Systematic acceleration of phonons in a turbulent medium

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(Submitted July 12, 1973; resubmitted December 27, 1974)
Zh. Eksp. Teor. Fiz. **68**, 1797–1805 (May 1975)

Fermi's idea regarding statistical acceleration of charged particles is employed in the case of systematic acceleration of phonons in a turbulent medium. The sound-field Hamiltonian with the corresponding canonical variables is determined. The Hamiltonian of the interaction between the sound field and the turbulence is calculated and the probabilities for transition of phonons from a given element of wave vector space are found. The obtained expressions permit an estimate of the shift (increase) of the frequency of sound propagating in a turbulent stream. The energy spread of phonons, which results in broadening of the spectral line, is also found. The frequency shift is not appreciable against the background of the spectral line broadening) but in principle it can probably be observed.

PACS numbers: 52.35.J

1. INTRODUCTION

Even back in 1949 E. Fermi, dealing with the problem of the origin of cosmic rays, advanced the idea of statistical acceleration of charged particles moving among random magnetic fields.[1] Fermi's idea was further developed and extended, in particular, to the case of acceleration of neutral particles (photons, neutrinos) as they move in a plasma.

To our knowledge, no such analysis was carried out in acoustics, although the concept of the phonon or quantum of elastic perturbation can be extended, under certain conditions, not only to solids but also to gases and liquids. The action of various perturbing factors can cause the number of phonons in a given state to change with time; the phonons can arrive at and depart from a given element of momentum space. The transition probability will be determined by the perturbing factors. It may turn out that as a result of the action of external random perturbations the phonon distribution function, and consequently also the average phonon energy, will change with time in monotonic fashion. In particular, the average energy can increase. In this case we can speak of phonon acceleration.

It is obvious that the major part of the phonon-acceleration problem consists of calculating the probability of their transition from a given element of momentum space. To determine this probability it is necessary to determine the phonon creation and annihilation operators, for which purpose it is necessary to determine correctly the Hamiltonian of the sound field with the corresponding canonical variables. From this Hamiltonian there should follow the equations of motion of the medium.

In our problem we disregard dissipation and the nonlinearity of the sound field. The factor that disturbs the phonon motion will henceforth be taken to be a field of turbulent pulsations.

2. DETERMINATION OF THE HAMILTONIAN

If we neglect the thermal-conductivity and dissipation processes and assume that the liquid is barotropic, then the equations of continuity and motion retain the total energy of the medium:

$$\mathscr{H}=\int dx\left(\frac{1}{2}\rho v^2+\varepsilon(\rho)\right). \tag{2.1}$$

We use here the following standard notation: ρ is the density of the medium, $\mathbf{v}$ is the vibrational velocity in Eulerian coordinates, $\epsilon(\rho)$ is the density of the internal energy of the medium. The sound field is determined by the change of the density ρ from its equilibrium value ρ_0. This field is potential, and $\mathbf{v}$ is a gradient of a certain scalar function, $\mathbf{v} = \nabla\varphi$.

The hydrodynamic equations for the case in question can be written in the form of functional derivatives of $\mathscr{H}$:

$$\frac{\partial\rho}{\partial t}=\frac{\delta\mathscr{H}}{\delta\varphi},\quad \frac{\partial\varphi}{\partial t}=-\frac{\delta\mathscr{H}}{\delta\rho}. \tag{2.2}$$

Thus, the pair ρ and φ is canonically conjugate. For the sound field, ρ plays the role of the canonical coordinate and φ that of the canonical momentum.[1)] We shall henceforth neglect the nonlinearity of the sound field. The Hamiltonian of the sound field takes in the considered approximation the form

$$\mathscr{H}_0=\frac{1}{2}\int dx\left(\rho_0(\nabla\varphi)^2+\frac{c^2}{\rho_0}\rho^2\right). \tag{2.3}$$

Here c^2 is the square of the adiabatic velocity of sound.

