Anisotropic cosmological Bianchi type V model in the general (axially symmetric) case with moving matter

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A homogeneous axially symmetric cosmological model of the Bianchi type V is considered within the framework of general relativity. The model belongs to a class of anisotropic models with 4-velocities that are not orthogonal to the invariant manifolds (homogenous spaces) V_3. Matter possesses a velocity and a nonvanishing 4-acceleration (except for dust matter). A transition from a synchronous system, with temporal geodesic lines orthogonal to the sapcelike V_3, to a comoving system reveals the presence of horizon surfaces and possible incompleteness of the initial synchronous system. This leads to the necessity of introducing also semi-geodesic systems with spacelike geodesics which are orthogonal to non-spacelike invariant manifolds V_3. A qualitative analysis of the Einstein equations is presented. A manifestation of the hydrodynamic specific features of continuous motion of matter in the presence of 4-acceleration is the double-valued character of the solution (the presence of limiting curves). An analysis of the motion of matter having the symmetry under consideration in Galilean space-time is also presented.

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1. INTRODUCTION

Relativistic cosmology and general relativity theory (GRT) deal with homogeneous anisotropic models for which, by definition, the metric admits of a three-parameter group G_3 of motions acting on three-dimensional invariant manifolds V_3 (the group criterion of homogeneity[1,2]). In accordance with the physical meaning of the cosmological model, it is usually assumed here that the homogeneous spaces V_3 with length element dl^2 are spacelike manifolds orthogonal to the lines of the time τ. The metric in the synchronous system is then written in the form (c is the speed of light, and the notation of [2] is used)

$$ds^2=(cd\tau)^2-dl^2. \tag{1.1}$$

In this article we consider an anisotropic model with homogeneous space V_3 of constant negative curvature, which admits of a group of motion of Bianchi type V.[1,2] Axial symmetry with respect to x^1 is assumed, and in (1.1) we have [1-3]

$$dl^2=R_1^2(\tau)(dx^1)^2-e^{-2x^1}R_2^2(\tau)[(dx^2)^2+(dx^3)^2]. \tag{1.2}$$

The source of the gravitational field is the hydrodynamic energy-momentum tensor of an ideal liquid

$$T_{ik}=(e+p)u_iu_k-pg_{ik},\quad u^iu_i=1. \tag{1.3}$$

It turns out to be of importance that the 4-velocity u^i in (1.3) is not orthogonal to V_3 in the model under consideration; the components u^4 and u^1 are different from zero in the system (1.1), (1.2) and the synchronous system (1.1), (1.2) in which the matter moves does not coincide with the comoving system. It is important to note for comparison that in Bianchi model I[1,2] the system (1.1) is always comoving and that in general there exists among the homogeneous anisotropic models an extensive class for which the 4-velocity u^i is orthogonal to V_3.[3,4]

We consider the equation of state of the matter in the form (e is the internal energy and p is the pressure).

$$p=(\gamma-1)e \tag{1.4}$$

with a constant γ in the interval $1\le\gamma\le 2$. From the conservation laws $(T_i^k)_{;k}=0$ for (1.3) we obtain after contracting with the 4-velocity

$$(e^{1/\gamma}u^k)_{;k}=0 \tag{1.5a}$$

and when this expression is taken into account we obtain also

$$u^ku_{i;k}=-[(\gamma-1)/\gamma](u_iu^k-\delta_i^k)\partial\ln e/\partial x^k. \tag{1.5b}$$

According to (1.5b), the 4-acceleration $u^ku_{i;k}$ vanishes for dustlike matter ($p = 0$, $\gamma = 1$), but differs from zero at $1 < \gamma \le 2$. This circumstance, together with the fact that the synchronous and comoving systems do not coincide, is one of the decisive features of the considered model. In the kinematic breakdown[5]

$$u_{i;k}=\Theta(g_{ik}-u_iu_k)+q_{ik}+\omega_{ik}+u_ku^lu_{i;l}; \tag{1.6}$$
$$q_{ik}=q_{ki},\quad \omega_{ik}=-\omega_{ki}$$

the nonvanishing quantities in this case are the isotropic dilatation Θ, the shear-tensor components q_1^1 and $q_2^2 = q_3^3 = -q_1^1/2$, and the 4-acceleration (1.5b) (there is no rotation), which are the anisotropy factors.

A comparison of the system (1.1) with the comoving system reveals the incompleteness of the synchronous system, an incompleteness which is completely absent in models in which u^i is orthogonal V_3 and which is a characteristic feature of the investigated model.[6]

The Bianchi type V model was considered by a number of workers. The case of the model of type V with u^i orthogonal to V_3 was considered by Heckmann and Schucking.[5,7] For dustlike matter, the exact solution of the GRT equations for (1.1), (1.2) was obtained by Farnsworth.[8] Questions of the asymptotic behaviors near the singularity during the isotropization process in models of type V and the anisotropy of the relict background were considered in [9-11]. In this paper it is considered the general case of matter with an equation of state (1.4). The presence of a horizon and the incompleteness of the synchronous system are discussed (Sec. 2). A qualitative analysis is presented of the gravitation equations in the comoving system (Sec. 3). A limiting line is shown to exist and to lead to ambiguity in the continuous solutions in the corresponding regions, this being a consequence of the hydrodynamic phenomena in the matter moving in the system (1.1). The dynamics of trial matter with (1.4), moving with the investigated symmetry against the background of a vacuum model of type V and reducing to a Galilean spacetime, is also considered (Sec. 4).

The general relations in anisotropic models with 4-velocity not orthogonal to V_3 were discussed in [12]. Questions connected with the inevitability of a singularity in cosmological solutions and with its possible character are discussed in [2,13-15,6].

