Heat transfer in type-II superconducting alloys

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An expression for heat flow in superconducting alloys is obtained on the basis of nonstationary superconductivity theory. It is shown that the energy dissipation mechanism proposed by Clem (1968) does not operate in the mixed state. The Ettingshausen and Nernst effects are investigated near the critical temperature and in weak fields $H \ll H_{c2}$. The anisotropy of the thermal conductivity in the mixed state near T_c and at low temperatures is also considered.

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1. INTRODUCTION

Kinetic phenomena in type-II superconducting alloys are presently the subject of intensive experimental and theoretical research. Particularly detailed studies were made, in the mixed state, of the energy dissipation accompanying the motion of Abrikosov vortices[1] and leading to a finite resistivity ρ_f of the sample (see the references in[2]). On the other hand, effects connected with heat transfer (thermal conductivity, the effects of Nernst and Ettingshausen, and others) in the mixed state have not been sufficiently well investigated theoretically, despite the fact that they have a number of features that distinguish them from the normal metal. In particular, experiments reveal, in dirty alloys, an anisotropy of the thermal conductivity with respect to the magnetic field direction,[3] and large values of the Ettingshausen and Nernst effects.[4]

The Ettingshausen effect is the name given to the appearance, in the presence of a magnetic field H_{0z}, of a temperature gradient $\nabla_y T$ perpendicular to the electric current j_x, under the condition that the heat flux $\mathbf{q}$ is equal to zero, while the Nernst effect is the appearance of an electric field E_x perpendicular to the temperature gradient $\nabla_y T$ under the condition $j = 0$. These effects are described by the coefficients $\alpha_{xy}(H)$ and $\beta_{yx}(H)$ in the expressions

$$j_i = \sigma_f E_i - \alpha_{ik}(\mathbf{H}) T^{-1} \nabla_k T, \quad q_i = -K \nabla_i T + \beta_{ik}(\mathbf{H}) E_k, \tag{1}$$

where $\sigma_f = \rho_f^{-1}$, and K is the thermal conductivity. Phenomenological models were proposed to calculate these coefficients[5, 6]. The first attempt at a microscopic study of the Ettingshausen and Nernst effects in the region of magnetic fields close to the upper critical field H_{c2} were made by Caroli and Maki.[7] Their result, however, has the important shortcoming that the entropy carried by the moving vortex does not vanish as $T \to 0$. Further attempts[8] to eliminate this shortcoming have led to kinetic coefficients $\alpha_{ik}(H)$ and $\beta_{ik}(H)$ in (1) that did not satisfy the Onsager relations:

$$\alpha_{ik}(\mathbf{H}) = \beta_{ki}(-\mathbf{H}).$$

This circumstance was pointed out in[9].

De Lange[10] calculated the entropy flux in the mixed state by differentiating, with respect to the temperature, the free-energy flux obtained by Schmid[11] for the phenomenological Ginzburg-Landau equation as a function of the time. The heat flux given in[10] did not contain the term that results from the deviation of the distribution function from equilibrium. This effect, as will be shown below, plays an important role in heat-transfer processes.

The method proposed in this paper makes it possible, on the basis of the stationary superconductivity theory developed by Gor'kov and Éliashberg,[12] to calculate for superconducting alloys in the mixed state the kinetic coefficients α_{ik} and β_{ik} as well as the thermal conductivity K. The concrete calculations are performed for temperatures close to critical in the field range $H_{c1} < H_0 \ll H_{c2}$.

We consider below also the question of the additional energy dissipation in the mixed state on the basis of a mechanism proposed by Clem.[13] The corresponding contribution to the dissipation was phenomenologically taken into account in[14]. It is shown in the present paper that the local changes of the energy (entropy) density, which occur when the vortex filaments move, do not lead to the appearance of local (in a region on the order of the filament dimension ξ) temperature gradients and make no additional contribution to the expression for the conductivity in comparison with that obtained in the isothermal technique (eg., in[15, 16]).

Since we are dealing below with dirty alloys, we consider those parts of the thermal conductivity and of the other kinetic coefficients which are due to electron-impurity collisions.

2. THE ENERGY BALANCE EQUATION

It is necessary first to define uniquely such quantities as the energy density, the energy flux, the heat flux, etc.; it is convenient here to write them in terms of the Green's functions used to express the electric current j and the self-consistency equation and to formulate the kinetic equations. We introduce, in accord with[17], the complete Green's functions

$$\mathcal{G}_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-) = \begin{pmatrix} G & F \\ -F^+ & \bar{G} \end{pmatrix},$$

where $\epsilon_\pm = \epsilon \pm \omega/2$ and $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$. Then

$$\mathbf{j}_\omega = -\frac{e}{m}\int\left\{\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)G_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-) - \left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\bar{G}_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\right\}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d\varepsilon}{4\pi i}, \quad \frac{\Delta_\omega}{|g|} = \int F_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d\varepsilon}{4\pi i}.$$

The functions $\mathcal{G}_{\epsilon_+,\epsilon_-}$ satisfy kinetic equations[17] of the following form in matrix notation:

$$\{\mathcal{G}_0^{-1}\mathcal{G} - \mathcal{G}\mathcal{G}_0^{-1}\} - \{\Sigma^R\mathcal{G} - \mathcal{G}\Sigma^A\} + \{\mathcal{G}^R\Sigma - \Sigma\mathcal{G}^A\} = 0. \tag{2}$$