We take the Fourier transforms of the canonical variables

$$\{\varphi,\rho\}=\sum_{\mathbf{k}}\left(\frac{2}{V}\right)^{1/2}\{\varphi_{\mathbf{k}},\rho_{\mathbf{k}}\}\cos\mathbf{k}\mathbf{x}, \tag{2.4}$$

where V is the normalization volume. Then

$$\mathscr{H}_0=\frac{1}{2}\sum_{\mathbf{k}}\left(\frac{1}{2}\rho_0 k^2\varphi_{\mathbf{k}}^2+\frac{1}{2}\frac{c^2}{\rho_0}\rho_{\mathbf{k}}^2\right), \tag{2.5}$$

i.e., in momentum space the canonical variables are the Fourier components φ_k and ρ_k, with φ_k the canonical momentum and ρ_k the canonical coordinate.

In the case of quantization φ_k and ρ_k are operators for which we can write corresponding commutation relations. The matrix elements of the operator $\mathscr{H}_0$ will take a diagonal form if

$$\begin{gathered}\rho_{\mathbf{k}}=(\hbar\rho_0/2Vc^2)^{1/2}\omega^{1/2}(a_{\mathbf{k}}+a_{\mathbf{k}}^*),\\ \varphi_{\mathbf{k}}=-i(\hbar c^2/2\rho_0V)^{1/2}\omega^{-1/2}(a_{\mathbf{k}}-a_{\mathbf{k}}^*),\end{gathered} \tag{2.6}$$

where ω = ck is the frequency of the k-th mode of the sound field, and the nonzero matrix elements of the operators a_k and a_k^*, which have the meaning of phonon creation and annihilation operators, are equal to

$$(n-1|a_{\mathbf{k}}|n)=n^{1/2}e^{-i\omega t},\quad (n|a_{\mathbf{k}}^*|n+1)=(n+1)^{1/2}e^{i\omega t}. \tag{2.7}$$

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3. TRANSITION PROBABILITIES

Since the sound wave propagates in a turbulent medium with velocity pulsations $\mathbf{u}$, we have $\mathbf{v} = \nabla\varphi + \mathbf{u}$ and div $\mathbf{u} = 0$. In this case the equations of motion retain the Hamiltonian in which these pulsations are taken into account:

$$\mathcal{H}=\mathcal{H}_0+\sum_{m=1}^{4}\mathcal{H}_{int}^{(m)}=\mathcal{H}_0+\int dx\left(\rho u\nabla\varphi+\rho_0 u\nabla\varphi+\frac{1}{2}\rho u^2+\frac{1}{2}\rho_0 u^2\right). \quad (3.1)$$

We can immediately leave out the last term of (3.1), since allowance for it will mean only a shift of the energy-reference level.

Of the remaining three terms of the interaction Hamiltonian, we can retain only the first. Indeed, in the calculation of the matrix elements of the interaction operators, and then of the transition probabilities, the corresponding expressions acquire δ functions due to the mismatches in the frequency and in the wave vectors. For the transition probability per unit time, corresponding to the operator

$$\mathcal{H}_{int}^{(2)}=\int d\mathbf{x}\rho_0\mathbf{u}\nabla\varphi,$$

we obtain

$$W(\mathcal{H}_{int}^{(2)})\sim\delta(\mathbf{k}-\boldsymbol{\varkappa})\delta(\omega-\Omega).$$

Here $\mathbf{k}$ is the wave vector of the corresponding state of the sound field, $\kappa = 2\pi/l$, l is a certain scale of the turbulent-pulsation field, and Ω is the pulsation frequency corresponding to this scale. It is obvious that in order of magnitude we have $\Omega \sim \langle u_c^2\rangle^{1/2} l^{-1}$. In order for $W(\mathcal{H}^{(2)})$ to be different from zero we must have $\mathbf{k} = \boldsymbol{\kappa}$ and $\omega = \Omega$, i.e., the relation $\langle u_c^2\rangle \ll c^2$ must be satisfied. But $\langle u_c^2\rangle \sim c^2$; this causes matrix element of the operator $\mathcal{H}_{int}^{(2)}$ which is of interest to us to vanish.

