Phase dynamics of particles in a microtron and the problem of stochastic instability of nonlinear systems

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(Submitted September 4, 1974)
Zh. Eksp. Teor. Fiz. **68**, 1601–1613 (May 1975)

The properties of nonlinear phase oscillations in a microtron have been studied systematically with a single theoretical method. Resonance and high-current instabilities of the phase motion are described, the cause of appearance of the two-humped distribution of the density of electron bunches is indicated, and the physical nature of the limitations in the energy and current of the accelerated particles is clarified. A mechanical model of a microtron is described. For the case of a microtron it is shown how the transition occurs in a nonlinear system (as the nonlinearity is increased) from the dynamic to the stochastic regime of oscillations.

PACS numbers: 29.20.H, 84.80.D

The phase motion of the particles in a microtron is extremely unique, since a microtron is the only device which employs a nonlinear resonance with a variable multiplicity of interaction.[1] Many features of the phase oscillations were already observed in the first studies,[2, 3] and subsequently it has been possible[4-6] to investigate the phase motion more completely, using the new method[5] of solving the difference equations describing this motion. This investigation has been continued in the present work: We have determined the factors leading to loss of particles accelerated in a microtron and limiting their energy; we have explained why the region of phase stability has such a queer shape, why its size is so anomalously small, and other items. As a whole, the properties of the phase motion can now be considered adequately studied, and the article sums up the work in this area. For completeness and unity, we have mentioned briefly the results obtained previously.

Study of the phase dynamics of particles is necessary not only to increase the efficiency of a microtron. As noted by Zaslavskiĭ and Chirikov[7] the variable-multiplicity resonance used in a microtron corresponds from a physical point of view to the limit of appearance of stochastic instability in nonlinear oscillatory systems. We can, therefore study in detail in the case of the microtron the transition from dynamic to stochastic properties, and also better understand the cause of appearance of stochastic behavior. In addition, study of the integral nonlinear resonances observed in a microtron[5] may turn out to be useful in solution of the problem of small denominators which arises in the general theory of stability of dynamic systems.[8]

1. SOLUTION OF THE NONLINEAR PHASE EQUATIONS

In derivation of the phase equations, we will assume the accelerating gap to be infinitely narrow, since the effect of its finite width is felt only in the first few turns and then will be taken into account by introduction of a correction factor. We will discuss the following quantities: the accelerating field phase φ_n for the n-th passage of the particles through the accelerating gap and the dimensionless energy Γ_n (the total energy U of the particles, relative to their rest energy U_0) which describe the particles after this passage. In the approximation indicated, the quantities φ_{n+1} and Γ_{n+1} are expressed in terms of φ_n and Γ_n by means of difference equations or, alternatively, point transformations of the following form (see for example ref. 9):

$$\varphi_{n+1}=\varphi_n+2\pi\Gamma_n/\Omega,\quad \Gamma_{n+1}=\Gamma_n+A\cos\varphi_{n+1}, \tag{1}$$

where the parameter Ω is equal to the ratio of the guiding magnetic field H to the cyclotron field $H_0 = m_0c\omega/e$ (ω is the frequency of the accelerating voltage), and the constant coefficient A is the ratio of the accelerating-field amplitude to the quantity U_0/e.

The system of equations (1) has a particular solution whith describes the resonance acceleration of equilibrium particles (resonance with variable multiplicity):

$$\Gamma_{s,n}=\Omega[m+g(n-1)],\quad \Delta\Gamma_s=g\Omega=A\cos\varphi_s,$$
$$\varphi_{s,n}=\varphi_s+2\pi(n-1)[m+{}^1\!/_2 g(n-2)], \tag{2}$$

where φ_s is the equilibrium phase measured from the closest maximum of the accelerating field, and the fixed integers m and g express (in units of $T_0 = 2\pi/\omega$) respectively the duration of the first revolution around the resonator and the change in the duration from revolution to revolution.