2. HOMOGENEITY CONDITION, COMOVING SYSTEM, AND INCOMPLETENESS OF THE SYNCHRONOUS SYSTEM

The group criterion of homogeneity (Sec. 1) leads in the considered case of the Bianchi type V model to a metric in the synchronous sytem in the form (1.1), (1.2), for which the component R_{01} of the Einstein equations with the tensor (1.3) yields

$$cR_{01}=-2(h_1-h_2)=(8\pi k/c^3)(e+p)u_0u_1,\quad h_1=\dot{R}_1/R_1,\quad h_2=\dot{R}_2/R_2 \tag{2.1}$$

(the dot denotes differentiation with respect to the time τ). It follows from (2.1) that $u^1 = 0$ only at $R_1(\tau) = R_2(\tau)$ in (1.2), i.e., for the open Friedmann model. In the anisotropic case with $h_1 \neq h_2$ we have $u^1 \neq 0$, i.e., the matter moves in the system (1.1).

For the 4-velocity components in (1.1) and (1.2) we have

$$\begin{gathered}u^0=u_0=\operatorname{ch}\varphi,\quad u_1=R_1\operatorname{sh}\varphi,\quad \varphi=\varphi(x^0);\\ V=u_1/u_0=R_1\operatorname{th}\varphi;\quad (v/c)^2=\operatorname{th}^2\varphi.\end{gathered} \tag{2.2}$$

From (1.5) with i = 1 we obtain for the metric (1.1), (1.2),

$$R_1\operatorname{sh}\varphi=K\left\{|R_1|R_2^2\operatorname{ch}\varphi\exp\left[2\int(\operatorname{th}\varphi/R_1)\,dx^0\right]\right\}^{\gamma-1},\qquad K=\text{const.} \tag{2.3a}$$

$$e=K_0/(R_1\operatorname{sh}\varphi)^{\gamma/(\gamma-1)},\quad K_0=\text{const.} \tag{2.3b}$$

Let us transform from the synchronous system (1.1), (1.2), to the comoving system y^i with $u^1 = 0$ (since there is no rotation we have $g_{0\alpha} = 0$ in the comoving system). The conditions $u^1 = 0$ and $g_{01} = 0$ are satisfied at $y^{2,3} = x^{2,3}$ and under the transformations with allowance for (2.3a) (excluding the permissible transformations of each of the coordinates)

$$y^0=Kx^1+\int KV^{-1}(x^0)\,dx^0,\quad y^1=x^1+\int R_1^{-2}(x^0)V(x^0)\,dx^0. \tag{2.4}$$

By virtue of (2.4), the coefficients of the metric (1.1), (1.2) depend on the quantity

$$\xi=y^0-Ky^1,\quad x^0=x^0(\xi). \tag{2.5}$$

The arbitrary constant K characterizes the velocity in the synchronous system; at K = 0 the synchronous system is comoving (the open Friedmann model).

The metric in the comoving system, when account is taken of (2.5), takes the form

$$\begin{gathered}ds^2=T^2(\xi)(dy^0)^2-X^2(\xi)(dy^1)^2-e^{-2y^1}Y^2(\xi)[(dy^2)^2+(dy^3)^2],\\ u_i=T\delta_i^0.\end{gathered} \tag{2.6}$$

The isotropic lines with y^2 = const and y^3 = const are given for (2.6) by the equations

$$\begin{gathered}T\,dy^0=\pm X\,dy^1,\quad d\xi=(1\mp\mu^{-1/2})\,dy^0,\\ \mu=X^2/K^2T^2\geqslant 0.\end{gathered} \tag{2.7}$$

According to (2.7), for the values $\xi = \xi^*$ with $\mu = 1$, the direction of the coordinate line $\xi = \xi^*$ coincides with the direction of one of the families of the isotropic lines; these coordinate lines are tangent to the light cones and make up the surface of the horizon.[15]

The transformation formulas relating the metrics (1.1), (1.2), with (2.6) take the form

$$\begin{gathered}R_1^2=X^2-K^2T^2,\quad R_2=Y\exp\int(-VR_1^{-2})\,dx^0,\\ V^2=R_1^2K^2T^2/X^2,\quad (v/c)^2=1/\mu,\\ X=R_1\operatorname{ch}\varphi,\quad d\xi/dx^0=V/KT^2=\pm[(\mu-1)/\mu]^{1/2}/T.\end{gathered} \tag{2.8}$$

These formulas are meaningful only at $\mu \geq 1$. Thus, the initial synchronous system (1.1) and (1.2) covers only the region $\mu \geq 1$ of the metric (2.6) and turns out to be (generally speaking) incomplete. It is essentially this circumstance that follows from geometric considerations and is not connected with the actual solution of the GRT equations.

In the range of values $0 \leq \mu \leq 1$ of the comoving system (2.6) the coordinate x^0 acquires in accordance with (2.8) a spacelike character. The invariant manifolds (homogeneous spaces) V_3 on which the group of motions G_4V acts are no longer spacelike. This circumstance is an essential feature of homogeneous models with invariant manifolds V_3 that are not orthogonal to the 4-velocity. For $0 \leq \mu \leq 1$, in analogy with the synchronous system (1.1) for $\mu \geq 1$, there exists a metric

$$ds^2=R_0^2(x^1)(dx^0)^2-(dx^1)^2-e^{-2x^0}R_2^2(x^1)[(dx^2)^2+(dx^3)^2] \tag{2.9}$$

with a dependence on the spatial variable x^1. The region $\mu \geq 1$ covered by the synchronous system will be called the T-region, and the region $0 \leq \mu \leq 1$ covered by the metric (2.9) will be called the X region.[1)] For the metric (2.9) we have in analogy with (2.2)

$$\begin{gathered}u^0=\operatorname{ch}\lambda/R_0,\quad u_1=\operatorname{sh}\lambda,\quad \lambda=\lambda(x^1),\\ V=\operatorname{th}\lambda/R_0,\quad (v/c)^2=\operatorname{th}^2\lambda,\end{gathered} \tag{2.10}$$

and integration of (1.5) leads to the relations

$$\begin{gathered}R_0\operatorname{ch}\lambda=K_1\{R_0R_2^2\operatorname{sh}\lambda\exp[2\textstyle\int(R_0\operatorname{th}\lambda)^{-1}dx^1]\}^{\gamma-1},\quad K_1=\text{const},\\ e=K_0/(R_0\operatorname{ch}\lambda)^{\gamma/(\gamma-1)},\quad K_0=\text{const.}\end{gathered}$$