Here

$$\mathcal{G}_{0\varepsilon}^{-1}(\mathbf{p}) = \begin{pmatrix} -\varepsilon + \dfrac{(\mathbf{p}-e\mathbf{A}/c)^2}{2m} - \mu + e\varphi, & -\Delta \\ \Delta^*, & \varepsilon + \dfrac{(\mathbf{p}+e\mathbf{A}/c)^2}{2m} - \mu + e\varphi \end{pmatrix},$$

$\mathbf{A}$ and φ are the vector and scalar potentials, and μ is the chemical potential which, generally speaking, differs somewhat from E_F. The abbreviated notation such as $\{\Sigma\mathscr{G}\}$ denotes the convolution

$$\{\Sigma\mathscr{G}\}=\int\Sigma_{\varepsilon_+,\varepsilon'}(\mathbf{k}_1)\mathscr{G}_{\varepsilon',\varepsilon_-}(\mathbf{p}_+-\mathbf{k}_1,\mathbf{p}_-)\frac{d\varepsilon'}{2\pi}\frac{d^3\mathbf{k}_1}{(2\pi)^3}.$$

The self-energy part is

$$\Sigma=\frac{1}{2\pi\nu(0)\tau}\int\mathscr{G}(\mathbf{p}_+,\mathbf{p}_-)\frac{d^3\mathbf{p}}{(2\pi)^3},$$

where τ is the free-path time and $\nu(0) = mp_F/2\pi^3$ is the state density on the Fermi surface. The kinetic equation (2) contains the regular functions $\mathscr{G}^R$ and $\mathscr{G}^A$, defined from the equation

$$\{\mathscr{G}_0{}^{-1}\mathscr{G}^{R(A)}-\mathscr{G}^{R(A)}\mathscr{G}_0{}^{-1}\}-\{\Sigma^{R(A)}\mathscr{G}^{R(A)}-\mathscr{G}^{R(A)}\Sigma^{R(A)}\}=0.$$

Equation (2), just as the usual kinetic equation, contains the laws of particle-number conservation (continuity equation) and energy conservation. The former is given by

$$\partial Ne/\partial t+\operatorname{div}\mathbf{j}=0,$$

where N is the density of the number of electrons and e is the electron charge. We note that in a metal, owing to the electroneutrality condition, it is necessary to use in lieu of the continuity equation the relation

$$\operatorname{div}\mathbf{j}=0. \tag{3}$$

Since (2) contains the excitation energy reckoned from the chemical potential,

$$\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2-\mu,$$

the energy-conservation law is formulated for the quantity $\tilde{\mathscr{E}} = \mathscr{E} - \mu N$, where $\mathscr{E}$ is the excitation-energy density, in the following manner:

$$\frac{\partial\tilde{\mathscr{E}}}{\partial t}+\operatorname{div}\mathbf{j}_{\mathscr{E}}=\mathbf{jE}-N\frac{\partial\mu}{\partial t}, \tag{4}$$

where

$$\mathscr{E}=-\int\varepsilon\{G_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)-\bar{G}_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\}\frac{d\varepsilon}{4\pi i}\frac{d^3\mathbf{p}}{(2\pi)^3}+\frac{|\Delta|^2}{|g|}-Ne\varphi, \tag{5}$$

$$\begin{aligned}\mathbf{j}_{\mathscr{E}}=-\frac{1}{m}\int\varepsilon\Big\{&\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)G_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\\&+\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\bar{G}_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\Big\}\frac{d\varepsilon}{4\pi i}\frac{d^3\mathbf{p}}{(2\pi)^3}-\mathbf{j}\varphi,\end{aligned} \tag{6}$$

and $\mathbf{E} = -\partial\mathbf{A}/c\partial t - \nabla\varphi$. We have changed over in (4) to a coordinate representation in $\mathbf{p}_+ - \mathbf{p}_- = \mathbf{k}$ and $\epsilon_+ - \epsilon_- = \omega$. To derive (4) it suffices to multiply the matrix equation (2) by ϵ, take the trace and integrate it with respect to $d\epsilon/4\pi i$ and $d^3p/(2\pi)^3$. The integral for the collisions with the impurities (the last two brackets in (2) then vanishes upon integration with respect to $d^3p/(2\pi)^3$.

Using the electroneutrality condition (3), we can rewrite the balance equation (4) for the energy $\mathscr{E}$ in the form

$$\partial\mathscr{E}/\partial t+\operatorname{div}\mathbf{j}_{\mathscr{E}}=\mathbf{jE}$$

or for the total energy$^{1)}$ $\mathscr{E}_t = \mathscr{E} + \mathbf{H}^2/8\pi$:

$$\partial\mathscr{E}_t/\partial t+\operatorname{div}\mathbf{j}_{\mathscr{E}t}=0,\quad \mathbf{j}_{\mathscr{E}t}=\mathbf{j}_{\mathscr{E}}+(c/4\pi)[\mathbf{E}\times\mathbf{H}].$$

It is thus clear that $\mathscr{E}$ is the density of the internal energy, and $\mathbf{j}_{\mathscr{E}}$ is the flux of the internal energy.