We can analogously disregard the matrix elements of the term

$$\mathcal{H}_{int}^{(3)}=\int d\mathbf{x}\frac{1}{2}\rho u^2,$$

which differ from zero if the length λ of the propagating monochromatic sound wave is much larger than the characteristic scales l in the turbulent pulsations. Indeed, from the expression

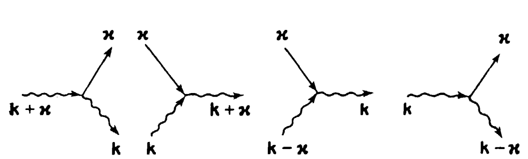
$$W(\mathcal{H}_{int}^{(3)})\sim\delta(\mathbf{k}-\boldsymbol{\varkappa}_1\pm\boldsymbol{\varkappa}_2)\delta(\omega-\Omega_1\pm\Omega_2)$$

it follows that in order of magnitude we have $\omega \sim \langle u_c^2\rangle^{1/2} l^{-1}$, whence $c \sim \langle u_c^2\rangle^{1/2}\lambda l^{-1}$. But even if this condition is satisfied, the influence of $\mathcal{H}_{int}^{(3)}$ is manifest only in the fact that the natural frequency of the corresponding mode is shifted by an amount proportional $\langle u^2\rangle$.

Thus, retaining in the interaction Hamiltonian the most essential term

$$\mathcal{H}_{int}^{(1)}=\int d\mathbf{x}\rho\mathbf{u}\nabla\varphi, \quad (3.2)$$

and using relations (2.4) and (2.6), we can easily show that all the possible interaction processes are described by the corresponding diagrams (Fig. 1). The diagrams that describe the simultaneous creation and annihilation of a phonon pair are obviously forbidden under the assumptions made above.



We have in mind a second-quantization description of the wave system. It follows therefore from the definition of the S matrix that the quantity

$$S^{(1)}=-\frac{i}{\hbar}\int_0^t dt'\mathcal{H}_{int}(t') \quad (3.3)$$

describes in first-order approximation the probability amplitude for the creation or vanishing of elementary field excitation during a time t. The standard calculation procedure enables us to find the probability of the transition of a phonon from an initial state, as a result of interaction with turbulent pulsation of scale $(\boldsymbol{\kappa}, \Omega)$; these probabilities are given by the expressions

$$W_\pm=2\pi c^{-2}\omega\omega_\pm n_{i\pm}u_i(\boldsymbol{\varkappa},\Omega)u_j^*(\boldsymbol{\varkappa},\Omega)n_{j\pm}\delta(\Delta\omega). \quad (3.4)$$

It is assumed here that

$$\mathbf{u}(\mathbf{x},t)=\sum_{\boldsymbol{\varkappa},\Omega}\mathbf{u}(\boldsymbol{\varkappa},\Omega)\exp i(\boldsymbol{\varkappa}\mathbf{x}-\Omega t), \quad (3.5)$$

the argument of the δ function is $\Delta\omega = \Omega - \mathbf{c}\cdot\boldsymbol{\kappa}$, and $\mathbf{n}_\pm$ is a unit vector defined by the condition $\mathbf{n}_\pm = (\mathbf{k} + \boldsymbol{\kappa})/|\mathbf{k} \pm \boldsymbol{\kappa}|$.

4. KINETIC EQUATION

The time variation of the phonon distribution function is due to the fact that the phonons arrive at a given momentum-space element and leave it with a certain probability. We can write an equation for the change of the distribution function of the number of phonons f_k with time. It must be borne in mind that the phonon goes over into the state $\mathbf{k}$ from states $\mathbf{k} + \boldsymbol{\kappa}$, and vice versa with corresponding probability.

Summing over all possible values of κ and Ω, we obtain

$$\frac{\partial f_{\mathbf{k}}}{\partial t}=-\sum_{\boldsymbol{\varkappa},\Omega}W_-(f_{\mathbf{k}}-f_{\mathbf{k}-\boldsymbol{\varkappa}})+W_+(f_{\mathbf{k}}-f_{\mathbf{k}+\boldsymbol{\varkappa}}). \quad (4.1)$$