The transformation formulas

$$y^0=Kx^0+\int KV(x^1)\,dx^1,\quad y^1=x^0+\int R_0^{-2}(x^1)V^{-1}(x^1)\,dx^1 \tag{2.11}$$

lead to the metric (2.6) with $\xi = y^0 - Ky^1$; we then have

$$\begin{gathered}R_0^2=K^2T^2-X^2,\quad R_2=Y\exp\int\frac{dx^1}{(-R_0^2V)},\\ \left(\frac{v}{c}\right)^2=(VR_0)^2=\mu,\quad X=R_0\operatorname{sh}\lambda,\quad \frac{d\xi}{dx^1}=-\frac{1}{KT^2V}.\end{gathered} \tag{2.12}$$

Returning to the comoving system (2.6) with (2.5) we obtain from (1.5), taking (1.3) and (1.4) into account,

$$e=K_2/(XY^2)^\gamma,\quad K_2=\text{const}, \tag{2.13a}$$

$$T=(XY^2)^{\gamma-1}. \tag{2.13b}$$

3. ANALYSIS OF THE GRAVITATION EQUATIONS

We consider the GRT equations (with allowance for (1.3) and (1.4)

$$R_i^k-(R/2)\delta_i^k=(8\pi k/c^4)T_i^k \tag{3.1}$$

in the comoving system (2.6). The components of (3.1) with i = k = 0 and i = k = 1 have in this case the respective forms[17] (the prime and the dot denote differentiation with respect to y^1 and y^0/c, respectively)

$$\frac{1}{c^2T^2}h_2(2h_1+h_2)+\frac{1}{X^2}(-2\lambda_2'+2\lambda_2\lambda_1-3\lambda_2^2)=\frac{8\pi k}{c^4}e, \qquad (3.2)$$

$$\frac{1}{c^2T^2}(2\dot{h}_2+3h_2^2-2h_2h_0)-\frac{1}{X^2}(\lambda_2^2+2\lambda_0\lambda_2)=\frac{8\pi k}{c^4}(-p). \qquad (3.3)$$

In (3.2) and (3.3), taking (2.5) into account, we have

$$\lambda_1\equiv\frac{\partial\ln X}{\partial y^1}=-\varkappa h_1,\quad h_1\equiv c\frac{\partial\ln X}{\partial y^0},\quad \lambda_2\equiv\frac{\partial\ln(e^{-\nu}Y)}{\partial y^1}=-(1+\varkappa h_2),$$

$$h_2\equiv c\frac{\partial\ln Y}{\partial y^0},\quad c\varkappa\equiv K. \qquad (3.4)$$

The component of (3.1) with i = 0 and k = 1 reduces to the form

$$cR_{01}\equiv2[\lambda_2(h_1-h_2)+\lambda_0h_2-\dot{\lambda}_2]=0,$$

$$\lambda_0\equiv\frac{\partial\ln T}{\partial y^1}=-\varkappa h_0,\qquad h_0\equiv c\frac{\partial\ln T}{\partial y^0}. \qquad (3.5)$$

The analysis of (3.2), (3.3), (3.5), and (2.13) completes the analysis of (3.1).[17,18]

From (3.5) with allowance for (3.4) and (2.13b) we get

$$h_1=\{c\varkappa(dh_2/d\xi)+[1-(2\gamma-3)\varkappa h_2]h_2\}/(1+\gamma\varkappa h_2). \qquad (3.6)$$

Setting up the combination $(\gamma-1)T_0^0+T_1^1$, which vanishes by virtue of (1.3) and (1.4), we obtain, taking (3.2)–(3.4) and (2.7) into account,

$$c\varkappa\frac{d\eta}{d\xi}=\frac{3(2-\gamma)\mu-[(7\gamma-6)+2(5\gamma-4)\eta+(3\gamma-2)\eta^2]}{2[\mu-(\gamma-1)]},$$

$$\eta\equiv1/\varkappa h_2,\quad \mu\equiv(X/c\varkappa T)^2. \qquad (3.7)$$

From the definition of μ and (2.13b) we obtain

$$(d\mu/2\mu d\eta)(cd\eta/d\xi)=h_1-h_0=(2-\gamma)h_1-2(\gamma-1)h_2,$$

from which we get, taking (3.6) and (3.7) into account, the usual equation

$$\begin{aligned}&\eta(\eta+\gamma)\{3(2-\gamma)\mu-(\eta+1)[(3\gamma-2)\eta+(7\gamma-6)]\}d\mu\\&=-2\mu\{[\gamma(3\gamma-2)+2(3\gamma-4)\eta]\mu-[(2-\gamma)(3\gamma-2)\eta^2\\&\quad-2(2\gamma^2-7\gamma+4)\eta+\gamma(3\gamma-2)]\}d\eta.\end{aligned} \qquad (3.8)$$

Finally, (3.2) with allowance for (3.4), (2.5), and (3.7) leads to the expression

$$(8\pi k/c^4)X^2e=3[\mu-(1+\eta)^2]/\eta(\eta+\gamma). \qquad (3.9)$$

Equations (3.8), (3.7), (2.13b), and (3.9) complete in this case the GRT system of equations (3.1).