For practical purposes it is more convenient to use not the energy $\mathscr{E}$ but the quantity $\tilde{\mathscr{E}}$. The point is that the Green's functions $\mathscr{G}$ are expressed in terms of the variables T and μ, while $\tilde{\mathscr{E}}$ is precisely a function of the independent variable μ, with $\partial\tilde{\mathscr{E}}/\partial\mu = -N$. We therefore rewrite the balance equation (4) in the form

$$(\partial\tilde{\mathscr{E}}/\partial t)_\mu+\operatorname{div}\mathbf{j}_{\mathscr{E}}=\mathbf{jE}. \tag{7}$$

In superconductors in the mixed state, the processes of energy dissipation and heat transfer are accompanied by the motion of the vortex structure. All the quantities, including the energy density, contain then an explicit time dependence. Therefore the time derivative of $\tilde{\mathscr{E}}$ contains, besides the usual term with the heat capacity

$$T\frac{\partial S}{\partial T}\frac{\partial T}{\partial t}=C\frac{\partial T}{\partial t},$$

also an additional term $(\partial\tilde{\mathscr{E}}/\partial t)_{\mu,T}$, where the differentiation of the explicit dependence on t takes place:

$$\left(\frac{\partial\tilde{\mathscr{E}}}{\partial t}\right)_\mu=C\frac{\partial T}{\partial t}+\left(\frac{\partial\tilde{\mathscr{E}}}{\partial t}\right)_{\mu,T}.$$

The right-hand side of (7) describes the work performed by the electromagnetic field on the excitations in the superconductor. This expression, generally speaking, is not positive-definite, namely, in the mixed state it contains the term $\mathbf{j}_0\cdot\mathbf{E}$, where $\mathbf{j}_0$ is the current that circulates around the equilibrium vortex filaments. To determine the heat flux $\mathbf{q}$ with the aid of (7) it is necessary to represent the balance equation in such a form that its right-hand side contains the positive-definite dissipation function

$$\frac{1}{T}\left(\frac{\partial\tilde{\mathscr{E}}}{\partial t}\right)_\mu+\operatorname{div}\left(\frac{\mathbf{q}}{T}\right)=\frac{\mathbf{j}_1\mathbf{E}}{T}-\frac{\mathbf{q}\nabla T}{T^2},$$

where $\mathbf{j}_1 = \mathbf{j} - \mathbf{j}_0$. It is clear that for this purpose it is necessary to subtract from the internal-energy flux $\mathbf{j}_{\mathscr{E}}$ the vector $\mathbf{j}'_{\mathscr{E}}$, which satisfies the equation

$$\operatorname{div}\mathbf{j}_{\mathscr{E}}{}'=\mathbf{j}_0\mathbf{E}.$$

The heat flux is thus $\mathbf{q} = \mathbf{j}_{\mathscr{E}} - \mathbf{j}'_{\mathscr{E}}$.

3. EXPRESSION FOR THE HEAT FLUX IN THE MIXED STATE NEAR THE CRITICAL TEMPERATURE

Consider a sample placed in an external magnetic field H_0 in the range $H_{c1} < H_0 \ll H_{c2}$. In this field region, the vortex filaments can be regarded as isolated. A transport current j_{tr} flows through the sample perpendicular to the magnetic field, and in addition a temperature gradient may be maintained along the sample. Just as before, we assume that the Ginzburg-Landau parameter $\kappa \gg 1$, and confine ourselves to corrections of first order in the vortex-filament velocity $\mathbf{v}_L$ and in the temperature gradient. In this approximation, the time derivatives $\partial/\partial t$ are replaced by spatial derivatives $-\mathbf{v}_L\nabla$ of the static quantities.

To calculate the heat flux it suffices to confine oneself in (2) and (6) to the principal terms in T_c/μ, and therefore we can change over in all the expressions to Green's functions integrated with respect to[17] $\zeta = p^2/2m - \mu$:

$$\hat{g}(\mathbf{v}_F,\mathbf{r})=\int\mathscr{G}(\mathbf{p},\mathbf{r})\frac{d\zeta}{\pi i}=\begin{pmatrix} g & f\\ -f^+ & \bar{g}\end{pmatrix}.$$

Now (2) takes the form

$$\begin{aligned}&-i\mathbf{v}_F\nabla\hat{g}_{\varepsilon_+,\varepsilon_-}-\varepsilon_+\sigma_z\hat{g}_{\varepsilon_+,\varepsilon_-}+\hat{g}_{\varepsilon_+,\varepsilon_-}\sigma_z\varepsilon_-+\{\hat{H}\hat{g}-\hat{g}\hat{H}\}_{\varepsilon_+,\varepsilon_-}\\&-\frac{i}{2\tau}\{\hat{g}_0{}^R\hat{g}-\hat{g}\hat{g}_0{}^A\}_{\varepsilon_+,\varepsilon_-}+\frac{i}{2\tau}\{\hat{g}^R\hat{g}_0-\hat{g}_0\hat{g}^A\}_{\varepsilon_+,\varepsilon_-}=0,\end{aligned} \tag{8}$$

where

$$\hat H = -\frac{e}{c}\mathbf{v}_F\mathbf{Q}_\omega\sigma_z + e\tilde\varphi_\omega - i\sigma_y\Delta_\omega, \quad \hat g_0(\mathbf{r}) = \int \hat g(\mathbf{v}_F, \mathbf{r})\frac{d\Omega_{\mathbf{v}_F}}{4\pi}.$$

We use here the gauge-invariant combinations

$$|\Delta|, \quad \tilde\varphi=\varphi+\dot\chi/2e, \; \mathbf{Q}=\mathbf{A}-c\nabla\chi/2e,$$

χ being the phase of the order parameter. The absolute-value sign of Δ will henceforth be omitted. Equation (8) must be supplemented by the relations[16]

$$\int \hat g^{R(A)}_{\varepsilon,\varepsilon_1}\hat g^{R(A)}_{\varepsilon_1,\varepsilon'}\frac{d\varepsilon_1}{2\pi}=2\pi\delta(\varepsilon-\varepsilon');$$
$$\int \{\hat g^{R}_{\varepsilon,\varepsilon_1}\hat g_{\varepsilon_1,\varepsilon'}+\hat g_{\varepsilon,\varepsilon_1}\hat g^{A}_{\varepsilon_1,\varepsilon'}\}\frac{d\varepsilon_1}{2\pi}=0. \tag{9}$$

