Kinetic properties of the *B* phase of superfluid He^3

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The temperature dependence of the viscosity, thermal conductivity, and the absorption coefficients of zero sound is found on the basis of the Balian–Werthamer state with unity momentum. The calculations are carried out by means of the kinetic equation for quasiparticle excitations in the superfluid state, which was found by the *u-v* transformation method. It is shown that the results obtained depend insignificantly on the choice of model and the value of orbital momentum of the Cooper pairs. The temperature dependence found is in agreement with the existing experimental data.

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1. DERIVATION OF THE KINETIC EQUATION FOR EXCITATIONS

The experimentally observed transition to the superfluid state [1] in He^3 is an interesting example of the formation of Cooper pairs with nonzero momentum. At the present time there are already a number of measurements of the kinetic characteristics of the superfluid phase: measurement of the viscosity, [2] thermal conductivity, [3] damping of zero sound, [4] and the propagation of fourth sound. [5] The experiments have shown the existence of two superfluid phases of He^3: phases A and B. On the basis of study of the kinetic properties it is possible to obtain information on the nature of the spectrum of excitations and their interactions, and therefore the theoretical discussion is of interest.

In the present work we investigate the properties of the B phase of superfluid He^3. We will use as a working model the Balian-Werthamer state. [6] To find the kinetic coefficients over a wide range of temperatures $T \lesssim \Delta_0$, it is possible to use the kinetic equation in which the u-v transformation has been used to make the transition to quasiparticle creation and annihilation operators. Here it is assumed that the following conditions are satisfied for the external perturbation:

$$\omega \ll \Delta_0, \quad qv_F \ll \Delta_0 \quad (\hbar=1). \tag{1}$$

In the Balian-Werthamer model it is necessary to use a generalized u-v transformation:

$$a_{\mathbf{p}s}=u^{\mathbf{p}}_{ss'}\alpha^{\mathbf{p}}_{s'}+v^{\mathbf{p}}_{ss'}\alpha^{+(-\mathbf{p})}_{s'}, \quad \overset{+}{a}_{\mathbf{p}s}=v^{*\mathbf{p}}_{ss'}\alpha^{-\mathbf{p}}_{s'}+u^{*\mathbf{p}}_{ss'}\alpha^{+\mathbf{p}}_{s'}; \tag{2}$$

the summation is carried out over the repeated spin indices s′. The u-v transformation matrix has the form

$$U=\begin{pmatrix} u_{\mathbf{p}} & v_{\mathbf{p}}(\hat{\mathbf{k}}\boldsymbol{\sigma})\sigma_y \\ -v_{\mathbf{p}}(\hat{\mathbf{k}}\boldsymbol{\sigma}^*)\sigma_y & -u_{\mathbf{p}} \end{pmatrix}, \tag{3}$$

where $\boldsymbol{\sigma}$ is the Pauli matrix and $\hat{\mathbf{k}}$ is a unit vector in the $\mathbf{p}$ direction,

$$u_{\mathbf{p}}^2={}^1/_2(1+\xi/\varepsilon), \quad v_{\mathbf{p}}^2={}^1/_2(1-\xi/\varepsilon), \quad \varepsilon=(\xi^2+\Delta^2)^{1/2}. \tag{3'}$$

The energy ξ is measured from the Fermi surface and includes Fermi-liquid effects; the dependence of Δ on temperature is the same as in the BCS model [6]:

$$1=|g|\int_{\Delta}^{\tilde{\omega}} \frac{d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}(1-2n_\varepsilon). \tag{4}$$

Knowing the Hamiltonian of the interaction in the normal state, we can determine the Hamiltonian of the interaction of quasiparticles in the superfluid state by means of a u-v transformation, and by this means we can also determine the collision integral. [7,8] Thus, for the quasiparticle distribution function the following equation is valid:

$$\frac{\partial n}{\partial t}+\frac{\partial n}{\partial \mathbf{r}}\frac{\partial H}{\partial \mathbf{p}}-\frac{\partial n}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{r}}=I(n). \tag{5}$$

For the condition (1) the corrections associated with deviation of Δ from equilibrium in Eq. (4) can be neglected. [9]

Equation (5) permits a solution corresponding to two-liquid hydrodynamics (cf. refs. 10 and 11). We shall seek the solution of (5) in the form $n = n_0 + n_1$, where

$$n_0=\left[\exp\left(\frac{\varepsilon+\mathbf{p}\mathbf{v}_s-\mathbf{p}\mathbf{v}_n}{T}\right)+1\right]^{-1} \tag{6}$$

and $n_1 \ll n_0$. Use of the conservation laws permits reduction of Eq. (5) to the form [10]

$$\begin{aligned} &\frac{n_0(1-n_0)}{T}\left\{\left(\frac{1}{T}\frac{\partial T}{\partial \rho}\varepsilon-\frac{\partial \varepsilon}{\partial \rho}\right)\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)\right. \\ &+\left[\frac{1}{T}\left(\frac{\partial T}{\partial \rho}\rho+\frac{\partial T}{\partial S}S\right)\varepsilon-\frac{\partial \varepsilon}{\partial \rho}-\frac{1}{3}\frac{\partial \varepsilon}{\partial \mathbf{p}}\mathbf{p}\right]\operatorname{div}\mathbf{v}_n \\ &+\frac{\nabla T}{T}\left(\mathbf{p}\frac{ST}{\rho_n}-\varepsilon\frac{\partial \varepsilon}{\partial \mathbf{p}}\right)-\frac{1}{2}\left(\frac{\partial \varepsilon}{\partial p_i}p_k-\frac{1}{3}\delta_{ik}\frac{\partial \varepsilon}{\partial \mathbf{p}}\mathbf{p}\right) \\ &\left.\times\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\operatorname{div}\mathbf{v}_n\right)\right\}=I(n_1). \end{aligned} \tag{7}$$

Helium-helium and helium-phonon collisions contribute to the collision integral. In addition, since most experiments are carried out near the melting point of He^3, scattering by fluctuation formations of the solid phase is also possible. We shall find below an explicit form of the collision integral for all of these processes and shall determine the temperature dependence of the kinetic coefficients.