Eq. (3.8) for $\eta(\mu)$ at $1<\gamma<2$ cannot be integrated in final form. The field of the integral curves (3.8) for dust (γ = 1) and for the ultrarelativistic equation of state e = 3p (γ = 4/3) are shown in Figs. 1 and 2. The picture of the integral curves for the remaining values of γ is shown qualitatively in Fig. 3 of the Appendix.[2)] By definition we have $\mu\geq0$. The variation of ξ along the integral curves is determined by formula (3.7); the arrows in Figs. 1–3 correspond to the direction of increasing $\eta\xi$.[3)] The character of the singular points A, B, C, D, E, F, G, H, E and of the infinitely remote singular points, and the asymptotic behavior of the solutions near these points, are indicated in the Appendix.

According to (3.9), the value e = 0 corresponds to the vacuum solution ("vacuum parabola" in the (η,μ) plane).

$$\mu=(1+\eta)^2. \qquad (3.10)$$

According to (3.9) the region with e > 0 (shown shaded in Figs. 1–3) is bounded by the solution (3.10) and by the integral straight lines $\eta=0$ and $\eta=-\gamma$. On the vacuum parabola (3.10) are located the singular points A and H.

The straight line μ = 1 separates on the integral

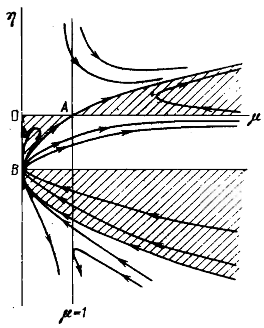

FIG. 1

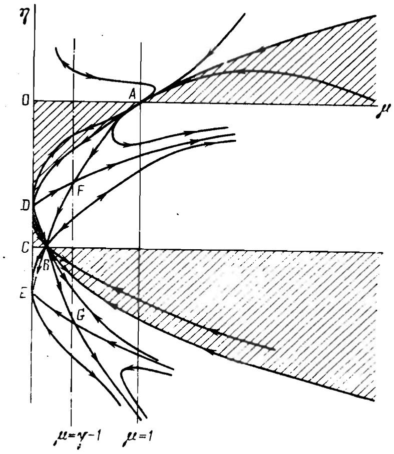

FIG. 2

FIG. 1. Picture of the integral curves of Eq. (3.8) for dustlike matter (γ = 1).

FIG. 2. Picture of the integral curves of Eq. (3.8) for matter with an ultrarelativistic equation of state (γ = 4/3).

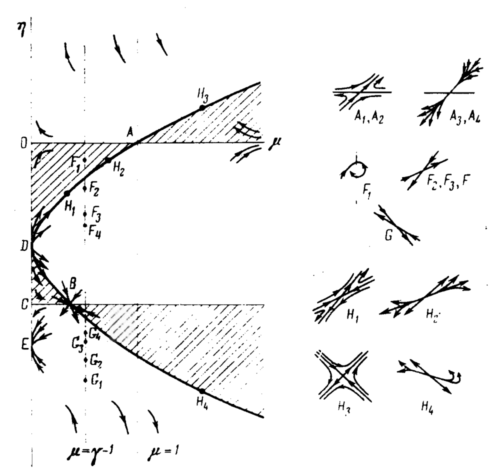

FIG. 3. Diagram of singular points of the Eq. (3.8) at $1<\gamma<2$. The subscript 1 pertains to the interval $1<\gamma<10/9$, subscript 2 pertains to $10/9<\gamma<6/5$, subscript 3 to the interval $6/5<\gamma<4/3$, and 4 to the interval $4/3<\gamma<2$.

curve the T region with $\mu\geq1$ from the X region with $0\leq\mu\leq1$. At the points μ = 1, with the exception of the point A, we have (at $1\leq\gamma<2$) according to (3.7)–(3.9)

$$X,\ Y,\ T\to\text{const},\ \xi\to\text{const},$$

and from the T-region side, according to (2.8), we have[4)]

$$R_1\sim x^0\to0,\quad R_2\sim(x^0)^\alpha,\quad \alpha=-2(\eta^*+\gamma)/[4\gamma+(3\gamma-2)\eta^*],$$
$$|v|\to c,\quad e\to\text{const}\neq0.$$

At the singular point A lying on the intersection of μ = 1 with (3.10) we have $6/5<\gamma<2$

$$X\approx c\varkappa T\sim Y^{2(\gamma-1)/(2-\gamma)}\sim|\xi-\xi_0|^{2(\gamma-1)/(6-5\gamma)},$$
$$e\sim|\xi-\xi_0|^{-2\gamma/(6-5\gamma)}\to0, \qquad (3.11a)$$

and near the point A on the T-region side in the system (1.1), (1.2) we have

$$R_1\sim R_2\sim x^0,\quad \pm v/c\approx\mu^{-1/2}\approx1-2\eta,$$
$$x^0\sim\eta^{(2-\gamma)/2(6-5\gamma)},\quad e\sim(x^0)^{-4\gamma/(2-\gamma)}. \qquad (3.11b)$$

In the interval $6/5<\gamma<2$ we obtain $|x^0|\to\infty$ and the asymptotic form of the flat Friedmann model for the metric, but with $|v| \to c$. The behavior of the solution at $\mu = 1$ at the point A differs essentially in that the parameter x^0 becomes infinite at $6/5 < \gamma < 2$.

Near the singular point H located on (3.10) we have in accord with (3.7)–(3.9)

$$X \approx c\varkappa \mu_H^{1/2} T \sim Y^{2(\gamma-1)/(2-\gamma)}, \quad Y \approx \exp\left[(3\gamma-4)(\xi-\xi_0)/(6-5\gamma)c\varkappa\right]. \tag{3.12a}$$

The point H is a node at $10/9 < \gamma < 6/5$ (in which case H is in the X region) and at $4/3 < \gamma < 2$ (in which case H is in the T region, see Fig. 3).