We separate in the complete $\hat g$-function the part containing the equilibrium distribution function[2)] $n_0(\hat\epsilon) = 1/2(1-\tanh(\epsilon/2T))$:

$$\hat g=\hat g^{R}_{\varepsilon_+,\varepsilon_-}\operatorname{th}\frac{\varepsilon_-}{2T}-\operatorname{th}\frac{\varepsilon_+}{2T}\hat g^{A}_{\varepsilon_+,\varepsilon_-}+\hat g_1, \tag{10}$$

where $\hat g_1$ describes the deviation of the distribution function from equilibrium. Substituting (10) in (8) and (9), we obtain, accurate to the principal terms in ω:

$$-i\mathbf{v}_F\nabla\hat g_1-\varepsilon(\sigma_z\hat g_1-\hat g_1\sigma_z)+\{\hat H\hat g_1-\hat g_1\hat H\}-\frac{i}{2\tau}\{\hat g_0{}^R\hat g_1-\hat g_1\hat g_0{}^A\}+\frac{i}{2\tau}\{\hat g^R\hat g_{10}-\hat g_{10}\hat g^A\}$$
$$=-2\frac{\partial n_0}{\partial\varepsilon}\left\{-i(\mathbf{v}_F\nabla T)\frac{\varepsilon}{T}(\hat g^R-\hat g^A)+\omega(\hat g^R\hat H_\omega-\hat H_\omega\hat g^A)\right\}, \tag{11}$$

$$\hat g^R\hat g_1+\hat g_1\hat g^A=0. \tag{12}$$

In dirty alloys, the Green's functions $\hat g(\mathbf{v}_F, \mathbf{r})$ are approximately isotropic in the directions of the vector $\mathbf{v}_F$, and can therefore be represented in the form[18]

$$\hat g_1(\mathbf{v}_F, \mathbf{r})=\hat g_{10}(\mathbf{r})+\frac{\mathbf{v}_F}{v_F}\mathbf{g}_1(\mathbf{r}), \quad |\mathbf{g}_1|\sim\frac{l}{\xi}g_{10}.$$

The succeeding calculations are analogous to those of[16]. Averaging once Eqs. (11) and (12) with respect to the directions of the vector $\mathbf{v}_F$, and then averaging the same equations but multiplied by $\mathbf{v}_F$, we get

$$\hat{\mathbf{g}}_1=-l\{\hat g_0{}^R\partial\hat g_{10}+\hat g_{10}\partial\hat g_0{}^A\}-2\frac{\partial n_0}{\partial\varepsilon}l\left\{\frac{\varepsilon}{T}\nabla T(1-\hat g_0{}^R\hat g_0{}^A)\right.$$
$$\left.-i\omega\frac{e}{c}\mathbf{Q}_\omega(\sigma_z-\hat g_0{}^R\sigma_z\hat g_0{}^A)\right\}. \tag{13}$$

Here $l = v_F\tau$ is the mean free path and

$$\partial\hat g_0=\begin{pmatrix}\nabla g_0, & (\nabla-2ie\mathbf{Q}/c)f_0\\ -(\nabla+2ie\mathbf{Q}/c)f_0{}^+, & \nabla\bar g_0\end{pmatrix}.$$

Proceeding now to concrete calculations as $T \to T_c$, we note first that (8) yields, in the principal approximation in $T_c - T$,

$$\tilde{\mathscr{E}}=\tilde{\mathscr{E}}_n-\nu(0)\Delta^2, \tag{14}$$

where $\tilde{\mathscr{E}}_n = \mathscr{E}_n - \mu N$ corresponds to the normal metal:

$$\tilde{\mathscr{E}}_n=2\int\varepsilon n_0(\varepsilon)\nu(\varepsilon)d\varepsilon.$$

Subtracting from (14) the value of $\tilde{\mathscr{E}}$ at large distances from the vortex $\tilde{\mathscr{E}}_n - \nu(0)\Delta_\infty^2$, we obtain the density of the energy connected with the presence of a single vortex:

$$\mathscr{E}_0=\nu(0)(\Delta_\infty{}^2-\Delta^2). \tag{14'}$$

When calculating the heat flux (6) as $T \to T_c$, the part containing the regular functions $\hat g^R$ and $\hat g^A$ admits of expansion in powers of Δ/T. Direct calculations show that in the approximation that we need in Δ/T, this part makes no contribution to $\mathbf{j}_\mathscr{E}$. It is also easy to verify that $\mathbf{j}'_\mathscr{E}$ is also small. Thus,

$$\mathbf{q}=-\frac{\nu(0)}{4}\int\varepsilon\mathbf{v}_F(g_1(\mathbf{v}_F, \mathbf{r})+\bar g_1(\mathbf{v}_F, \mathbf{r}))\frac{d\Omega_{\mathbf{v}_F}}{4\pi}d\varepsilon$$
$$=-\frac{\nu(0)v_F}{12}\int\varepsilon\operatorname{Sp}(\hat{\mathbf{g}}_1)d\varepsilon. \tag{15}$$