2. SCATTERING BY FLUCTUATION FORMATIONS OF THE SOLID PHASE

In view of the low value of surface tension of He^3 near the melting point, scattering by fluctuation formations of the solid phase begins to play an important role. Each of these formations is a macroscopic object, i.e., its mass $M \gg m^*_{\mathrm{He}}$ and its size $R \gg a$ (a is the atomic size), and therefore the scattering is elastic and classical. Since spherical formations of the solid phase have the least surface energy, their contribution to the scattering will be maximal and we have the following expression valid for the scattering cross section,

$$\sigma=\sigma_{\mathrm{cl}}=\pi R^2. \tag{8}$$

The Hamiltonian of the interaction in the normal state for this process has the form

$$H'=\sum_{\mathbf{p},\mathbf{p}',s} V_{\mathbf{p}\mathbf{p}'}a_{\mathbf{p}s}\overset{+}{a}_{\mathbf{p}'s}. \tag{9}$$

With the aid of the u-v transformation we go over to the Hamiltonian of the interaction in the superfluid state. Assuming the matrix element $V_{pp'}$ isotropic, we find the collision transport time:

$$\frac{1}{\tau_{tr}}=\frac{1}{2\tau_{tr}^{(0)}}\left(1+\frac{\xi^2+\Delta^2/3}{\varepsilon^2}\right)\frac{\varepsilon}{|\xi|}, \tag{10}$$

where $\tau_{tr}^{(0)}$ is the transit time in the normal state:

$$1/\tau_{tr}^{(0)}=nv_F\sigma. \tag{11}$$

The density n of formations of the solid phase with radius R has a Boltzmann distribution:

$$\begin{gathered}n(R)=n_0(P,\ T)\exp\{-F(R)/T\},\\ F(R)={}^4/_3\pi R^3(f_{so}(P,\ T)-f_{li}(P,\ T))+4\pi R^2\mathfrak{S};\end{gathered} \tag{12}$$

f is the free energy density. Averaging the transit time (11) by means of the distribution (12), near the melting point we obtain

$$\left\langle\frac{1}{\tau_{tr}^{(0)}}\right\rangle=n_0(P,T)v_F\left(\frac{T}{4\mathfrak{S}}\right)\left[1-\frac{3[f_{so}(P,T)-f_{li}(P,T)]}{\sqrt{\pi}\,T}\left(\frac{T}{4\pi\mathfrak{S}}\right)^{1/2}\right].$$

The quantity $n_0(P,\ T)$ is closely related to the Lindemann melting parameter:

$$n_0(P,\ T)\sim(\langle\Delta R^2\rangle-\langle\Delta R^2\rangle_{cr})^{-1/2},$$

where $\langle\Delta R^2\rangle_{cr}$ corresponds to melting. For an estimate of $n_0(P,\ T)$ we can use the results of ref. 12.

Calculating the flux of energy Q and momentum Π_{ik} in the usual way, [8,10] we find the thermal conductivity and viscosity due to collisions of helium excitations (a) with solid formations (s):

$$\begin{gathered}\varkappa_{as}=\frac{2}{3}\frac{p_F^3\langle\tau_{tr}^{(0)}\rangle}{\pi^2m^*}\Phi_1(T),\\ \Phi_1(T)=-T^{-1}\int_\Delta^\infty 3\frac{(\varepsilon^2-\Delta^2)}{(3\varepsilon^2-\Delta^2)}\varepsilon^2\frac{\partial n_0}{\partial\varepsilon}d\varepsilon;\end{gathered} \tag{13}$$

$$\begin{gathered}\eta_{as}=\frac{p_F^4v_F\langle\tau_{tr}^{(0)}\rangle}{15\pi^2}\Phi_2(T),\\ \Phi_2(T)=-\int_\Delta^\infty\frac{\varepsilon^2-\Delta^2}{3\varepsilon^2-\Delta^2}\frac{\partial n_0}{\partial\varepsilon}d\varepsilon.\end{gathered} \tag{14}$$

Equations (13) and (14) can be used to estimate the values κ_{as} and η_{as} also under nonequilibrium conditions if we consider n in Eq. (11) as an experimental parameter. Direct use of Eq. (14) for the viscosity under these conditions can lead to an incorrect result, since in the principal method of viscosity measurement—the rotating disk method—a strong perturbation of the medium occurs.

3. THE HELIUM-HELIUM INTERACTION

In the region where fluctuation formation of the solid phase can be neglected, the main contribution to the collision integral (7) is from helium-helium collisions. In estimating the contribution of collisions of helium with phonons it is necessary to take into account that the ratio of the Fermi velocity to the velocity of sound is $v_F/s \lesssim 0.1$ (refs. 4, 13, 14), and therefore the ratio of the momentum of thermal photons to the Fermi momentum is $q/p_F \sim Tv_F/\epsilon_F s \lesssim 10^{-4}$. If we keep in mind that the process $n(1-n')N$ is forbidden by the conservation laws, we find that the contribution of collisions of helium with photons becomes appreciable only for $T \lesssim 10^{-4}$ K and in the region $10^{-4}\text{ K} \lesssim T \lesssim 10^{-3}$ K it is not necessary to take it into account.