The properties of the phase oscillations have been studied previously in the linear approximation[2, 3] and we will not discuss them here. In order to find the nonlinear solution, following ref. 5, we will introduce the complex variable

$$W_n=\frac{1}{2}\left(1+i\,\mathrm{tg}\frac{\nu}{2}\right)\psi_n+\pi(1-i\,\mathrm{ctg}\,\nu)\eta_n=|W_n|\exp(i\theta_n),$$
$$\psi_n=\varphi_n-\varphi_{s,n},\quad \eta_n=(\Gamma_n-\Gamma_{s,n})/\Omega,\quad \cos\nu=1-\pi\,\mathrm{tg}\,\varphi_s \tag{3}$$

(ν is the frequency of linear oscillations, $0 < \nu < \pi$). The choice of W_n indicated is determined by the fact that in the linear approximation the solution has the very simple form $e^{i\nu n}W_0$, i.e., the representative point moves along a circle.

We will use Eqs. (1)–(3) to express W_{n+1} in terms of W_n; we will then apply this procedure successively, beginning with n = 0, and represent the solution W_n in the form of the sum of the linear term indicated above and nonlinear terms depending on $W_0, W_1, \ldots, W_{n-1}$. Then we will take into account the smallness of the nonlinear terms (for oscillations of small amplitude) and find the solution (i.e., the dependence of W_n on n and W_0) by the method of successive approximations.

If we limit ourselves to the third approximation and retain only the quadratic and cubic terms in the power expansion, the solution has a singularity for two values of the frequency ν: for $\nu = \pi/2$ and $\nu = 2\pi/3$. Far from these frequencies the nonlinearity leads only to a shift of the frequency and center of the oscillations, but does not destroy the stability of small oscillations. However, near these frequencies the situation is qualitatively different: In the power expansion of the quantity W_n, additional resonance terms appear whose amplitude increases in proportion to n and which lead to a buildup in the phase oscillations and a number of other consequences. In these resonance cases the solution is sought as follows. Calculating the increment $W_{n+3} - W_n$ (for $\nu \approx 2\pi/3$) or $W_{n+4} - W_n$ (for $\nu \approx \pi/2$) and taking into account, in view of the quasiperiodic nature of the motion, the smallness of these increments, we will replace the increments by differentials; the solution of the differential equations obtained also determines the shape of the phase trajectories.

For $\nu = 2\pi/3 + \delta$ ($|\delta| \ll 1$), retaining the linear term due to detuning of the frequency δ and the quadratic resonance term ($\propto \overline{W}^2$), we find the phase trajectories

$$|W|^2[3^{3/2}\delta/2\pi+|W|\cos 3\theta]=\text{const}, \tag{4}$$

along which the representative points drift (in the theory of point transformations such curves are usually called invariants, since they transform into themselves on reflection).

If $\nu = \pi/2 + \delta$ ($|\delta| \ll 1$), then there are no quadratic resonance terms, but the quadratic nonlinearity in the second approximation and the cubic nonlinearity in the first approximation result in an additional resonance term ($\propto \overline{W}^3$) whose action, when the nonlinear correction to the frequency is taken into account, determines the phase trajectories:

$$|W|^4[\pi^2+1-(\pi^2-{}^1/_3)\cos 4\theta]-4\delta|W|^2=\text{const}. \tag{5}$$

2. RESONANCE INSTABILITY OF NONLINEAR PHASE OSCILLATIONS

The principal nonlinear effect, determined by Eqs. (4) and (5) and first observed by Melekhin and Luganskiĭ[4, 5], is the instability of phase oscillations near resonance values of φ_S. It follows from Eq. (4) that for $\nu = 2\pi/3$ ($\delta = 0$) the phase trajectories are open and extend to infinity for oscillations with arbitrarily small initial amplitude; consequently, these oscillations are unstable. Since the values of φ_S are small in the microtron ($\varphi_S < 32°$), the quadratic nonlinearity is large and the instability due to it develops rather rapidly. Integration of the equations shows that the number of turns in which the amplitude of phase oscillations rises substantially amounts to $n \sim 3/|W_0|$ ($|W_0|$ is the initial amplitude of the oscillations), i.e., 10–20 turns for the main mass of the particles. The results of a numerical calculation of the initial system (1) confirm these conclusions.