Equation (3.7) shows that on going through the straight line

$$\mu = \gamma - 1, \tag{3.13}$$

which is located at $1 < \gamma < 2$ in the X region $0 < \mu < 1$, the direction of variation of $\kappa\xi$ is reversed (with the exception of the singular points F at $10/9 < \gamma < 2$ and G), so that passage through (3.13) in the direction of further increase of $\kappa\xi$ is impossible. This gives rise to a mathematical ambiguity (one ξ corresponds to two values of η), due to the fact that the solution has two sheets. The equations of the acoustic characteristics with y^2 = const and y^3 = const take in the comoving system (2.6), with allowance for (2.5), the form ($\omega = c(dp/de)^{1/2}$ is the speed of sound)[20]

$$(\omega/c)\, T dy^0 = \pm X dy^1, \quad K[(\gamma-1)^{1/2} \mp \mu^{1/2}] dy^1 + (\gamma-1)^{1/2} d\xi = 0. \tag{3.14}$$

According to these expressions, at the points $\xi = \xi^*$ with (3.13) the coordinate line $\xi = \xi^*$ has the same direction as the acoustic characteristics; the envelope of these lines in the (y^0, y^1) plane forms a limiting line (see [21]) through which the solution cannot be continued. A similar situation obtains in Newtonian gas dynamics of nonstationary motions; the presence of a limiting line is interpreted there as an indication of the onset of a discontinuity.[20]

The presence of a limiting line raises the question of the possibility of constructing solutions that have the investigated symmetry and are separated by discontinuity surfaces. In the T region with metric (1.1), (1.2), owing to the dependence of the hydrodynamic quantities on τ, introduction of a discontinuity should cause the law governing its propagation to take the form $\tau = \tau^*$ = const, which is physically impossible (in particular, conjugation with the Friedmann solution is impossible on the discontinuity). In the X region with metric (2.9), the structure of the motion admits of introduction of an (immobile) discontinuity surface at $x = x^*$ = const. It is necessary to satisfy on this surface the conservation laws $[T^{ik} n_k] = 0$,[20] which take in this case, in the system (2.9) with allowance for (2.10), the form (the states on opposite sides of the discontinuity are designated by the subscripts 1 and 2)

$$e_1(\gamma \operatorname{ch}^2\lambda_1 - 1) = e_2(\gamma \operatorname{ch}^2\lambda_2 - 1), \quad e_1 \operatorname{sh}\lambda_1 \operatorname{ch}\lambda_1 = e_2 \operatorname{sh}\lambda_2 \operatorname{ch}\lambda_2.$$

Therefore, when (3.9), (2.12), and the continuity of R_0 are taken into account, it follows that the signs of e_1 and e_2 on the discontinuity are equal, and also that

$$\mu_1\mu_2 = (\gamma-1)^2, \quad v_1 v_2 = \omega^2,$$

so that the discontinuity in the (η, μ) plane can go from the side $\gamma - 1 < \mu < 1$ of (3.13) to the other side. In fact, however, by virtue of the structure of the field of the integral curves, according to Figs. 2 and 3, it is inevitable when a discontinuity surface is introduced in the X region that the solution becomes discontinuous in the region $e > 0$, owing to the onset of the ambiguity.

Let us consider the solution for dust ($p = 0$, $\gamma = 1$). For dust, the 4-acceleration is equal to zero, $u_i = \partial\Phi/\partial x^i$, the matter moves along geodesics, the speed of sound is $\omega = 0$, and the line (3.13) is absent. According to (2.13b) we have $T = 1$ in (2.6), so that the comoving system (2.6) is synchronous, but has a dependence on ξ and there is a difference between the T region ($\mu \geq 1$) and the X region ($0 \leq \mu \leq 1$). The solution for dust, obtained by integrating (3.3) in the comoving system (2.6),[8] takes the form[18]

$$\begin{gathered} \xi = F_0(\pm \operatorname{sh}\eta - \eta), \quad Y = F_0(\pm \operatorname{ch}\eta - 1), \\ XY = -Y^2 + KF_0(\mp \operatorname{sh}\eta), \quad e = -3c^4 F_0/4\pi k X Y^2. \end{gathered} \tag{3.15}$$

In terms of the variables (η, μ), the integral (3.8) for dust takes the form

$$\mu^{1/2}\eta[\mu - (1+\eta)^2] = k_0(1+\eta)^3, \quad k_0 = \text{const}.$$

The picture of the integral curves is shown in Fig. 1. The point B in Fig. 1 corresponds to a singularity (in the X region) ($e \to \infty$) with

$$\begin{gathered} Y \sim \exp(-\xi/c\varkappa),\ \xi \to 0;\ X \sim \xi \ \text{ if } \ k_0 \neq -(4/27)^{1/2}, \\ X \sim \xi^2 \ \text{ if } \ k_0 = -(4/27)^{1/2} \ (3\mu \approx (1+\eta)^2[1 \pm 2(-1-\eta)^{1/2}]); \end{gathered}$$

the values $\eta \to 0$ and $\mu \to \infty$ correspond to a singularity with the asymptotic form (A.1) (Kasner singularity of the "filament" type); the values $\mu \alpha \eta^2 + \text{const}\cdot\eta$ correspond as $\eta \to \pm\infty$ to asymptotic isotropization with the asymptotic form (A.2). According to Fig. 1, two types of regimes are possible with a region where the parameter variation is $-\infty < \eta\xi < +\infty$. One corresponds to motion along the integral curve from $\kappa\xi = -\infty$ in the region $e > 0$ with intersection $\mu = 1$ into a singular point B, and then in the region $e < 0$ to $\mu \to \infty$, $\eta \to -0$ and from $\eta \to \infty$, $\eta \to +0$ in the region $e > 0$ to the state $\eta\xi \to \infty$; we have in this case asymptotic isotropization in the region $e > 0$. This regime corresponds to the upper sign in (3.15). The second regime (lower sign in (3.15)) corresponds to motion from $\eta\xi = -\infty$ in the region $e < 0$ to $\eta \to -\infty$, $\mu \to$ const with a possible landing in the region OAB and to motion from $\eta = +\infty$, μ = const to $\eta\xi \to +\infty$ in the region $e < 0$.