Equation (4.1) becomes particularly simple if the condition $|\boldsymbol{\kappa}| \ll \mathbf{k}$ is satisfied. Then we get from (4.1) by an elementary series expansion

$$\frac{\partial f_{\mathbf{k}}}{\partial t}=\frac{\partial}{\partial k_i}D_{ij}\frac{\partial}{\partial k_j}f_{\mathbf{k}}, \quad (4.2)$$

$$D_{ij}=\sum_{\boldsymbol{\varkappa},\Omega}\varkappa_i\varkappa_j W_-=\frac{2\pi}{c^2}\omega^2\sum_{\boldsymbol{\varkappa},\Omega}\varkappa_i\varkappa_j n_k u_k(\boldsymbol{\varkappa},\Omega)u_l^*(\boldsymbol{\varkappa},\Omega)n_l\delta(\Omega-\mathbf{c}\boldsymbol{\varkappa}).$$

If we are interested in the time variation of the mean value of a certain function A_k, with

$$\langle A_{\mathbf{k}}\rangle=\int\frac{d\mathbf{k}}{(2\pi)^3}f_{\mathbf{k}}A_{\mathbf{k}},$$

then we easily obtain from (4.2), by integrating by parts,

$$\frac{\partial}{\partial t}\langle A_{\mathbf{k}}\rangle=\left\langle\frac{\partial}{\partial k_j}D_{ij}\frac{\partial}{\partial k_i}A_{\mathbf{k}}\right\rangle. \quad (4.3)$$

It must be borne in mind in the foregoing analysis that we always stipulated the requirement that the turbulent pulsation velocities be small, $\langle u^2\rangle^{1/2} \ll c$. In addition, it was assumed that the nonlinear effects connected with the finite amplitudes of the sound waves are negligibly small. It was also assumed that the field of the turbulent pulsations is specified and the sound waves do not perturb it. Only in this case can the components D_{ij} be regarded as constant. The equation (4.1) obtained under these assumptions is quite general and remains effective also at long durations of the process, when the transitional methods (e.g., the method of geometric acoustics) cannot be used.

5. CALCULATION OF PHONON ACCELERATION IN A TURBULENT MEDIUM

We apply the results to the case of sound propagation in an isotropic turbulent stream. We change over to a continuous distribution, and furthermore take into account the fact that in this case, from (3.5), under the condition div u = 0, we can use in the expression (3.4) for the transition probability the tensor

$$B_{ij}=u_i(\varkappa,\Omega)u_j(\varkappa,\Omega)=4\pi^3\left(\delta_{ij}-\frac{\varkappa_i\varkappa_j}{\varkappa^2}\right)\frac{E(\varkappa,\Omega)}{\varkappa^2}. \qquad (5.1)$$

Here $E(\kappa, \Omega)$ is the energy density of the turbulent pulsations, and satisfies the relation

$$^1/_2\langle u^2\rangle=\int d\varkappa\, d\Omega E(\varkappa,\Omega).$$

Straightforward but cumbersome calculations yield for the components D_{ij}

$$D_{11}=\frac{\pi}{c^3}\omega^2\int d\varkappa\, d\Omega\varkappa E(\varkappa,\Omega)\int d\cos\theta\cos^2(\theta)\,(1-\cos^2\theta)\,\delta\left(\cos\theta-\frac{\Omega}{c\varkappa}\right)$$
$$=\frac{\pi}{c^5}\omega^2\int d\varkappa\, d\Omega\frac{\Omega^2}{\varkappa}E(\varkappa,\Omega),\quad D_{12}=0,\quad D_{21}=0. \qquad (5.2)$$

Here $\cos\theta$ is reckoned from the phonon propagation velocity.

Noting that the components D_{ij} of interest to us are proportional to the square of ω, that the phonon propagation direction coincides with κ, and that $\mathbf{c} = (c,0,0)$, we obtain

$$\frac{\partial}{\partial t}\langle\omega\rangle=\left\langle\frac{\partial}{\partial k_j}D_{ij}\frac{\partial\omega}{\partial k_i}\right\rangle=\left\langle c\frac{\partial}{\partial k_j}D_{ij}\right\rangle=\frac{2\pi}{c^3}\int d\varkappa\, d\Omega\frac{\Omega^2}{\varkappa}E(\varkappa,\Omega)\langle\omega\rangle. \qquad (5.3)$$