The Hamiltonian of the helium-helium interaction has the form

$$H'=\sum_{\substack{\mathbf{p},\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\\ s,s'}}V(\mathbf{p}_2,\mathbf{p}_1;\mathbf{p},\mathbf{p}_3)\overset{+}{a}_{\mathbf{p}_2s'}\overset{+}{a}_{\mathbf{p}_1s}a_{\mathbf{p}_3s}a_{\mathbf{p}s'}. \tag{15}$$

After use of the u-v transformation, each of the terms a^+a^+aa results in 16 terms. Processes corresponding to $\alpha\alpha\alpha\alpha$ and $\alpha^+\alpha^+\alpha^+\alpha^+$ are forbidden by the conservation laws. Taking into account the permissible symmetry relations leads to the result that in each of the remaining three groups $\alpha^+\alpha^+\alpha^+\alpha$, $\alpha\alpha\alpha\alpha^+$, and $\alpha^+\alpha^+\alpha\alpha$ there will be 24 terms.

In explicit calculation of the helium-helium collision integral we will rest substantially on the fact that all properties of superfluid Fermi systems can be formulated [9,15,16] in terms of Green's functions integrated over ξ. Omitting the cumbersome but essentially straightforward calculations, we obtain

$$\begin{aligned}I_{aa}=&\frac{2\pi}{p_Fv}\left(\frac{mp_F}{2\pi^2}\right)^2\iiint_\Delta^\infty\frac{\varepsilon_1\varepsilon_2\varepsilon_3\,d\varepsilon_1\,d\varepsilon_2\,d\varepsilon_3}{(\varepsilon_1^2-\Delta^2)^{1/2}(\varepsilon_2^2-\Delta^2)^{1/2}(\varepsilon_3^2-\Delta^2)^{1/2}}\int\frac{d\hat{\mathbf{k}}_1}{4\pi}\int\frac{d\hat{\mathbf{k}}_2}{4\pi}\\ &\times\int\frac{d\hat{\mathbf{k}}_3}{4\pi}\delta\left(\frac{\mathbf{p}}{p_F}+\frac{\mathbf{p}_3}{p_F}-\frac{\mathbf{p}_1}{p_F}-\frac{\mathbf{p}_2}{p_F}\right)\{{}^1/_3M_1[(1-n)n_1n_2n_3-n(1-n_1)(1-n_2)(1-n_3)]\\ &\times\delta(\varepsilon-\varepsilon_1-\varepsilon_2-\varepsilon_3)+M[n_1n_2(1-n)(1-n_3)-(1-n_1)(1-n_2)nn_3]\delta(\varepsilon+\varepsilon_3-\varepsilon_1-\varepsilon_2)\\ &+M_2[n_1(1-n)(1-n_2)(1-n_3)-(1-n_1)nn_2n_3]\delta(\varepsilon+\varepsilon_2+\varepsilon_3-\varepsilon_1)\}.\end{aligned} \tag{16}$$

Here

$$\begin{aligned}M=&2|V_{\mathbf{p}-\mathbf{p}_2}|^2\left[\left(1+\frac{\Delta^4(\hat{\mathbf{k}}_1\hat{\mathbf{k}}_3)(\hat{\mathbf{k}}\hat{\mathbf{k}}_2)}{\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3}\right)-\Delta^2\left(\frac{(\hat{\mathbf{k}}\hat{\mathbf{k}}_2)}{\varepsilon\varepsilon_2}+\frac{(\hat{\mathbf{k}}_1\hat{\mathbf{k}}_3)}{\varepsilon_1\varepsilon_3}\right)\right]\\ &+2|V_{\mathbf{p}-\mathbf{p}_1}|^2\left[\left(1+\frac{\Delta^4(\hat{\mathbf{k}}\hat{\mathbf{k}}_1)(\hat{\mathbf{k}}_2\hat{\mathbf{k}}_3)}{\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3}\right)-\Delta^2\left(\frac{(\hat{\mathbf{k}}_2\hat{\mathbf{k}}_3)}{\varepsilon_2\varepsilon_3}+\frac{(\hat{\mathbf{k}}\hat{\mathbf{k}}_1)}{\varepsilon\varepsilon_1}\right)\right]\\ &+2|V_{\mathbf{p}_1+\mathbf{p}_2}|^2\left[\left(1+\frac{\Delta^4(\hat{\mathbf{k}}_1\hat{\mathbf{k}}_2)(\hat{\mathbf{k}}\hat{\mathbf{k}}_3)}{\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3}\right)-\Delta^2\left(\frac{(\hat{\mathbf{k}}\hat{\mathbf{k}}_3)}{\varepsilon\varepsilon_3}+\frac{(\hat{\mathbf{k}}_1\hat{\mathbf{k}}_2)}{\varepsilon_1\varepsilon_2}\right)\right]\\ &-V_{\mathbf{p}-\mathbf{p}_2}V_{\mathbf{p}-\mathbf{p}_1}\left[1+\frac{\Delta^4}{\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3}\{(\hat{\mathbf{k}}\hat{\mathbf{k}}_2)(\hat{\mathbf{k}}_1\hat{\mathbf{k}}_3)+[\hat{\mathbf{k}}\hat{\mathbf{k}}_2][\hat{\mathbf{k}}_1\hat{\mathbf{k}}_3]\}-\Delta^2L\right]\\ &-V_{\mathbf{p}-\mathbf{p}_2}V_{\mathbf{p}_1+\mathbf{p}_2}\left[1+\frac{\Delta^4}{\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3}\{(\hat{\mathbf{k}}_2\hat{\mathbf{k}}_1)(\hat{\mathbf{k}}\hat{\mathbf{k}}_3)+[\hat{\mathbf{k}}_2\hat{\mathbf{k}}_1][\hat{\mathbf{k}}\hat{\mathbf{k}}_3]\}-\Delta^2L\right]\\ &-V_{\mathbf{p}-\mathbf{p}_1}V_{\mathbf{p}_1+\mathbf{p}_2}\left[1+\frac{\Delta^4}{\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3}\{(\hat{\mathbf{k}}\hat{\mathbf{k}}_1)(\hat{\mathbf{k}}_3\hat{\mathbf{k}}_2)+[\hat{\mathbf{k}}\hat{\mathbf{k}}_1][\hat{\mathbf{k}}_3\hat{\mathbf{k}}_2]\}-\Delta^2L\right],\\ L=&\frac{\hat{\mathbf{k}}\hat{\mathbf{k}}_2}{\varepsilon\varepsilon_2}+\frac{\hat{\mathbf{k}}\hat{\mathbf{k}}_1}{\varepsilon\varepsilon_1}+\frac{\hat{\mathbf{k}}\hat{\mathbf{k}}_3}{\varepsilon\varepsilon_3}+\frac{\hat{\mathbf{k}}_1\hat{\mathbf{k}}_3}{\varepsilon_1\varepsilon_3}+\frac{\hat{\mathbf{k}}_1\hat{\mathbf{k}}_2}{\varepsilon_1\varepsilon_2}+\frac{\hat{\mathbf{k}}_2\hat{\mathbf{k}}_1}{\varepsilon_2\varepsilon_1};\end{aligned}$$