Let us consider the solution for an equation of state $e = 3p$ ($\gamma = 4/3$, Fig. 2). Here the 4-acceleration differs from zero; the solution can be continued through the straight line (3.13) along separatrices that pass through the singular points F and G and are located in the region $e < 0$. The continuous solution with $e > 0$ can be constructed in a region bounded by the parabola (3.10) and $\eta = 0$ with $\mu > 1$, located in the T region. In this case ξ varies along the integral curve within finite limits; as $\eta \to 0$ and $\mu \approx \text{const}\cdot\eta^{-2}$ we have a Kasner singularity with asymptotic form (A.1); at the point A we have the asymptotic form (3.11b) as $x^0 \to \infty$ for $\kappa > 0$ and as $x^0 \to -\infty$ for $\kappa < 0$.

Proceeding to the general case with $1 < \gamma < 2$, we note that the character of the solution (see Fig. 3) depends essentially on the equation of state. In the interval $1 < \gamma < 6/5$, the point A is a saddle; the intersection between the integral curves and the line $\mu = 1$ occurs outside the point A at finite ξ, but with further displacement in the region $e > 0$ a crossing of (3.13) takes place. It is possible to construct in the region $e > 0$, $\mu > 1$, $\eta > 0$ a continuous solution from $\mu = \infty$, $\eta = +0$ (Kasner singularity of the "filament" type) to $\mu \approx \eta^2$; $\kappa\xi \to +\infty$ (isotropization to the Friedmann open model). At $6/5 < \gamma < 4/3$ the point A is a node and H is a saddle. Continuous solutions of three types are then possible in the region $e > 0$, $\mu > 1$, $\eta > 0$: from $\mu = \infty$, $\eta = +0$ to $\mu \approx \eta^2$ (analogous to the solution at $1 < \gamma < 6/5$), from $\mu = \infty$, $\eta = +0$ to the point H (3.12a) with $\kappa\xi \to \infty$ and

$$|x^0|\to\infty,\quad R_1\sim x^0,\quad R_2\sim (x^0)^{(2-\gamma)/(4-3\gamma)},\quad |v/c|\to\mu_H^{-1/2}, \tag{3.12b}$$

and from $\mu = \infty$, $\eta = +0$ to the point A with finite ξ and (3.11a); only the first of these regimes undergoes isotropization to the Friedmann model. At $\gamma > 4/3$ it is possible to construct a continuous solution (which does not undergo isotropization) in the region $e > 0$, $\mu > 1$, $\eta > 0$, analogous to the case $\gamma = 4/3$. We note also that at $1 < \gamma < 2$ it is possible to construct continuous solutions with transition from the X region (from the point D) into the T region along separatrices that pass through the singular points F at $10/9 < \gamma < 2$ and G, but these solutions (Fig. 3) are located in the region $e < 0$ (at $e < 0$ for $10/9 < \gamma < 6/5$ it is possible also to construct solutions from the point H in the X region with transition to the T region).

Let us consider also the case of an equation of state $e = p$ ($\gamma = 2$). Then (3.13) coincides with $\mu = 1$. At $\gamma = 2$ we have in accordance with (2.13)

$$Y^4=\mu^{-1}(c\varkappa)^{-2},\quad e=(3a_0c^2/8\pi k\varkappa^2)T^{-2},\quad a_0=\text{const}. \tag{3.16}$$

Taking (3.16) into account, expression (3.9) is an integral of (3.8). We confine ourselves to solutions with $e > 0$ ($a_0 > 0$), putting in (3.16) $a_0 = (b^2 - 1)^{-1}$ with $b = \text{const} > 1$. The integral of (3.8) is then written in the form

$$\mu=(b^2-1)(1+\eta)^2[b^2-(1+\eta)^2]^{-1},$$

and integration of (3.7) yields an expression for $\xi(\eta)$ in the form

$$(1+\eta)^2(b-1-\eta)^{b-1}(b+1+\eta)^{-b-1}=\exp[4(\xi-\xi_0)/c\varkappa].$$

Integration of (3.6) determines $X(\eta)$. In Fig. 3, E coincides with C and B coincides with G and H at $\gamma = 2$. There are two types of solutions in the T region: with $b - 1 \geq \eta \geq 0$ and with $-2 \leq \eta \leq -b - 1$ (and two types of motion in the X region: with $0 \leq \eta \leq -1$ and with $-1 \leq \eta \leq -2$). For $b - 1 \geq \eta \geq 0$ we then have at $\eta = b - 1$ the singularity ($\kappa\xi \to -\infty$)

$$\mu \sim \exp\left[-\frac{4\xi}{(b-1)c\varkappa}\right],\quad X \sim \exp\left[\frac{(b-2)\xi}{(b^2-1)c\varkappa}\right],$$

$$\pm x^0 \sim \exp\left[\frac{3b\xi}{(b^2-1)c\varkappa}\right],\quad R_1\sim (x^0)^{(b-2)/3b},\quad R_2\sim (x^0)^{(b+1)/3b}, \tag{3.17}$$

and at $\eta = 0$ and $\mu = 1$ we have ($\xi \to \xi_1$, $x^0 \to \text{const}$, $d\xi/d\eta = 0$):

$$\mu-1\sim(\xi_1-\xi)^{1/2}\sim x^0,\quad X=c\varkappa T\sim(\xi_1-\xi)^{-1/4},$$
$$R_1\approx\text{const},\quad R_2\sim Y\approx Y_0,\quad |v|\to c,\quad e\to 0.$$