If the phase φ_S is slightly shifted from the resonance value ($\delta \neq 0$), then, as follows from Eq. (4), there is a closed separatrix passing through special points of the saddle type and located at the most a distance $3^{1/2}\delta/\pi$ from the center of oscillations (in the W plane); the minimum size is a factor of two smaller. Inside the separatrix the phase trajectories are closed and the oscillations are stable, and outside it the trajectories extend to infinity as before. With increase of $|\delta|$ the inner region of stable oscillations is extended; for $|\delta| \approx 0.3$ the size of this region coincides with the total size of the

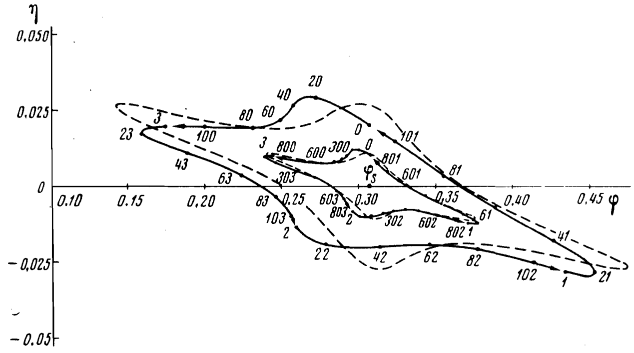

FIG. 1. Phase trajectories for $\nu = \pi/2$ ($\varphi_S = 0.308$, $\varphi = \varphi_S + \psi$). Solid curves—numerical calculation by computer, dashed curves—plotted from Eq. (5) and drawn through points No. 80 (outer curve) and No. 600.

region of phase stability, and with further change of φ_S the stable region changes in size relatively slowly. This quantity also determines the zone of resonance instability. For $\nu = \pi/2$ the pattern of oscillations is somewhat different. As a result of the stabilizing action of the cubic nonlinearity, the phase trajectories remain closed, but the effect of the quadratic nonlinearity leads to a strong beating (see Fig. 1).

We see that in the discrete system being considered a resonance buildup of oscillations occurs without external perturbations only under the influence of the intrinsic nonlinearity, and the resonance is integral, i.e., it arises in the case when the period of the phase oscillations is equal to a whole number of turns k. As this number increases, the strength of the resonance drops rapidly and for oscillations of small amplitude only the cases k = 3 and k = 4 previously discussed are important. However, with increase of the amplitude of oscillations, the resonance perturbation is strengthened and resonances of much higher order begin to appear. The theory developed above is not suitable for large amplitudes, but for physical reasons it is clear that integral resonances should lead to instability also for oscillations with large amplitude.

The existence of this effect is confirmed by numerical calculations, and in fact it explains the anomalously small size (in comparison with other accelerators) of the region of phase stability corresponding to stable oscillations for a fixed value of φ_S. This fact was noted already by Kolomenskiĭ[2] but the reason for it was not understood. A calculation carried out by us showed that for φ_S equal to 0.20, 0.25, and 0.308 (the period of linear oscillations $2\pi/\nu$ in these cases is 5.2, 4.6, and 4) the period of oscillations at the limit of the stable region is respectively 7, 6, and 5. For this reason the boundary of the region is not a completely defined curve (separatrix) as in other systems described by differential equations, but some transitional resonance zone. The size of the stable region increases with increasing number of turns since resonances of higher order begin to appear and, more precisely, it is necessary to consider the dynamic phase aperture of the microtron (cf. ref. 10).

The constant loss of particles from the stable phase region as the result of resonance buildup of oscillations can explain why in a microtron with thirty orbits[9] an appreciable falloff of current is observed in further orbits, practically independent of the means of injection of particles. Only 2.5–3% of the particles injected into the resonator are accelerated to a final energy of 30

MeV. Fortunately, the particles which leave the stable phase region are rapidly lost in the resonator walls as the result of buildup of vertical oscillations and therefore load the resonator comparatively little. Altogether about 25% of the power of the high-frequency generator in 30-MeV microtrons is used in acceleration to the final beam energy. However, this effect nevertheless limits the accelerated beam power substantially and leads to the necessity of providing a high current density from the cathode.

3. HIGH-CURRENT INSTABILITY IN A MICROTRON

The loss of particles in acceleration by the applied field rises sharply when the values of φ_s lie inside one of the resonance zones ($\nu \approx \pi/2$ and $\nu \approx 2\pi/3$). In Fig. 2 we have shown "current-voltage" characteristics of a microtron obtained by numerical calculation, i.e., the dependence of the accelerated current (which is proportional to the number of representative points N on the n-th orbit located near the center of the stable region) on the resonator voltage, which is uniquely associated with the value of φ_s (in Eq. (1) the coefficient $A \sim 1/\cos\varphi_s$) The calculation was carried out with the system of equations (1) and the initial distribution of representative points in the phase plane was assumed uniform. It is evident that the shape of the curves in the n-th orbit depends on the resonator width L; for L = 0 in the vicinity of the two resonance values there are dips which become deeper with an increase in the number of orbits.