Hence

$$\langle\omega\rangle=\omega^0\exp\left[\left(\frac{2\pi}{c^3}\int d\varkappa\, d\Omega\frac{\Omega^2}{\varkappa}E(\varkappa,\Omega)\right)t\right]. \qquad (5.4)$$

These formulas solve our problem. Speaking in classical language. They describe a monotonic shift (increase) of the frequency with time as a function of the stream parameters. We note that the assumption $\kappa \ll k$, which leads to (5.3) and (5.4), is not a principal one and is made only because it is desired to simplify the computational part of the problem. If we do not make this assumption, then the expression for the average energy change can be obtained directly from (4.1). Multiplying (4.1) by ω, integrating this expression with respect to κ, and changing the integration variable $\mathbf{k}$, we can reduce this expression to the form

$$\langle\dot\omega\rangle=\int\left[-\sum_{\varkappa,\Omega}(\omega-\omega_-)W_-+(\omega-\omega_+)W_+\right]f_k\frac{dk}{(2\pi)^3}. \qquad (5.5)$$

Using the law of energy conservation in each individual collision, $\omega-\omega_\mp=\pm\Omega$, we obtain

$$\frac{\partial}{\partial t}\langle\omega\rangle=\left\langle\sum_{\varkappa,\Omega}\Omega(W_+-W_-)\right\rangle. \qquad (5.6)$$

The physical meaning of this expression is quite clear, if it is recognized that W_+ and W_- are the probabilities of the phonon acquiring or losing an energy Ω.

Substituting here the values of $W_\pm$ from (3.4) and changing over to a continuous distribution, we obtain

$$\frac{\partial}{\partial t}\langle\omega\rangle=\left\langle\int\frac{d\varkappa\, d\Omega}{(2\pi)^4}\Omega\frac{2\pi}{c^2}\omega(B^+\omega_+-B^-\omega_-)\delta(\Delta\omega)\right\rangle.$$

If $|\kappa| \ll k$, then here $B^\pm = n_i^\pm B_{ik}n_k^\pm \approx n_iB_{ik}n_k = B$, $\omega_+-\omega_-=2\Omega$ and consequently

$$\frac{\partial}{\partial t}\langle\omega\rangle=\int\frac{d\varkappa\, d\Omega}{(2\pi)^4}\Omega\frac{2\pi}{c^2}2\Omega B\langle\omega\rangle=\frac{2}{(2\pi)^3c^2}\int d\varkappa\, d\Omega\Omega^2n_iB_{ij}n_j\langle\omega\rangle.$$

Since

$$n_iB_{ij}n_j=4\pi^3(1-\cos^2\theta)E(\varkappa,\Omega)/\varkappa^2,$$

it follows that

$$\frac{\partial}{\partial t}\langle\omega\rangle=\frac{2\pi}{c^2}\int d\varkappa\, d\Omega\Omega^2E(\varkappa,\Omega)\int d\cos\theta(1-\cos^2\theta)\delta(\Omega-c\varkappa\cos\theta)\langle\omega\rangle$$
$$\approx\frac{2\pi}{c^3}\int d\varkappa\, d\Omega\frac{\Omega^2}{\varkappa}E(\varkappa,\Omega)\langle\omega\rangle.$$

This expression is exactly equal to (5.3).

Naturally, if the velocity pulsations of the turbulent medium are independent of the time, i.e., $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})$, no statistical acceleration occurs. Then, indeed, it follows from the definition of the correlation tensor $B_{ik}(\mathbf{x}; t) = B_{ik}(\mathbf{x})$ that $\Omega = 0$. In this case the integral in (5.5) vanishes and $\partial\langle\omega\rangle/\partial t = 0$, i.e., $\langle\omega\rangle = \text{const}$.