At temperatures close to T_c and in the region of fields $H_0 \ll H_{c2}$, the retarded and advanced functions are connected by the relations[15]

$$\hat g_0{}^R=\hat g=\begin{cases}-\hat g_0{}^A, & |\varepsilon|>\Delta\\ \hat g_0{}^A, & |\varepsilon|<\Delta\end{cases}; \quad \hat g=\begin{pmatrix}g; & f\\ -f; & -g\end{pmatrix}=(\varepsilon^2-\Delta^2)^{-1/2}\begin{pmatrix}\varepsilon; & \Delta\\ -\Delta; & -\varepsilon\end{pmatrix}. \tag{16}$$

With the aid of (16) we readily find from (12) and (13) that $\operatorname{Sp}(\hat{\mathbf{g}}_1) = 0$ at $|\epsilon| < \Delta$ and that at $|\epsilon| > \Delta$ we have

$$\operatorname{Sp}(\hat{\mathbf{g}}_1)=8l\left\{-\frac{\partial n_0}{\partial\varepsilon}\frac{\varepsilon}{T}\nabla T+\nabla n_1(\varepsilon)\right\},$$

where the quantity

$$n_1(\varepsilon)=-(f_1+f_1{}^+)/8f$$

has the meaning of the deviation of the distribution function $n(\epsilon)$ from the equilibrium Fermi function. The equation for n_1 is also obtained from (11), (12), and (16), and takes the form

$$D\nabla^2 n_1(\varepsilon)-\frac{\partial n_0}{\partial\varepsilon}\frac{\varepsilon}{T}D\nabla^2 T=-\frac{\partial n_0}{\partial\varepsilon}\frac{\partial}{\partial t}(\varepsilon^2-\Delta^2)^{1/2}, \tag{17}$$

where $D = lv_F/3$ is the diffusion coefficient. This equation should be compared with Eq. (20) of[15]. Finally,

$$\mathbf{q}=4\nu(0)D\int_\Delta^\infty\varepsilon\left\{\frac{\partial n_0}{\partial\varepsilon}\frac{\varepsilon}{T}\nabla T-\nabla n_1(\varepsilon)\right\}d\varepsilon. \tag{18}$$

It is easy to verify with the aid of (14), (17), and (18) that the energy balance equation, in the approximation linear in the vortex velocity (in this case $\partial T/\partial t = 0$)

$$(\partial\tilde{\mathscr{E}}/\partial t)_{\mu,T}+\operatorname{div}\mathbf{q}=0, \tag{19}$$

is identically satisfied regardless of the value of the temperature gradient, and in particular also at $\nabla T = 0$. This circumstance is due to the relative long time of homogeneous energy relaxation in our case. Indeed, when the vortex filament moves past a given point of the sample, excitations are produced in the vicinity of this point and have an excess energy in comparison with the excitations at larger distances from the filament. This excess energy can by distributed in two ways: part of it will flow away from the given point of the sample, owing to diffusion of the excitation, while the remainder, owing to inelastic electron-electron and electron-phonon collisions, will lead to a local heating of the excitations. However, the energy relaxation times τ_ϵ corresponding to the inelastic collisions are quite large in comparison with diffusion relaxation time $t_D \sim \xi^2/D$ (τ_ϵ is of the order Θ_D^2/T^3 for electron-phonon collisions and of the order of E_F/T^2 for electron-electron collisions). For this reason, the predominant part of the excess energy is diverted by diffusion of the excitations. It is therefore possible to neglect the processes of homogeneous energy relaxation in the kinetic equation (2).

At the same time, the foregoing means that in our case, when $\tau_\epsilon \gg t_D$, the excitations do not manage to become thermalized over the characteristic scales of the problem, which are of the order of ξ. It follows therefore that no local temperature gradients can arise. We emphasize that the absence of local temperature gradients (in a region of the order of the filament dimension ξ), neglecting homogeneous energy relaxation processes, is not connected with the assumption that T is close to T_c, but takes place at arbitrary temperatures. The energy balance is then ensured by diffusion of the excitations, i.e., Eq. (19) can be satisfied at arbitrary temperature, as can be readily verified with the aid of (2), at a constant temperature.

Thus, if the same temperature is maintained on the sample boundaries, then the existence of heat fluxes does not lead to the additional energy dissipation that would arise in the presence of local temperature gradients on account of the term $-\mathbf{q}\cdot\nabla T/T$ in the dissipation function. Consequently there is no energy-dissipation mechanism in the large-τ_ϵ approximation proposed by Clem[13].

Let us explain the foregoing using as an example the motion of vortices near the critical temperature. Assume that the same temperature is maintained on the sample boundaries. From (17) we get

$$\nabla n_1=-\frac{1}{4TD}\,\mathrm{ch}^{-2}\frac{\varepsilon}{2T}\left[(\varepsilon^2-\Delta^2)^{1/2}-(\varepsilon^2-\Delta_\infty^2)^{1/2}\right]\mathbf{v}_L \tag{20}$$

(we have written down the expression for ∇n_1 only at $\epsilon > \Delta_\infty$, since only values $\epsilon \sim T$ will be of importance to us from now on). The boundary conditions are chosen such that the heat flux at large distances from the vortex must vanish when the average temperature gradient is zero. With the aid of (18) we obtain

$$\mathbf{q}=\mathbf{v}_L\nu(0)\int_\Delta^\infty\frac{\varepsilon}{T}\,\mathrm{ch}^{-2}\frac{\varepsilon}{2T}\left[(\varepsilon^2-\Delta^2)^{1/2}-(\varepsilon^2-\Delta_\infty^2)^{1/2}\right]d\varepsilon=\mathbf{v}_L\mathscr{E}_0, \tag{21}$$

where the energy density $\mathscr{E}_0$ is given by (14′). Thus, the heat flux transports precisely the "proper" energy of the vortex, i.e., as the vortex moves it effects a heat "convection" of sorts, in accordance with the fact that the diffusion mechanism ensures complete removal of the excess excitation energy.