The finite width of the resonator can be taken into account (Fig. 2b) by introducing a so-called flight factor $2\sin(\Delta\varphi/2)/\Delta\varphi$, where $\Delta\varphi$ is the angle of flight of the particles across the resonator. In this case the left dip almost disappears as the result of the rapid increase in loss of particles with small φ_s (in the initial orbits the condition of resonance acceleration is not satisfied for them), but the right-hand dip remains as before. An experiment carried out in a microtron with thirty orbits[4] showed that in the first type of acceleration, where the resonator width is small,[9] the measured current-voltage characteristics (in different orbits) are very close

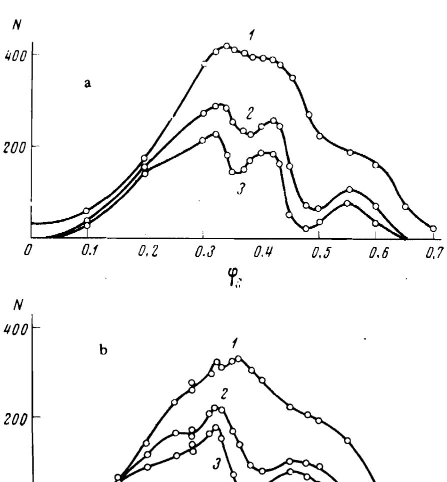

FIG. 2. Current-voltage characteristics of a microtron: a–L = 0, for curves 1, 2, and 3 n is respectively 8, 15, and 28; b–L = 1.45 c/ω, for curves 1, 2, and 3 n is respectively 8, 15, and 25.

to the curves of Fig. 2a, and in the second type of acceleration, where the resonator width and the flight angle are large, the results are close to Fig. 2b.

The appearance of the dips in the characteristics is not so important at first glance, but in practice this leads to serious consequences. The left slope of the dips corresponds to a negative differential conductivity of the accelerated beam. In the case of electronic apparatus it is well known that negative resistance leads to instability at sufficiently large currents. A calculation made by the author and an experiment carried out by Luganskiĭ and the author[4] showed that in fact these "large" currents are extremely small and in the thirtieth orbit the cubic term amounts to only several milliamperes (in the pulse). For the currents of ~100 mA achieved in practice, if the phase falls on the negative slope of the characteristic, the instability develops rapidly and leads to disruption of the acceleration regime. Here the electron loading of the resonator decreases sharply, the field strength in it rises several times, and as a result high-frequency breakdown occurs.

For this reason in pulsed microtrons with 15–20 turns or more it is generally not possible to achieve phases φ_s lying to the right of the first maximum (in order of increasing φ_s) of the volt-ampere characteristic. As a result, even without this the narrow region of equilibrium phases which are stable in the linear approximation (32.5°) narrows further by about a factor of two, and the range of achievable values of accelerating voltage decreases in this case by a factor of four. If we further take into account that for small φ_s the capture of particles is too small (see Fig. 2), the actually achievable limits are $\Delta\varphi_s \sim 3°$ and $\Delta A/A \sim 1\%$, which leads to quite severe technical difficulties in a microtron with a large number of turns.

4. SUBDIVISION OF THE PHASE PLANE NEAR RESONANCE VALUES OF φ_s

The nonlinearity leads to an additional curious feature of the phase motion. In experiments carried out by Bykov[11], a two-humped radial phase distribution of the density of electron bunches accelerated in the microtron was observed. This phenomenon does not fit into the framework of theoretical ideas, disappears on changing the adjustment of the accelerator,[12] and in general could be considered some kind of accident, if it were not observed repeatedly and in subsequent experiments of other workers.[9] An explanation of this effect is contained in Eq. (5). It follows from this equation that for small positive values of δ the W plane is broken up into individual regions. In addition to stable points W = 0 around which closed phase trajectories exist, there are four unstable points of the saddle type ($|W|^2 = 3\delta/(3\pi^2 + 1)$, $\theta = (1/4)\pi(2m + 1)$) and four stable points of the center type ($|W|^2 = 3\delta/2$, $\theta = \pi m/2$), in the vicinity of which there also exist stable closed trajectories. For larger amplitudes Eq. (5) describes closed trajectories which encompass the entire region described.