V_q is the Fermi-liquid scattering amplitude. To obtain M_2 it is necessary to make the substitution $\epsilon_2 \leftrightarrow -\epsilon_2$ in M; M_1 and M_2 differ by the permutation $\epsilon \leftrightarrow \epsilon_1$ and $\hat{\mathbf{k}} \leftrightarrow \hat{\mathbf{k}}_1$.

In the further calculations we will use the fact that the BCS relation $T_c = 0.57\,\Delta_0$ is valid. Therefore for the B phase over the entire temperature region we have $\Delta/T \gtrsim 3.5$ and consequently the quantity $n_0(\epsilon) \sim e^{-\epsilon/T}$ is a small parameter.

The process $(1-n)n_1n_2n_3$ has a threshold $\epsilon \geq 3\Delta$, and therefore its contribution to the coefficients of thermal conduction and first viscosity are of order $e^{-3\Delta/T}$ and can be neglected. In the case of thermal conduction the correction n_1 to the distribution function has the form

$$n_1=n_0(1-n_0)\cos\theta\frac{\partial T}{\partial x}\varphi(\varepsilon,\theta). \tag{17}$$

We can assume that $\varphi(\epsilon,\ \theta) = \text{const}$ (cf. ref. 16). The term $\mathbf{p}ST/\rho_n$ in Eq. (7) gives a relative contribution $\sim T/\epsilon_F$, and need not be taken into account. Eventually we obtain:

$$\varkappa_{aa}\approx\frac{2\pi}{3}\left(\frac{T}{\Delta}\right)^2\frac{v_F^3}{m\Delta}\left(1+\frac{\Delta}{T}\right)^2e^{-2\Delta/T}\frac{1}{V_0^2F_1(\Delta/T)},$$

$$F_1\left(\frac{\Delta}{T}\right)\equiv\frac{\sqrt{T\Delta}}{\varepsilon_F}\left\{\frac{3}{2}e^{-2\Delta/T}\left(\frac{T}{\Delta}+\frac{T^2}{\Delta^2}+\frac{T^3}{2\Delta^3}\right)\right.$$

$$+\frac{7}{3}\int_0^\infty \exp\left(-\frac{2\Delta}{T}\operatorname{ch} y\right)\frac{y^2}{\operatorname{sh} y}dy\Bigg\}+\frac{4}{3}\int_0^\infty \frac{y^2}{\operatorname{ch} y}\exp\left(-\frac{3\Delta}{T}\operatorname{ch} y\right)dy$$

$$=\frac{\sqrt{T\Delta}}{\varepsilon_F}\left(\frac{3}{2}I_1+\frac{7}{3}I_2\right)+\frac{4}{3}I_3. \qquad (18)$$

For the condition $\Delta/T \gg 1$ we have

$$I_2=e^{-2\Delta/T}T/2\Delta, \quad I_3=e^{-3\Delta/T}(2\pi T/3\Delta)^{1/2}/4\pi.$$

In the calculations we have retained only the zero harmonic of the amplitude $V_0 \sim F_S^0$.