4. THE EFFECTS OF ETTINGSHAUSEN AND NERNST

Expression (21) yields the heat-flux density in the vicinity of an isolated moving vortex. The heat flux averaged over the sample is equal to

$$\mathbf{q}=\mathbf{v}_L n_L\int_{V_0}\mathscr{E}_0 d^2r,$$

where $n_L = B/\Phi_0$ is the density of the vortex filaments and $\Phi_0 = \pi c/e$ is the magnetic-flux quantum. The integration extends over the unit cell of the vortex-filament lattice. The quantity

$$S_d=\frac{\nu(0)}{T}\int_{V_0}(\Delta_\infty^2-\Delta^2)\,d^2\mathbf{r}$$

has here the meaning of the energy transported by one vortex of unit length. Expressing the velocity $\mathbf{v}_L$ in terms of the electric field

$$\mathbf{E}=B_0c^{-1}[\mathbf{n}_H\times\mathbf{v}_L],$$

we get

$$\mathbf{q}=-cTS_d[\mathbf{n}_H\times\mathbf{E}]/\Phi_0,$$

where $\mathbf{n}_H$ is a unit vector in the direction of the magnetic field.

Assume now that, under the conditions of the experiment, a constant temperature difference is maintained on the sample boundaries. Then a temperature gradient satisfying the condition $\mathrm{div}(K\nabla T) = 0$ is established in the sample. With the aid of (17) and (18) we get

$$\mathbf{q}=-cTS_d[\mathbf{n}_H\times\mathbf{E}]/\Phi_0-K\nabla T, \tag{22}$$

where the thermal conductivity K takes as $T \to T_c$ the form

$$K=\frac{\nu(0)D}{T^2}\int_\Delta^\infty\varepsilon^2\,\mathrm{ch}^{-2}\frac{\varepsilon}{2T}\,d\varepsilon. \tag{23}$$

We defer the discussion of this formula to the next section.

We now assess the effect of the presence of the temperature gradient on the value of the transport current. To this end we use the method developed earlier in[15, 16, 19]. We represent Δ, **j**, and **Q** in the form

$$\Delta=\Delta_0(T(\mathbf{r}-\mathbf{v}_Lt),\ \mathbf{r}-\mathbf{v}_Lt)+\Delta_1,$$
$$\mathbf{j}=\mathbf{j}_0(T(\mathbf{r}-\mathbf{v}_Lt),\ \mathbf{r}-\mathbf{v}_Lt)+\mathbf{j}_1$$

etc., where Δ_0 and $\mathbf{j}_0$ correspond to the static quantities. Linearizing the equations for Δ and for the current in terms of Δ_1 and $\mathbf{Q}_1$ we obtain, exactly as in[16],

$$c^{-1}\int(\mathbf{j}_1(\mathbf{a}\nabla)\mathbf{Q}_0-(\mathbf{a}\nabla)\mathbf{j}_0\mathbf{Q}_1)\,d^2\mathbf{r}=-4\nu(0)\int(\mathbf{a}\nabla)\Delta_0d^2\mathbf{r}\int_\Delta^\infty fn_1(\varepsilon)\,d\varepsilon, \tag{24}$$

where **a** is an arbitrary constant vector. The left-hand side reduces to an integral over a surface remote from the core of the vortex and is expressed with the aid of the Londons' equation in terms of the two-dimensional vector **u**(**r**) of the deformation of the vortex lattice[19]:

$$c^{-1}\int(\mathbf{j}_1(\mathbf{a}\nabla)\mathbf{Q}_0-(\mathbf{a}\nabla)\mathbf{j}_0\mathbf{Q}_1)\,d^2\mathbf{r}=n_L^{-1}\Big\{(c_{11}-c_{66})\mathbf{a}\nabla\,\mathrm{div}\,\mathbf{u}$$
$$+c_{66}\mathbf{a}\nabla^2\mathbf{u}+c_{44}\mathbf{a}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big\}, \tag{25}$$

where c_{ik} are the elastic moduli obtained by Labusch[20] for a triangular lattice of vortex filaments.

To connect the lattice deformation with the transport current and with the temperature gradient, we write down the Gibbs thermodynamic potential of the system of vortex filaments in a given field H (cf.[19]):

$$\mathscr{F}=\mathscr{F}_0+\int\Big\{\Big[\frac{1}{2}(c_{11}-c_{66})\left(\frac{\partial u_i}{\partial x_i}\right)^2+c_{66}\left(\frac{\partial u_k}{\partial x_i}\right)^2+c_{44}\left(\frac{\partial u_i}{\partial z}\right)^2\Big]$$
$$-\frac{(\mathbf{H}-\mathbf{H}_0)}{4\pi}\mathbf{B}-S(T(\mathbf{r})-T_0)\Big\}\,d^3\mathbf{r}$$

(i, k = 1, 2). The last term describes the influence of the temperature gradient. The entropy per unit volume near T_c is obviously equal to

$$S=-\nu(0)\Delta_\infty^2/T+n_LS_d.$$

The sought relations can be obtained from the condition that the potential be minimal with respect to the variation δu_i of the deformation vector. Then

$$\delta n_L=-n_{L0}\,\partial(\delta u_i)/\partial x_i.$$

As a result we get

$$-\Big[(c_{11}-c_{66})\nabla\,\mathrm{div}\,\mathbf{u}+c_{66}\nabla^2\mathbf{u}+c_{44}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]$$
$$+\frac{n_{L0}\Phi_0}{4\pi}\Big[\frac{\partial\mathbf{H}}{\partial z}-\nabla(H_z-H_0)\Big]-n_{L0}S_d\nabla T=0.$$