These stable fourfold points (which are reproduced every four turns) serve as new centers of attraction in the phase plane; as a result a nonmonotonic radial phase distribution of the electron bunch density arises. Luganskiĭ and Melekhin[6] carried out detailed numerical calculations which showed that if the means of filling the phase plane (at the moment of injection) is taken into account, a two-humped radial phase distribution arises

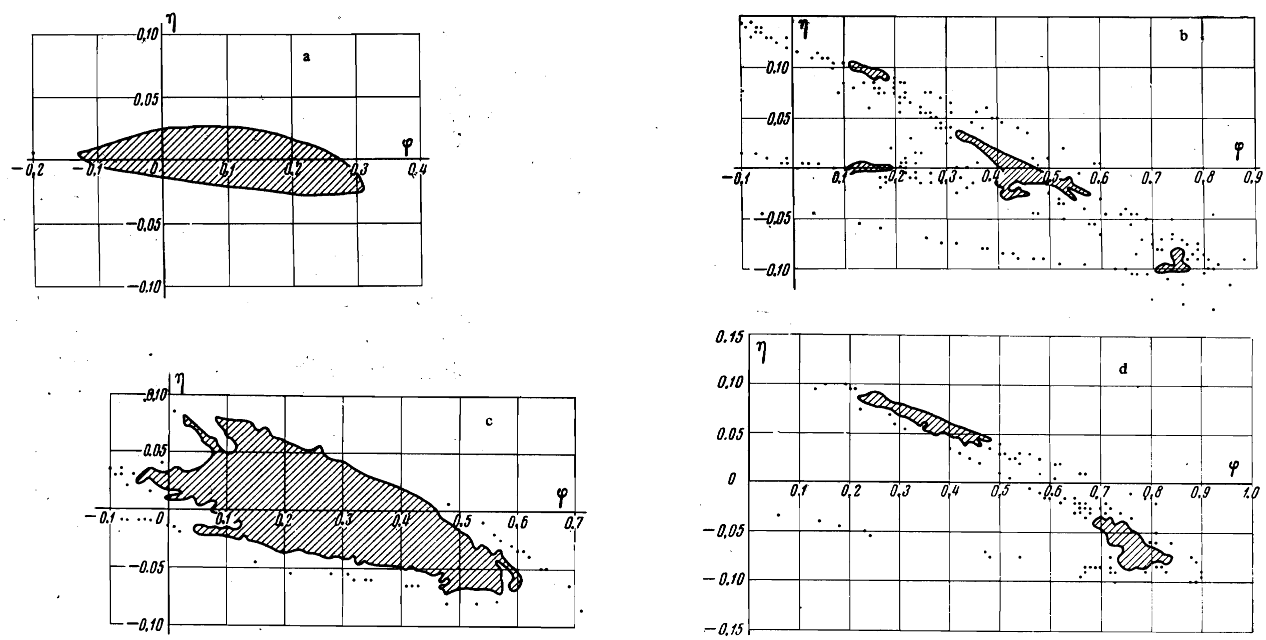

FIG. 3. Phase-stability region calculated for 30 turns: a—$\varphi_s = 0.15$, b—$\varphi_s = 0.30$, c—$\varphi_s = 0.445$, d—$\varphi_s = 0.594$.

in subsequent orbits, for which the shape and other characteristics are close to that observed experimentally. This structure of the bunches occurs, in agreement with the conclusions of the theory, only in a narrow region of φ_s ($\nu = \pi/2 + \delta$, for small $\delta > 0$). It is interesting to note that these values of φ_s correspond to the left maximum of the curve of Fig. 2a. The two-humped behavior has been observed quite regularly, in spite of the smallness of the region of φ_s, just because it arises in adjusting the accelerator for maximum current. With further increase of φ_s, the fourfold points approach the boundary of the stable region and become unstable together with their regions of attraction. This also leads to appearance of a dip in the volt-ampere characteristics.