For the viscosity we have $n_1 \propto P_2(\cos\theta)$. The process $nn_2n_3(1-n_1)$ gives a contribution a factor $e(\Delta/T)$ times smaller than scattering and can be omitted, and therefore

$$\eta_{aa}\approx\frac{\pi}{6}\left(\frac{\varepsilon_F T}{\Delta^2}\right)\frac{v_F{}^3}{\Delta}e^{-2\Delta/T}\frac{1}{V_0{}^2F_2(\Delta/T)},$$

$$F_2(\Delta/T)={}^3/_4I_1+{}^{15}/_4I_2+3I_4, \qquad (19)$$

$$I_4=\int_0^\infty \exp\left(-\frac{2\Delta}{T}\operatorname{ch} y\right)y\operatorname{ch} y\,dy;$$

$$\text{for } \Delta/T\gg 1 \quad I_4=e^{-2\Delta/T}T/2\Delta.$$

Equations (18) and (19) have been derived for the temperature region $T \sim \Delta_0$ with the possibility of their future extrapolation to the region $T \ll \Delta_0$ but not to the region $T \gg \Delta(T)$. If we relate κ_a and η_a to the corresponding values in the normal phase, then the quantity $|V_0|^2$ drops out of Eqs. (18) and (19).[8] We note also that the choice of the correction in the form $n_1 \propto \xi$ in combination with integration with respect to ξ over the region $(-\infty, \infty)$, similar to the procedure[17] in the calculations for He^4, would lead to an incorrect result, since in the limit $\Delta \rightarrow 0$ the collision integral would not go over to the corresponding expression for the normal state.[18]

Below T_c the thermal conduction rises by a factor $\min\{e^{\Delta/T}, \epsilon_F/T\}$ and for temperatures $e^{\Delta/T} \gg \epsilon_F/8\Delta$ it takes on a dependence $\kappa_{aa} \propto T^{-3/2}$. The viscosity coefficient approaches a constant and in order of magnitude is the same as in the normal state near T_c. Since the total thermal conductivity and viscosity are determined by the relations

$$\varkappa_a{}^{-1}=\sum_i \varkappa_{ai}{}^{-1}, \quad \eta_a{}^{-1}=\sum_i \eta_{ai}{}^{-1},$$

for $T \lesssim 10^{-4}$ K the interaction of helium excitations with phonons becomes dominant. In view of this κ_a and η_a fall off exponentially with temperature as $e^{-\Delta/T}$. Near the melting point, if the scattering of helium by formations of the solid phase is dominant ($l_{as} \ll l_{aa}$), then equations (13) and (14) are valid for κ_a and η_a. For experiments in capillary tubes the rapid rise of the mean free path $\propto e^{\Delta/T}$ can lead to the result that a dominant role is played by collisions with the walls. Then the value of the kinetic coefficients decreases roughly by a factor $de^{-\Delta/T}/\xi_0$ in comparison with (18) and (19) (ξ_0 is the coherence length and d is the capillary diameter).

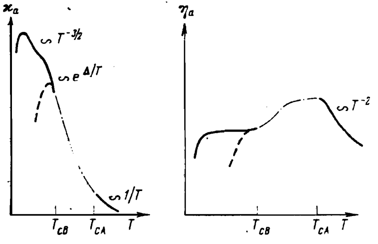


FIG. 1. Temperature dependence of the helium part of the coefficients of thermal conductivity and first sound (without inclusion of $\varkappa_{as}$ and η_{as}). The dashed portion shows the temperature dependence in capillary tubes.

The temperature dependence of the helium part of the viscosity and thermal conductivity is shown schematically in Fig. 1.

4. PHONON VISCOSITY AND THERMAL CONDUCTIVITY

For the phonon distribution function the following kinetic equation is valid:

$$\frac{\partial N}{\partial t}+\frac{\partial N}{\partial \mathbf{r}}\frac{\partial H_{ph}}{\partial q}-\frac{\partial N}{\partial \mathbf{q}}\frac{\partial H_{ph}}{\partial \mathbf{r}}=2\pi\int\frac{1}{2}|U|^2\left(1+\frac{\Delta^2(\hat{\mathbf{k}}\hat{\mathbf{k}}')}{\varepsilon\varepsilon'}\right)\delta(\varepsilon+\varepsilon'-\omega)$$

$$\times\delta(\mathbf{p}-\mathbf{p}'-\mathbf{q})[(N+1)(1-n)(1-n')-Nnn']d\tau\,d\tau' \qquad (20)$$

$$+2\pi\int\delta(\varepsilon+\varepsilon_3-\varepsilon_1-\varepsilon_2-\omega)\delta(\mathbf{p}+\mathbf{p}_3-\mathbf{p}_1-\mathbf{p}_2-\mathbf{q})\widetilde{M}[nn_3(1-n_1)(1-n_2)(N+1)$$

$$-(1-n)(1-n_3)n_1n_2N]d\tau\,d\tau_1\,d\tau_2\,d\tau_3.$$

The quantity $\widetilde{M}$ has a form similar to (16) if by V_q we understand the corresponding matrix element of the helium-phonon interaction.

We have not taken into account the processes $n(1-n')N(1+N')$ and the phonon-phonon interaction, which begin to play an important role only for $T \lesssim 10^{-5}$ K, and also the processes $n_1n_2n_3N(1-n)$, which contain an extra small factor $e^{-\Delta/T}$. In calculation of the first viscosity and thermal conductivity it is possible to omit also the first term in Eq. (20), since beyond the threshold $\omega \gtrsim 2\Delta$ it gives a contribution which falls off as $e^{-2\Delta/T}$.

