}$, $T \sim 10^4$ eV, $\mathscr{E} \sim 10^6$ eV we get for a deuterium plasma $l \sim 2$ m). It is important that in the regime considered the energy lost by the beam is dissipated due to Coulomb collisions and, hence, transferred to the main body of the electrons in the plasma.

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1) Tsytovich [3] had earlier considered the spherically symmetric problem.

2) Tsytovich and Kaplan [3,7] have stated that the set (18) and (19) has no stationary solutions for which $N^t \neq 0$. From the contents of the present section it will be clear that in fact such solutions exist.

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