The existence of stable multiple points in the phase plane of a microtron (near $\nu = 2\pi/3$ there are threefold points, and for $\nu \gtrsim \pi$ there are twofold points) was observed theoretically by Luganskii[13] and is illustrated in Fig. 3. In the general theory of point transformations, this island structure had already been described by Birkhoff[14] and is an important feature of nonlinear systems. The physical reason for appearance of stable and unstable multiple points is the compensation of a small linear detuning δ of the frequency ν (near resonance values of ν) by the nonlinear shift of the frequency $\Delta\nu$. In the linear approximation for resonance values of ν, all points of the phase plane are multiple and there are no phase trajectories at all. However, the action of the nonlinearity leads to a dependence of the oscillation frequency on the amplitude, the indicated degeneracy of the phase plane is removed, and it is broken up into individual regions.

This phenomenon is particularly interesting near the value $\nu = \pi$. The very appearance of a boundary at this point can be treated as a resonance buildup of oscillations even in the linear approximation. The nonlinearity decreases the phase oscillation frequency in a microtron, and therefore the boundary of the stable region is shifted somewhat to larger φ_s, the region becoming doubly connected and grouped around stable twofold points.[13] The variable W introduced above is not appropriate for investigation of this case, since the frequency ν becomes complex, but it is possible to find directly a particular solution of the system (1) corresponding to the twofold points if we limit ourselves to several terms of the expansion. In the quadratic approximation the solution has the form

$$\psi_n = 2/\pi - \operatorname{tg}\varphi_s + (-1)^n(\operatorname{tg}^2\varphi_s - 4/\pi^2)^{1/2},$$
$$\eta_n = \pi^{-1}(-1)^{n+1}(\operatorname{tg}^2\varphi_s - 4/\pi^2)^{1/2}. \qquad (6)$$

Since $\eta_{max} \sim 0.1$, the maximum value of stable φ_s departs from the linear limit by only 3°, i.e., the extension of the region of phase stability is small and is of interest more in principle than in practice.

5. ESTIMATES OF THE LIMITING NUMBER OF TURNS

We saw above that the resonance instability gradually (with increase of the number of turns) destroys the region of phase stability. This effect leads to a practical limitation of the number of turns in the microtron, in spite of the fact that in an ideal field equilibrium particles can be accelerated without limit. The nature of the limitations depends substantially on the value of the equilibrium phase. We shall describe how the size and shape of the region of phase stability gradually change with increase of the equilibrium phase φ_s (see Fig. 3).

For small φ_s (Fig. 3a) the frequency of the oscillations is small, nonlinear resonances cannot appear, and therefore the boundary of the region has a smooth shape and its size is close to the maximum achievable (the left boundary almost coincides with $\varphi = -\varphi_s$). For $\varphi_s = \arctan \pi^{-1}$ ($\nu = \pi/2$, Fig. 3b) with k = 5 resonance determines the boundary of the region. Its size is relatively small and the boundary is cut up, since the rate of growth of the oscillations depends not only on $|W_0|$ but also on the initial angle θ_0 in the W plane. For a particularly favorable initial position of the representative point, the oscillations do not rise initially, but decay. This also serves as the cause of appearance of isolated stable points outside the continuous region (in fact, of course, these are not points, but small regions whose size is less than the calculation cell).

With a further small increase of φ_s, the fourfold points described above arise, and for $\varphi_s = \arctan(3/2\pi)$ ($\nu = 2\pi/3$, Fig. 3c) as a result of the strong resonance instability the central region is very small, there are a number of isolated points, and isolated regions appear which surround the threefold stable points. In the linear boundary of the region ($\varphi_s = \arctan(2/\pi)$, $\nu = \pi$), as was noted above, the region is singly connected and rather large as the result of the stabilizing action of the nonlinearity, and for larger φ_s (Fig. 3d) two isolated regions appear which surround the twofold points of Eq. (6). The size of these regions decreases rapidly and for $\varphi_s = 35.5°$ they disappear.

The strengthening of the effect of nonlinear resonances observed with increase of φ_s appears also in the different nature of the drop in current with number of turns. In Fig. 4 we have shown the theoretical dependence obtained for an accelerating gap of zero width with a uniform initial distribution of representative points in phase space. In view of the approximations the drop in the first few orbits is not characteristic, but in the further orbits the approximations mentioned are unimportant. It can be seen that for small φ_s the number of accelerated particles is comparatively small (the stability region is small), and after that the current no longer drops and for large φ_s the capture of particles into the acceleration regime occurs more efficiently; however, the current falls off too rapidly.