For $-2 \leq \eta \leq -b - 1$ we have at $\eta = -b - 1$ the singularity ($\kappa\xi \to +\infty$), the asymptotic form near which is given by (3.17) with the substitution $b \to -b$ (< -1), and at $\eta = -2$ and $\mu = 1$ we have

$$\mu-1\sim\xi-\xi_1,\quad X\sim(\xi-\xi_1)^{-(2b^2+1)/2(b^2-1)},\ \pm x^0\sim(\xi-\xi_1)^{-(b^2+2)/2(b^2-1)},$$
$$R_1\sim x^0,\ R_2\sim (x^0)^{(b^2-1)/(b^2+2)}),\ |v|\to c.$$

The character of the isotropy of the relict radiation in a model of type V was discussed by Grishchuk, Doroshkevich, and Novikov[9] and also by Ellis and King.[6]

4. DYNAMICS IN GALILEAN SPACE-TIME

It is of interest to consider, against the background of the vacuum solution (3.10) corresponding to $e = 0$ in (3.1), the dynamics of (trial) matter in accordance with the considered group of motions.

In the region $\mu \geq 1$ for (1.1), (1.2) at $e = 0$ we obtain from (2.1) $h_1 = h_2$, which corresponds to the vacuum case for the open Friedmann model;[5] according to (3.1) we have then $R_1 = R_2 = \pm c\tau$. The transformation

$$e^{-x^1}=\operatorname{ch}\chi+\operatorname{sh}\chi\cos\theta,\quad x^2=e^{x^1}\operatorname{sh}\chi\sin\theta\cos\varphi,$$
$$x^3=e^{x^1}\operatorname{sh}\chi\sin\theta\sin\varphi \tag{4.1}$$

changes (1.2) into

$$dl^2=(c\tau)^2[d\chi^2+\operatorname{sh}^2\chi(d\theta^2+\sin^2\theta\, d\varphi^2)],$$

and the transformation[2]

$$t=\tau\operatorname{ch}\chi,\ r=c\tau\operatorname{sh}\chi,\ z^1=r\cos\theta,$$
$$z^2=r\sin\theta\cos\varphi,\quad z^3=r\sin\theta\sin\varphi \tag{4.2}$$

reduces (1.1), (1.2) to the Minkowski metric

$$ds^2=(cdt)^2-[(dz^1)^2+(dz^2)^2+(dz^3)^2]. \tag{4.3}$$

According to (4.2) we have

$$(c\tau)^2=(ct)^2-(z^1)^2-(z^2)^2-(z^3)^2\geqslant 0, \tag{4.4}$$

so that (4.1) and (4.2) transform the space (1.1), (1.2) into the cavity (4.4) of the light cone of the plane (ct, z^α). Analogously, in the region $0 \leq \mu \leq 1$ we obtain for the metric (2.9) at $e = 0$ (from (3.1) $R_0 = R_2 = \pm x^1$, while the transformation

$$e^{-x^0}=\operatorname{ch}\chi+\operatorname{sh}\chi\operatorname{ch}\vartheta,\quad ct=x^1\operatorname{sh}\chi\operatorname{ch}\vartheta,\quad z^1=x^1\operatorname{ch}\chi,$$
$$z^2=x^1\operatorname{sh}\chi\operatorname{sh}\vartheta\cos\varphi=x^1x^2e^{-x^0},\quad z^3=x^1\operatorname{sh}\chi\operatorname{sh}\vartheta\sin\varphi=x^1x^3e^{-x^0} \tag{4.5}$$

converts (2.9) into the metric (4.3), where in accord with (4.5)

$$-(x^1)^2=(ct)^2-(z^1)^2-(z^2)^2-(z^3)^2\leqslant 0, \tag{4.6}$$

so that the space (2.9) is transformed into the cavity (4.6) of the light cone of the plane (ct, z^α).

For trial matter with an equation of state (1.4), moving against the background of the metric with $e = 0$ in (3.1), the equations (2.3)–(2.5) and (2.8) are valid and lead to the metric (2.6) with (3.10), (2.13), while $\eta(\xi)$ and $Y(\xi)$ are given by (3.7) and (3.10). At $\gamma \neq 1, 4/3, 2$ we have the solution $\eta = \eta_H$ and $\mu = \mu_H$ (Fig. 3).

Consider dustlike trial matter with $\gamma = 1$. According to (2.13), (3.10), and (3.17) we have

$$\eta+1=\xi/c\varkappa,\ T=1,\ X^2=(c\varkappa)^2\mu=\xi^2,\ e=\text{const}/(\eta+1)\eta^2.$$

According to (2.8), (2.3a), and (2.12) at $\mu \geq 1$ we have $x^0 = |c\kappa|[\eta(\eta + 2)]^{1/2}$ for $\xi = c\kappa(\eta + 1) > 0$, $x^0 = -|c\kappa|[\eta(\eta + 2)]^{1/2}$ for $\xi < 0$, and at $0 \leq \mu \leq 1$ we have $(x^1)^2 = -(c\kappa)^2\eta(\eta + 2)$, $(x^1)^2 \leq (c\kappa)^2$.