Omitting the standard calculations (cf. ref. 8) and with allowance for the fact that the matrix element of the helium-phonon interaction has the form $|\widetilde{V}|^2 = |\widetilde{V}'|^2\omega$, we obtain

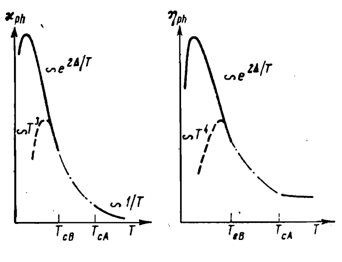
$$\varkappa_{ph\,a}\approx 4\pi^3\zeta(2)\left(\frac{T}{\Delta}\right)^2\frac{v_F{}^3}{m\Delta}\frac{1}{p_F{}^3\omega_D|\widetilde{V}'|^2\Phi(\Delta/T)}, \qquad (21)$$

$$\eta_{ph\,a}\approx\frac{12}{5}\pi^3\zeta(2)\left(\frac{T}{\Delta}\right)^2\frac{v_F{}^3}{m\Delta}\frac{T}{s^2}\frac{1}{p_F{}^3\omega_D|\widetilde{V}'|^2\Phi(\Delta/T)}, \qquad (22)$$

where $\omega_D \equiv p_F s$ is the Debye frequency and $\Phi(\Delta/T) = 3I_1 + {}^7/_3 I_2 + I_4$.

First of all let us note a curious fact, that $\kappa^{(0)}_{ph\,a} \sim \kappa^{(0)}_{aa} \sim 1/T$. Under these same conditions the viscosity $\eta^{(0)}_{ph\,a}$ is much less than the helium viscosity: $\eta^{(0)}_{ph\,a} \sim (T/\epsilon_F)^2(v_F/s)^2\eta^{(0)}_{aa}$. Below T_c the phonon viscosity and thermal conductivity rise rapidly as $e^{2\Delta/T}$ and become dominant ($\kappa = \kappa_a + \kappa_{ph}$, $\eta = \eta_a + \eta_{ph}$). In measurements in capillary tubes the rise in the mean free path according to a law $e^{2\Delta/T}$ leads to the result that scattering by the walls of the vessel begins to become important. Under these conditions κ_{ph} and η_{ph} begin to fall off with further decrease of temperature according to the well

FIG. 2. Temperature dependence of the phonon part of the coefficients of thermal conductivity and first sound (the dashed portion shows the temperature dependence in capillary tubes).



known laws $\kappa_{ph} \propto T^3$, $\eta_{ph} \propto T^4$. Near the melting point the mean free path of the phonons can be limited by scattering on formations of the solid phase, and here κ_{ph} and η_{ph} will drop with temperature as in the scattering of phonons by defects.[8]

The temperature dependence of the phonon kinetic coefficients is shown schematically in Fig. 2.

5. COEFFICIENTS OF SECOND VISCOSITY

As was shown by Khalatnikov,[10] the two-fluid hydrodynamics of a superfluid liquid leads to appearance of four second viscosity coefficients. Their physical nature is closely related to processes in which a change in the total number of Cooper pairs n_a and phonons N_{ph} occurs. Superfluid He^3 is a system with an undetermined number of excitations and therefore the pair distribution function $n_0 = [e^{\epsilon/T} + 1]^{-1}$ in equilibrium has a chemical potential equal to zero. Under these conditions the Khalatnikov method[10] is completely applicable. The second viscosity coefficients are expressed in terms of the constants of relaxation of the excitations to the equilibrium distribution. In the corresponding collision integrals it is necessary to substitute the nonequilibrium distribution functions

$$n=\left[\exp\left(\frac{\varepsilon-\mu_a}{T}\right)+1\right]^{-1},\quad N=\left[\exp\left(\frac{\omega-\mu_{ph}}{T}\right)-1\right]^{-1}$$

and to linearize the collision integrals, using the smallness of μ/T.

The change in the number of helium excitations

$$\dot{n}_a=-\Gamma_{aa}\mu_a \tag{23}$$

receives its main contribution from the process $n_1n_2n_3(1-n)$ and therefore

$$\Gamma_{aa}\approx\frac{8\pi}{3}V_0^2\frac{1}{p_Fv}\left(\frac{mp_F}{2\pi^2}\right)^3\frac{\Delta^3}{T}G\left(\frac{\Delta}{T}\right), \tag{24}$$

where

$$G(\Delta/T)=6I_5-{}^7/_3I_3,$$

$$I_5=\int_0^\infty \operatorname{ch}^3 y\exp\left(-\frac{3\Delta}{T}\operatorname{ch} y\right)dy;$$

$$\text{for } \Delta/T\gg1\quad I_5=(2\pi T/3\Delta)^{1/2}e^{-3\Delta/T}.$$

Correspondingly for phonons the rate of change

$$\dot{N}_{ph}=\Gamma_{ph\,a}(\mu_a-\mu_{ph}) \tag{25}$$

receives contributions from the processes $nn_3(1-n_1)(1-n_2)N$ and $N(1-n)(1-n')$:

$$\begin{gathered}\Gamma_{ph\,a}^{(1)}\approx8\pi^2|\tilde{V}'|^2\left(\frac{T}{2\pi s}\right)^3\frac{1}{p_Fv}\left(\frac{mp_F}{2\pi^2}\right)^3\Delta^3\Phi\left(\frac{\Delta}{T}\right),\\ \Gamma_{ph\,a}^{(2)}\approx8\pi\left(\frac{\Delta}{2\pi s}\right)^3\left(\frac{mp_F}{2\pi^2}\right)\frac{\omega_D|U'|^2}{\varepsilon_F}e^{-2\Delta/T}\operatorname{th}\left(\frac{\Delta}{2T}\right).\end{gathered} \tag{26}$$