At first glance it appears that it is possible, for a not too high beam intensity, to increase the number of turns significantly by going to small φ_s at which the frequency ν is small (curves 1 and 2 of Fig. 4). However, in this case an instability of another kind arises, due to various disturbing factors. The stability region for small φ_s is small, the tolerance on the accelerating voltage is quadratically small, and the phase oscillations in the microtron, as was noted above, are not damped, since the relativistic increase of the particle mass is compensated by the increase in the phase length of the turns. Therefore a slow diffusion of the representative points occurs under the influence of perturbations, and when they leave the stable region they are rapidly lost.

There is an additional interesting effect. Since the duration of the n-th turn increases in proportion to n, the phase oscillations, which are periodic in the number of turns, are aperiodic in time and slow down with increasing turn number. For small φ_s the oscillations become so slow that they hit a resonance with fluctuations of the field in the resonator, and as a result rapid loss of particles occurs (curve 4 in Fig. 4, which was obtained with inclusion of periodic pulsations (± 0.6%) of the accelerating voltage, 300 times slower than the accelerating frequency). When these factors are taken into account, the use of small φ_s gives no practical advantage.

The drop in current with increasing number of turns and the associated decrease in efficiency of the microtron, and also the decrease in the tolerances on the various parameters of the accelerator with increasing number of turns, leads to the result that a number of turns of the order of 100 must be considered the natural limit for a microtron. In the existing acceleration modes the maximum increment of energy per turn is ~1.5 MeV, and therefore an energy of ~150 MeV is the limiting value

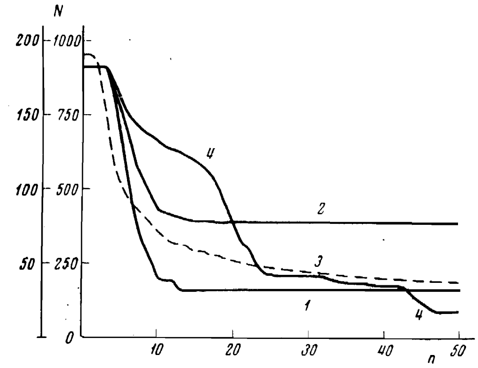

FIG. 4. Drop in current with number of turns in perturbed case (curve 1—$\varphi_s = 0.10$, 2—$\varphi_s = 0.15$, 3—$\varphi_s = 0.32$) and under the influence of a slow perturbation (4—$\bar{\varphi}_s = 0.15$). The right-hand scale of N is given only for curve 3.

for a microtron of the usual type. Further increase of energy is possible only in a transition to racetrack microtrons, in which a small linear accelerator is placed in the magnet gap and the energy increment per turn can be raised to 20–30 MeV and the final energy to 500–1000 MeV. In practice a microtron of the usual type with a continuous magnet is convenient to use up to 40–50 MeV, and then it is necessary to go to racetrack microtrons in order that the number of turns as before not exceed 30–40.

This limitation on the number of turns executed by the particles exists only in the microtron, the only accelerator with a variable multiplicity of acceleration. To achieve the microtron regime, two conditions are necessary: the accelerating field must be localized in the accelerating gap, as a result of which the energy accumulation has a discrete nature, and the magnitude of the energy increment per turn must be sufficiently large to provide the necessary change of phase duration of the turns. In view of the first condition, the phase motion is described by difference equations, and the high frequency of the phase oscillations, due to the second reason, does not permit these equations to be replaced by differential equations as is usually done in other accelerators.

It is appropriate to mention here that Turrin[15] uses as initial equations not difference equations, but the differential equations usual for all accelerators, and only then makes a transition to difference equations. It is just this procedure which constitutes an error leading to results which are inconsistent not only with all previous theoretical conclusions but also with direct measurements of the region of phase stability, carried out by Melekhin and Luganskiĭ[4].

6. MECHANICAL MODEL OF A MICROTRON

Let us discuss a mechanical model of a microtron which permits its place among other dynamic systems to be seen readily. Let an ideally elastic sphere bounce on a plane horizontal base which oscillates harmonically along the vertical. The duration of the flight of the sphere up and down in a gravitational force field is proportional to its initial vertical velocity, and the change in this velocity on reflection is equal to twice the velocity of the reflecting plane at the moment of collision, i.e., it depends harmonically on the phase of reflection. If the amplitude of oscillation of the reflector is small in comparison with the height of rise of the sphere and, for this reason, we can assume that the reflection occurs in one and the same plane but for different velocities of the reflector, and if we also neglect the deformation of the sphere and the base, then the difference equations describing the vertical oscillations of the sphere will coincide with equations (1).