Calculating curl **H** = $(4\pi/c)\mathbf{j}_{tr}$, we find

$$[\mathbf{n}_H\times\mathbf{a}]\mathbf{j}_{tr}=\frac{cS_d(\mathbf{a}\nabla T)}{\Phi_0}+\frac{c}{n_L\Phi_0}\Big[(c_{11}-c_{66})\cdot\mathbf{a}\nabla\,\mathrm{div}\,\mathbf{u}$$
$$+c_{66}\mathbf{a}\nabla^2\mathbf{u}+c_{44}\mathbf{a}\frac{\partial^2\mathbf{u}}{\partial z^2}\Big].$$

Comparing this expression with (24) and (25), we get ultimately

$$[\mathbf{n}_H \times \mathbf{a}]\mathbf{j}_{tr} = \frac{cS_d}{\Phi_0}(\mathbf{a}\nabla T) - \frac{4\nu(0)e}{\pi}\int (\mathbf{a}\nabla)\Delta d^2\mathbf{r}\int_\Delta^\infty \varepsilon n_1(\varepsilon)\,d\varepsilon.$$

The function $n_1(\epsilon)$ is determined from (20) and coincides exactly with that obtained in[15]. Thus, the complete expression for the transport current is

$$\mathbf{j}_{tr} = \sigma_f\mathbf{E} + cS_d[\mathbf{n}_H \times \nabla T]/\Phi_0, \tag{26}$$

where the conductivity is[3)]

$$\sigma_f = \beta(T)\sigma_n H_{c2}(T)/B_0, \quad \beta(T) = 1.1(1 - T/T_c)^{-1/2}.$$

We assume for the sake of argument that the electric field is parallel to the x axis and ∇T is parallel to the y axis; then relations (22) and (26) take the form (1), where the coefficients $\alpha_{xy}(\mathbf{H})$ and $\beta_{yx}(\mathbf{H})$ satisfy the Onsager relations:

$$\alpha_{xy}(\mathbf{H}) = \beta_{yx}(-\mathbf{H}) = (\mathbf{n}_H)_z cTS_d/\Phi_0.$$

The transported entropy is

$$S_d = \frac{\nu(0)\Delta_\infty^2\xi^2}{T}\int_{V_0}(1-\psi^2)\,d^2x, \tag{27}$$

where $\psi(x) = \Delta/\Delta_\infty$ and $x = \rho/\xi$. In the calculation of S_d it can be assumed, with logarithmic accuracy in d/ξ, that the elementary cell is round. At large x we have $\psi = 1 - 1/2x^2$. Thus,

$$S_d \approx a\nu(0)D\ln(d/\xi) \approx {}^1/_2 a\nu(0)D\ln(H_{c2}/B_0),$$

where $a = 2\pi^4/7\zeta(3) = 23.1$ (we have assumed for concreteness that the distance d between the vortices is smaller than the field penetration depth λ). Putting $E_F = 10^{4}\,^\circ\mathrm{K}$ and $l = 10^{-6}$ cm, we get

$$S_d = 0.6\ln(H_{c2}/B_0)\cdot 10^{-7}\ [\mathrm{erg/deg\cdot cm}]$$

The units of S_d were chosen for convenience in comparison with experiment.[4]

We call attention to the fact that the results of measurements of σ_f are greatly influenced by the heat transfer from the sample boundaries. Indeed, assume that no heat is removed: $q_y = 0$. In this case there arises a temperature gradient

$$\nabla_y T = -\alpha_{xy}E_x/K,$$

as a result of which we obtain

$$j_x = (\sigma_f + (\alpha_{xy})^2/KT)E_x.$$

That is to say, the quantity measured as the conductivity is

$$\sigma_f(q=0) = \sigma_f\left(1 + \frac{(\alpha_{xy})^2}{KT\sigma_f}\right) \approx \left(1 + 10\beta^{-1}(T)\frac{B_0}{H_{c2}}\ln^2\frac{H_{c2}}{B_0}\right)\sigma_f.$$

The increment to σ_f is small in weak fields $B_0 \ll H_{c2}$ and as $T \to T_c$, but if $H \sim H_{c2}$ and $T \lesssim T_c$ it may turn out to be comparable with σ_f.

In the case of ideal heat removal ($\nabla_y T = 0$), one measures σ_f directly. In the real situation it is probable that the intermediate case is realized, as a result of which, in strong fields $H \sim H_{c2}$, one measures a somewhat overestimated value of σ_f.

Expression (26) shows also that in the presence of a temperature gradient $\nabla_y T$ and at $j_{tr} = 0$ the vortex filaments go into motion at a velocity

$$v_{Ly} = -\frac{c^2 S_d}{\Phi_0 H_{c2}\beta\sigma_n}\nabla_y T,$$

directed from the hotter edge of the sample to the less heated one. This effect was recently observed experimentally.[1] It results in an additional "convective" heat flow. The total thermal conductivity $K_\perp(j_{tr} = 0)$ (in a plane perpendicular to the magnetic field) satisfies the relation

$$K_\perp(j_{tr}=0)/\sigma_f(q=0) = K_\perp/\sigma_f.$$

It follows from (27), at first glance, that the superconducting thermomagnetic phenomena do not vanish at $T = T_c$. It must be borne in mind, however, that (27) is valid if one assumes a widely-spaced lattice of vortex filaments ($d \gg \xi(T)$). At a fixed external field H_0, with increasing temperature the dimension of the vortex core increases, and this leads to an overlap of the vortex cores, i.e., the situation $H_0 \approx H_{c2}(T)$ sets in. With further increase of T, the entropy S_d vanishes as $H_{c2}(T) \to H_0$.