The further course of the calculations coincides completely with ref. 10 and therefore we will limit ourselves to presenting the answer:

$$\begin{gathered}\zeta_1=-\frac{1}{\Gamma_{aa}}\frac{\partial n_a}{\partial\rho}\left(n_a-\frac{\partial n_a}{\partial S}S-\frac{\partial n_a}{\partial\rho}\rho\right)-\frac{1}{\Gamma_{ph\,a}}\frac{\partial N_{ph}}{\partial\rho}\left(N_{ph}-\frac{\partial N_{ph}}{\partial S}S-\frac{\partial N_{ph}}{\partial\rho}\rho\right),\\ \zeta_2=\frac{1}{\Gamma_{aa}}\left(n_a-\frac{\partial n_a}{\partial S}S-\frac{\partial n_a}{\partial\rho}\rho\right)^2+\frac{1}{\Gamma_{ph\,a}}\left(N_{ph}-\frac{\partial N_{ph}}{\partial S}S-\frac{\partial N_{ph}}{\partial\rho}\rho\right)^2,\\ \zeta_3=\frac{1}{\Gamma_{aa}}\left(\frac{\partial n_a}{\partial\rho}\right)^2+\frac{1}{\Gamma_{ph\,a}}\left(\frac{\partial N_{ph}}{\partial\rho}\right)^2,\\ \zeta_1=\zeta_4.\end{gathered} \tag{27}$$

6. SOUND ABSORPTION

In discussing the question of damping of sound it is necessary to distinguish two limiting regions: $\Delta_0 \gg \omega \gg 1/\tau$ (zero sound) and $\omega\tau \ll 1$ (first sound), where τ is the relaxation time. (For $\omega \gtrsim \Delta$ it is necessary to take into account the change of Δ with time[9].) The absorption of zero sound can be discussed as the direct absorption of sound quanta by Cooper excitations. For $\omega \geq 2\Delta$ pair production becomes possible.[19] For the absorption coefficient γ we find by means of the first term in the collision integral (20) in the region $(\omega - 2\Delta)/\Delta \gg v_F/s$

$$\gamma_1=a\omega\frac{\omega^2+4\Delta^2}{\omega^2-4\Delta^2}\operatorname{th}\frac{\omega}{4T}. \tag{28}$$

In the limit $\omega \to 2\Delta$ we, like Pokrovskiĭ,[20] arrive at the expression

$$\gamma_1\sim\pi m^2s|U'|^2\Delta\operatorname{th}(\Delta/2T). \tag{29}$$

Another process contributing to sound absorption is helium-helium collisions $nn_3(1-n_1)(1-n_2)N$:

$$\gamma_2\sim\pi|\tilde{V}'|^2\omega e^{-\omega/T}(e^{\omega/T}-1)\frac{1}{p_Fv}\left(\frac{mp_F}{2\pi^2}\right)^3\Delta^3\Phi\left(\frac{\Delta}{T}\right). \tag{30}$$

The function $\Phi(\Delta/T)$ was defined in Eq. (21). Thus, for $1/\tau \ll \omega \ll \Delta$ the absorption coefficient of zero sound falls off rapidly with temperature as $e^{-2\Delta/T}$.

In the absorption of first sound a dominant role is played by relaxation processes. For that part of the absorption which is due to the first viscosity coefficient, as in the normal state,[13] we have the expression

$$\gamma_a\sim\eta_a,\qquad \gamma_{ph}\sim\eta_{ph}, \tag{31}$$

where γ_a and γ_{ph} are associated with the dissipation for helium and phonon excitations, respectively. Equation (31) can be obtained from the kinetic equations (7), (16), and (20) as was done in ref. 8.

CONCLUSION

We see that in calculation of κ, η, and γ only two facts are important: the existence of a gap isotropic spectrum (3′) and the existence of the small parameter v_F/s. A change in the value of the orbital momentum would lead only to a change in the numerical coefficients and would only slightly change the temperature dependence of the kinetic coefficients.

Liquid He^3 is a system with a strong Fermi-liquid interaction. The approximation in which the calculations were carried out corresponds to taking into account only the zero harmonic $F_s^0 \sim V_0$, and therefore the results will be valid only in order of magnitude. If we assume in the excitation energy ξ in Eq. (3′) that the mass is equal to the effective mass of helium, then further inclusion of corrections will change the results only insignificantly (cf. ref. 21). The explicit form of the left-hand side of Eq. (7) is valid for any isotropic interaction, since its derivation is based on use of Galilean invariance, the conservation laws, and thermodynamic relations.

In conclusion let us dwell on comparison of our calculations with experiment. Greytak et al.[3] experimentally measured the thermal resistance $R^* \equiv \Delta T/Q$. In transition from the normal state to phase B its value decreased by approximately a factor of 20. A theoretical estimate on the basis of Eq. (21) gives a rise of ~100 times. A quantitative difference may lie in the fact that the experiment[3] was carried out in capillary tubes. Thus, if we assume that the condition $\xi_0 e^{2\Delta/T} \gg d \gg \xi_0 e^{\Delta/T}$ is satisfied, then $\kappa = \alpha T^3 + \kappa_{aa}$ (κ_{aa} in accordance with Eq. (18)) and rises by approximately $e^3 \sim 27$ times. In addition it is necessary to take into account the decrease in Δ and the possibility of fluctuation formation of the solid phase.