For a small vibration frequency of the reflecting plane, the mechanical model corresponds to an ordinary cyclotron and there is a limiting height of rise of the sphere. When the frequency exceeds a critical value and the condition of resonance with variable multiplicity is satisfied (Eqs. (2) for $g = 1$), then the height of rise of the sphere begins to increase monotonically from collision to collision (of course, for an appropriate choice of initial conditions), which corresponds to the acceleration regime in a microtron.

With a further increase of the vibration frequency, the nonlinear resonances described above arise, and then linear oscillations with $g = 1$ become unstable but a narrow phase-stability region arises corresponding to $g = 2$, and so forth. If the frequency of oscillations is very high, there are a number of unstable equilibrium phases corresponding to resonances with variable multiplicity for $g = 1, 2, 3, \ldots$, the phase oscillations are mixed, and the motion has a random nature to a significant degree.

In a microtron a similar stochastic acceleration regime can in principle be achieved by decreasing the magnetic guide field H, increasing the accelerating frequency ω, or increasing the accelerating amplitude A (cf. ref. 7). In all of these cases the parameter $A/\Omega \approx g_{max}$ will be large and a number of unstable equilibrium phases arise which correspond to $g = 1, 2, \ldots, g_{max}^{-1}$. Near each of these phases the oscillations build up exponentially, which is a necessary condition for appearance of stochastic behavior. The numerical experiments carried out in the work of Zaslavskiĭ and Chirikov[7, 16], particularly on calculation of various mechanical models with use of an elastic sphere, actually demonstrate the approach to a stochastic nature of the oscillations in the presence of a strong nonlinearity corresponding in the case of a microtron to a large value of the parameter A/Ω; the value $A/\Omega \sim 1$ serves as the boundary of stochastic behavior.

It should be noted that the problem of appearance of stochastic behavior in a dynamic system cannot be considered definitively solved at the present time. As a rule, this problem does not permit analytic solution, and in numerical calculations there always remains the question of the role of rounding errors. While it was previously assumed[7] that the stochastic properties depend first of all on the magnitude of the nonlinearity, the recent work of Zakharov[17] and Manakov[18] has shown that in a number of problems the corresponding continuous systems are completely integrable and therefore the rate of approach to stochastic behavior of a discrete system depends only on its deviation from its continuous analog.

Phase oscillations in a microtron for a large value of g_{max} satisfy both conditions for appearance of stochastic behavior: The system is both substantially nonlinear and typically discrete, and far from the continuous limit. Therefore the onset of stochastic behavior in this case appears very likely. At the same time we should apparently observe partially ordered oscillations corresponding to new centers of attraction around multiple points of the type corresponding to Eq. (6). From what has been said it follows that a microtron as a physical system is a very appropriate object for theoretical and experimental study of the problem of appearance of stochastic instability.

It is possible to state in another way why the microtron acceleration regime occupies an intermediate position between the ordinary mode of resonance acceleration (for example, in an isochronous cyclotron) and the stochastic acceleration regime. In ordinary resonance, the particle energy U increases in proportion to the acceleration time t, and if the phases of passage of the particles through the resonator are random, then we have $U \propto t^{1/2}$. In a microtron the particles arrive only in the accelerating phases, as in the ordinary resonance method. However, since the total acceleration time in n turns is $t_n \propto n^2$ and the energy $U_n \propto n$ (see Eq. (2)), then we have $U \propto t^{1/2}$ as in the stochastic case. For just this reason a resonance with variable multiplicity leads in principle to an unlimited increase of the amplitude of oscillation of an nonisochronous oscillator when a periodic pulsed force acts on it.

In conclusion the author expresses his deep indebtedness to Academician P. L. Kapitza for his attention to this work and for its support. The author is extremely grateful to L. A. Vaĭnshteĭn and S. P. Kapitza for helpful advice and valuable critical remarks, to L. B. Luganskiĭ for his helpful cooperation in solution of a number of technical questions which arose in this article.

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Translated by C. S. Robinson
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