Obviously, the correlation tensor can be represented in the form

$$B_{ik}(\mathbf{x};\ t)=B_{ik}(\mathbf{x};\ t\neq0)+B_{ik}(\mathbf{x};\ t=0).$$

This notation corresponds to a representation of the spectral energy density in the form

$$E(\varkappa,\ \Omega)=E(\varkappa,\ \Omega=0)+E(\varkappa,\ \Omega\neq0)=E(\varkappa)\delta(\Omega)+E(\varkappa,\ \Omega\neq0). \qquad (5.7)$$

If there exists in the medium a regular flux u^r that carries away the isotropic turbulence without changing its state, then we can obviously write

$$E(\varkappa,\ \Omega)=E(\varkappa)\delta(\Omega-\mathbf{u}^r\boldsymbol{\varkappa})+E(\varkappa)\delta(\Omega)+E(\varkappa,\ \Omega\neq0). \qquad (5.8)$$

In this case

$$B_{ik}(\mathbf{x};\ t)=B_{ik}(\mathbf{x}-\mathbf{u}^rt;\ t=0)+B_{ik}(\mathbf{x};\ t=0)+B_{ik}(\mathbf{x};\ t\neq0). \qquad (5.9)$$

In real conditions the last two terms are very frequently omitted, and one confines oneself to the hypothesis of "frozen" turbulence. Then the calculation of the coefficient of interest to us leads to the expression

$$D_{11}=\frac{\omega^2}{2c^2}\int d\varkappa\varkappa^2\, d\Omega\, d\varphi\, d\cos\theta\cos^2\theta\sin^2\theta E(\varkappa)\delta(c\varkappa\cos\theta-\Omega)$$
$$\times\delta(\Omega-u^r\varkappa(\cos\theta\cos\theta_1+\sin\theta\sin\theta_1\sin\varphi\sin\varphi_1+\sin\theta\sin\varphi\sin\theta_1\cos\varphi_1))$$
$$=\frac{\omega^2}{2c^3}\int d\varkappa\varkappa E(\varkappa)\int d\varphi\, d\cos\theta\cos^2\theta\sin^2\theta$$
$$\times\delta\left[\cos\theta\left(1-\frac{u^r}{c}\cos\theta_1\right)-\frac{u^r}{c}\sin\theta\sin\theta_1\cos(\varphi-\varphi_1)\right].$$

We have introduced here spherical coordinates; the running angle θ is measured from the direction of $\mathbf{c}$ and the angle between the vectors $\mathbf{u}^r$ and $\mathbf{c}$ is equal to θ_1 = const.

It is easily seen that, accurate to terms of order $(u^r/c)^2 \ll 1$, the argument under the δ-function sign vanishes when

$$\cos\theta=\cos\left(\frac{\pi}{2}-\frac{u^r}{c}\sin\theta_1\cos(\varphi-\varphi_1)\right)$$

From this we obtain

$$D_{11}=\frac{\omega^2}{4c^3}\left(\frac{u^r}{c}\sin\theta_1\right)^2\int d\varkappa\varkappa E(\varkappa). \qquad (5.10)$$

In the general case, taking (5.9) into account, the solution of (5.3) takes the form

$$\langle\omega\rangle=\omega^0\exp\left[\left(\frac{1}{2c}\left(\frac{u^r}{c}\sin\theta_1\right)^2\int d\varkappa\varkappa E(\varkappa)+\frac{2\pi}{c^3}\int d\varkappa\, d\Omega\frac{\Omega^2}{\varkappa}E(\varkappa,\Omega)\right)t\right]. \qquad (5.11)$$

For small t, expression (5.11) can be expanded in a series. If we confine ourselves only to the first term in the argument of the exponential, leaving out the second term, then as the sound wave passes through a turbulent layer of thickness x, the average frequency of the sound wave should increase by the amount

$$\Delta\omega=\langle\omega\rangle-\omega_0=\frac{\omega^0}{2c}x\left(\frac{u^r}{c}\sin\theta\right)^2\int d\varkappa\varkappa E(\varkappa). \qquad (5.12)$$

This expression agrees qualitatively in form with the analogous expression obtained in the second geometrical-optics approximation for an electromagnetic wave passing through a turbulent layer of given thickness.[3]