The behavior of the viscosity $\eta = \eta_a + \eta_{ph}$ obtained in 3 and 4 is in good agreement with the recent experiments of Alvesalo et al.[2] Numerical estimates in Eqs. (19) and (22) show that for the equality $\eta_a \sim \eta_{ph}$ at $T \approx 1.2$ mK it is necessary, however, that the cross section for the helium-helium interaction be substantially greater than the cross section for the helium-phonon interaction: $V_0^2/p_F^3\omega_D|\tilde{V}'|^2 \gtrsim 10^2-10^3$. The data on sound absorption in the normal state in principle permit this level to achieved. Simultaneous measurement of the thermal conductivity could serve as a proof that just η_{pha} is responsible for the observed rapid rise.[2] Under these conditions $\kappa_{pha} \gg \kappa_{aa}$ and the dependence (21) is valid. For experiments in capillary tubes the following dependence is satisfied,

$$\eta=\beta T^4+\eta_{aa} \quad \text{for} \quad \xi_0 e^{2\Delta/T}\gg d\gg\xi_0 e^{\Delta/T},$$

$$\eta=\beta T^4+\tilde{\eta}_{aa} \quad \text{for} \quad \xi_0 e^{\Delta/T}\gg d,$$

where $\tilde{\eta}_{aa} \sim de^{-\Delta/T}\eta_{aa}/\xi_0$, and η_{aa} is defined by Eq. (19).

The damping coefficient of zero sound for $\omega < \Delta$ falls off rapidly with reduction of the temperature, as $\sim e^{-2\Delta/T}$. The temperature dependence of the absorption of first sound is determined by Eq. (31). Experiments on damping of zero sound[4] refer mainly to the A phase, but as the transition temperature to the B phase is approached a rapid decrease of the zero-sound absorption coefficient has actually been observed.

The authors recently learned of the work of Pethick et al.[22] who calculated κ_{aa} and η_{aa} in the τ approximation and of the work of Wall and Houghton[23] who calculated η_{aa} for $T \ll \Delta$.

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[1] D. D. Osheroff, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. **28**, 885 (1972). D. D. Osherov et al., Phys. Rev. Lett. **29**, 920 (1972).

[2] T. A. Alvesalo, Yu. D. Anufriyev, H. K. Collan, O. V. Lounasmaa, and P. Wennerström, Phys. Rev. Lett. **30**, 962 (1973). T. A. Alvesalo, H. K. Collan, M. T. Loponen, and M. C. Veuro, Phys. Rev. Lett. **32**, 981 (1974).

[3] T. J. Greytak, R. T. Johnson, D. N. Paulson, and J. C. Wheatley, Phys. Rev. Lett. **31**, 452 (1973).

[4] D. T. Lawson, W. J. Gully, S. Goldstein, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. **30**, 541 (1973); J. Low Temp. Phys. **15**, 169 (1974). D. N. Paulson, R. T. Johnson, and J. C. Wheatley, Phys. Rev. Lett. **30**, 829 (1973).

[5] H. Kojima, D. N. Paulson, and J. C. Wheatley, Phys. Rev. Lett. **32**, 141 (1974).

[6] R. Balian and N. R. Werthamer, Phys. Rev. **131**, 1553 (1963).

[7] B. T. Geĭlikman, Zh. Eksp. Teor. Fiz. **34**, 1042 (1958) [Sov. Phys.-JETP **7**, 721 (1958)].

[8] B. T. Geĭlikman and V. Z. Kresin, Kineticheskie i nestatsionarnye yavleniya v sverkhprovodnikakh (Kinetic and Nonstationary Phenomena in Superconductors), Nauka, 1972.

[9] G. M. Éliashberg, Zh. Eksp. Teor. Fiz. **61**, 1254 (1971) [Sov. Phys.-JETP **34**, 668 (1972)].

[10] I. M. Khalatnikov, Vvedenie v teoriyu sverkhtekuchesti (Introduction to the Theory of Superfluidity), Nauka, 1965.

[11] V. S. Shumeĭko, Zh. Eksp. Teor. Fiz. **63**, 621 (1972) [Sov. Phys.-JETP **36**, 330 (1972)].

[12] B. T. Geĭlikman, Fiz. Tverd. Tela **15**, 3293 (1973) [Sov. Phys.-Solid State **15**, 2194 (1973)].

[13] D. Pines and P. Nozières, The Theory of Quantum Liquids, Benjamin, New York, 1966; Russ. transl., Mir, 1967.

[14] L. R. Corruccini, J. S. Clarke, N. D. Mermin, and J. W. Wilkins, Phys. Rev. **180**, 225 (1969).

[15] G. Eilenberger, Z. Phys. **214**, 195 (1968).

[16] B. T. Geĭlikman and V. G. Orlov, Fiz. Tverd. Tela **15**, 2104 (1973) [Sov. Phys.-Solid State **15**, 1400 (1973)].

[17] I. A. Fomin, Zh. Eksp. Teor. Fiz. **60**, 1178 (1971) [Sov. Phys.-JETP **33**, 637 (1971)].

[18] A. A. Abrikosov and I. M. Khalatnikov, Usp. Fiz. Nauk **65**, 551 (1958).

[19] P. Wölfle, Phys. Rev. Lett. **31**, 1437 (1973).

[20] V. L. Pokrovskiĭ, Zh. Eksp. Teor. Fiz. **40**, 143 (1961) [Sov. Phys.-JETP **13**, 100 (1961)].

[21] O. B. Betbeder-Matibet and P. Nozières, Ann. Phys. (N.Y.) **51**, 392 (1969).

[22] C. J. Pethick, H. Smith, and P. Bhattacharyya, Phys. Rev. Lett. **34**, 643 (1975).

[23] O. Wall and A. Houghton, Phys. Lett. **50A**, 211 (1974).

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