5. THERMAL CONDUCTIVITY

The thermal conductivity K is measured directly in experiment either at a zero electric-field component perpendicular to the gradient temperature, or in the presence of pinning forces, i.e., at $v_L = 0$. This limitation pertains, obviously, only to the thermal conductivity in a plane perpendicular to the magnetic field.

The thermal conductivity of superconducting alloys in the mixed state in the region of strong fields ($H_0 \approx H_{c2}$) was calculated by Caroli and Cyrot.[22] In weak fields ($H_0 \ll H_{c2}$) near the critical temperature the thermal conductivity is given by (23). The same formula is valid also for a homogeneous superconductor at any temperature, since its derivation is based on relations (16) which are valid, of course, for a homogeneous superconductor at all temperatures. In this latter case, Eq. (23) was first derived by Geĭlikman[23] and by Bardeen et al.[24] A formula for the thermal conductivity of superconducting alloys in the mixed state, suitable at any temperature, is given (without proof) in[25]. There, however, the anisotropy of K with respect to the direction of the magnetic field was not investigated.

Near T_c we obtain from (23)

$$K = K_n - \nu(0)D\Delta^3/3T^2,$$

where $K_n = 2\pi^2\nu(0)DT/3$ is the thermal conductivity of the normal metal. This expression must be averaged over a surface perpendicular to the temperature gradient. Assume that initially $\nabla T \parallel H_0$; then the thermal conductivity is

$$K_\parallel = K_n - \frac{\nu(0)D\Delta_\infty^3}{3T^2}\left\{1 - n_L\int_{V_0}\left(1 - \frac{\Delta^3}{\Delta_\infty^3}\right)d^2\mathbf{r}\right\}.$$

For the expression in the curly bracket we obtain with logarithmic accuracy

$$1 - \frac{3}{4}\frac{B_0}{H_{c2}}\ln\frac{H_{c2}}{B_0}.$$

Let now $\nabla T \perp H_0$, By virtue of the equation $\operatorname{div}(K\nabla T) = 0$ we can choose any surface for the calculation of the heat flux. Taking a surface that passes everywhere far from the vortex cores, we get

$$K_\perp = K_n - \nu(0)D\Delta_\infty^3/3T^2,$$

with the ratio

$$\frac{K_\parallel - K_n}{K_\perp - K_n} = 1 - \frac{3}{4}\frac{B_0}{H_{c2}}\ln\frac{H_{c2}}{B_0}.$$

The strongest anisotropy of the thermal conductivity occurs at low temperatures ($T \ll \Delta$). Indeed, if the temperature gradient is perpendicular to the magnetic field H_0 then, choosing a surface that passes far from the vortex cores, we see that the thermal conductivity is given by formula (23) for a homogeneous superconductor at $T \ll \Delta$ and decreases exponentially with temperature. The electronic thermal-conductivity component due to collisions with phonons behaves similarly. In this case the decisive factor is the lattice thermal conductivity (phonon scattering by impurities), which is proportional to T^3.

Matters are different when the temperature gradient is parallel to $\mathbf{H}_0$. Now the main heat flux is through regions occupied by the vortex cores, where the thermal conductivity is of the order of the electronic thermal conductivity K_n of the normal metal, which decreases linearly with temperature.

From (13) with the aid of((12) we obtain at $T \ll \Delta$ (cf.[16])

$$\operatorname{Sp}\hat{g}_1=-l\left\{8\frac{\partial n_0}{\partial\varepsilon}\frac{\varepsilon}{T}\nabla T+\nabla\left(\frac{g_{10}-\bar{g}_{10}}{g}\right)\right\}g^2.$$

and (11) yields $\operatorname{div}\operatorname{Sp}\hat{\mathbf{g}}_1 = 0$. At $\nabla T \parallel H_0$ we can put $g_{10} - \bar{g}_{10} = 0$. Finally we get

$$K_{\parallel}(\rho)=\frac{\nu(0)D}{T^2}\left(\int_0^\infty \varepsilon^2\operatorname{ch}^{-2}\frac{\varepsilon}{2T}d\varepsilon\right)g^2\Big|_{\varepsilon=0}=K_n g^2_{\varepsilon=0}(\rho).$$

The function $g(\rho)$ is defined in[16]. It is the state density normalized to $\nu(0)$ and depends on the distance from the center of the filament. As $\rho \to 0$ we have $g_{\epsilon=0} = 1$, and at large ρ the function $g_{\epsilon=0}$ decreases exponentially over the distance ξ.

Averaging over a surface perpendicular to H_0, we get

$$K_{\parallel}=K_n\frac{B_0}{H_{c2}}\int g^2(x)\,x\,dx.$$

Using the results of Watts-Tobin and Waterworth,[26] who calculated the function $g(x)$, we can obtain the numerical value of the integral:

$$K_{\parallel}=0.62K_nB_0/H_{c2}.$$

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[1] The term with E^2 in the energy can be neglected

[2] Expression (10) is equivalent to resolving the complete $\hat{g}$-function into the regular and anomalous parts (see, e.g., [16,17]).

[3] The numerical factor 1.1 in $\beta(T)$ was obtained in [15] using an interpolation formula for $\Delta(\rho)$.

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