It is obvious that besides the acceleration effect, i.e., the effect of the frequency shift, there occurs within the framework of the statistical mechanism a phonon energy spread, which leads to broadening of the spectral line. The magnitude of this effect can be estimated from (4.3), where A_k must be replaced by ω^2. To estimate the spectral-line broadening and compare it with the possible shift $\Delta\omega = \langle\omega\rangle - \omega_0$, we introduce the quantity

$$\sigma=\langle\omega^2\rangle-\langle\omega\rangle^2.$$

It follows from (4.3) and (5.3) that if $D_{11} = \omega^2 B$, B = const, then we can obtain for an initially monochromatic signal

$$\langle\omega\rangle=\omega_0e^{2cBt}, \quad \langle\omega^2\rangle=\omega_0{}^2e^{6cBt},$$

where according to (5.11)

$$2cB=\frac{1}{2c}\left(\frac{u^r}{c}\sin\theta\right)^2\int d\varkappa\varkappa E(\varkappa). \qquad (5.13)$$

Consequently

$$\sigma=\langle\omega^2\rangle-\langle\omega\rangle^2=\omega_0{}^2(e^{6cBt}-e^{4cBt}),$$
$$(\Delta\omega)^2=(\langle\omega\rangle-\omega_0)^2=\omega_0{}^2(e^{2cBt}-1)^2. \qquad (5.14)$$

Hence

$$\chi^2=\sigma/(\Delta\omega)^2=e^{2cBt}/(1-e^{-2cBt}). \qquad (5.15)$$

In cases when the method of successive approximations can be used, i.e., if $2cBt \ll 1$, we have

$$\chi\approx(2cBt)^{-1/2}\gg1, \qquad (5.16)$$
$$\sigma^{1/2}\approx\omega_0(2cBt)^{1/2}. \qquad (5.17)$$

We note that expression (5.17) corresponds to the formula for the rms fluctuation of the phase for sound propagating in a turbulent medium.

Let us obtain numerical estimates. It is known[4] that in the inertial interval of isotropic turbulence we have $E(\kappa) = 0.76\gamma\bar{\epsilon}^{2/3}\kappa^{-5/3}$. Here $\bar{\epsilon}$ is the dissipation energy per unit time and per unit mass; the atmospheric layer next to the earth this quantity is of the order of 10^3 cgs units. Therefore

$$\int d\varkappa\varkappa E(\varkappa)=2.28\gamma\bar{\varepsilon}^{2/3}(2\pi/l_0)^{1/3}, \quad l_0\ll L_0.$$

The parameter γ, generally speaking, is subject to changes and increases with increasing wind velocity. We assume here $\gamma^{1/2} \sim 3$, which is close to the value given in [4]. Here l_0 is the internal turbulence scale, L_0 is the external scale of the flux, $l_0 \sim 1$ to 10^{-1} cm. If we use expressions (5.11) and (5.17) for small t, then estimates yield the following results: at a sound frequency $f \sim 2 \times 10^4$ Hz, a flux velocity $u^r \sim 20$ m/sec, and a process duration $t \sim 0.2$ sec, the line broadening in Hertz

$$\sigma_f^{1/2}=\frac{1}{2\pi}\sigma^{1/2}=\frac{1}{2\pi}(\langle f^2\rangle-\langle f\rangle^2)^{1/2}$$

is of the order of 10^2 Hz, whereas the shift of the mean value $\langle f\rangle$ from the initial f_0 yields a value on the order of 2–3 Hz. Against the background of such a line broadening, under the assumed conditions, the effect of statistical acceleration of the phonons is difficult to discern, but in principle it is observable. We note that it follows from (5.15) that the parameter χ^2 has a minimum that is reached at $t_* = \ln 2/2cB$ and is obviously equal to $\chi_{min} = 2$. This circumstance gives grounds, in principle, to hope to be able to observe experimentally the acceleration effect against the background of the statistical spread of the phonons in those cases when the condition $2cBt \ll 1$ is not satisfied.

The foregoing estimates cannot claim better than order-of-magnitude accuracy. Yet the values obtained for $\sigma^{1/2}$ are in satisfactory agreement with the available experimental data, and the values for $\Delta\omega$ do not contradict them.

It must be borne in mind that the method described in the paper can be applied to a few other problems of sound propagation in the field of random velocity pulsations, for example in the ocean, where the energy density of such pulsations can be quite large in a number of cases.

[1)]Concerning the definition of the canonical variables for hydrodynamic problems see, e.g., [2].

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Translated by J. G. Adashko
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