Let us examine the dynamics in the system (1.1), (1.2) from $x^0 = -\infty$ with increasing x^0. If $\kappa < 0$, i.e., the matter is gathered together by cosmological contraction, then as $x^0 = -\infty$ we have $\kappa\xi = +\infty$, $\eta = +\infty$, and at the instant $x^0 = 0$ we arrive at a singularity with $e = \infty$ (point A of Fig. 1), which corresponds in the system (4.3) to the light cone $\tau = 0$ (4.4). On the other hand if $\kappa > 0$, i.e., the matter moves apart in the course of contraction, then as $x^0 = -\infty$ we have $\kappa\xi = -\infty$, $\eta = -\infty$, and at $x^0 = 0$ we arrive at $\eta = -2$ and $\mu = 1$ with $e = \text{const}$ and with a possible continuation of the solution into the X region up to the singularity $\mu = 0$, $\eta = -1$, which corresponds in (4.3) to the hyperbola

$$(z^1)^2+(z^2)^2+(z^3)^2-(ct)^2=(c\varkappa)^2.$$

The Milne model corresponds to $\kappa = 0$.

We consider also the dynamics of matter with e = 3p and $\gamma = 4/3$. Continuous motion, according to Fig. 2, is possible in the region $\mu \geq 1$, $\eta \geq 0$; in this case

$$(c\tau)^2=\mathrm{const}\cdot\exp(-3\eta)\,(\eta+2)/\eta, \quad e=\mathrm{const}\cdot\eta^4\exp(6\eta).$$

A singularity with e = ∞ is reached at $\eta = +\infty$, corresponding in (4.3) to the light cone $\tau = 0$ (4.4), the inside cavity of which is occupied by the moving matter, and on the cone itself we have in the system (4.3)

$$(v^\alpha/c)'=z^\alpha/ct, \quad |v'|=c.$$

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APPENDIX

ASYMPTOTIC BEHAVIOR AT THE SINGULAR POINTS OF EQ. (3.8) (FIG. 3)

Taking (2.13b) and (3.7) into account we have

$$X^{2(2-\gamma)}\sim\mu Y^{4(\gamma-1)}, \quad T\sim\mu^{-1/2}X, \quad 1/\eta=c\varkappa d\ln Y/d\xi.$$

Point A – formulas (3.11a), (3.11b).

Point B (complex node):

$$\eta_B=-\gamma, \quad \mu_B=(\gamma-1)^2; \quad \eta-\eta_B\approx\mathrm{const}\cdot(\mu-\mu_B), \quad c\varkappa(\eta-\eta_B)\approx \approx-3(\gamma-1)(\xi-\xi^*).$$

Point C (saddle): $\eta_C = -\gamma$, $\mu_C = 0$.

Point D (node):

$$\eta_D=-1, \quad \mu_D=0: \quad \mu=\mathrm{const}\cdot(\eta+1)^2 \text{ с } \eta+1\sim\exp(2\xi/c\varkappa)$$

$$\text{и } \mu\approx-4(\gamma-1)(\eta+1)/3(2-\gamma) \text{ с } \eta+1\sim\exp(4\xi/c\varkappa), \quad \varkappa\xi\to-\infty;$$

$$R_0\sim T\approx\mathrm{const}, \quad R_2\sim Y\sim(x^1)^{-1/2}\to\infty.$$

Point E (node):

$$\eta_E=(6-7\gamma)/(3\gamma-2), \quad \mu_E=0; \quad \mu\approx\mathrm{const}\cdot(\eta-\eta_E)^{(6\gamma-4)/(7\gamma-6)}$$
$$\text{and } \mu\approx4(\gamma-1)(\eta-\eta_E)/3(7\gamma-6), \quad \varkappa\xi\to+\infty.$$

Points F (focus at $1 < \gamma < 10/9$, saddle at $10/9 < \gamma < 2$) and G (saddle at $1 < \gamma < 2$):

$$(3\gamma-2)\eta^2_{F,G}+2(5\gamma-4)\eta_{F,G}+\gamma(3\gamma-2)=0, \quad \eta_F>-\gamma,$$
$$\eta_G<-\gamma, \quad \mu_F=\mu_G=\gamma-1.$$

Point H (formulas (3.12a)):

$$\eta_H=(6-5\gamma)/(3\gamma-4), \quad \mu_H=(\eta_H+1)^2,$$

the directions of $d\eta/d\mu$ of the entrance k_1 (vacuum parabola) and k_2:

$$k_1=(4-3\gamma)/4(\gamma-1), \quad k_2=3(6-5\gamma)(2-\gamma)/4(\gamma-1)(3\gamma-2);$$

in the case of a node ($10/9 < \gamma < 6/5$ and $4/3 < \gamma < 2$)

$$\eta-\eta_H\approx k_2(\mu-\mu_H)+\mathrm{const}\cdot\mu^\alpha, \quad \alpha=(6-5\gamma)(4-3\gamma)/(2-\gamma)(9\gamma-10),$$
$$\eta-\eta_H\sim\exp[2(4-3\gamma)\xi/(6-5\gamma)c\varkappa];$$

at $1 < \gamma < 6/5$ and $4/3 < \gamma < 2$ we have $\kappa\xi \to -\infty$, at $6/5 < \gamma < 4/3$ we have $\kappa\xi \to \infty$, and at $6/5 < \gamma < 2$ in (1.1) we have (3.12b) along k_2.

Infinitely remote points:

1) $\eta = 0$, $1/\mu = 0$ (node):

$$\mu\approx\mathrm{const}\cdot\eta^{2(2-3\gamma)/3(2-\gamma)}, \quad \eta\approx3(2-\gamma)(\xi-\xi_0)/2c\varkappa,$$
$$R_1\sim R_2^{-1/2}\sim(x^0)^{-1/3}, \quad v/c\sim(x^0)^{(3\gamma-2)/3}, \quad e\sim(x^0)^{-\gamma},$$
$$\xi-\xi_0\sim(x^0)^{2-\gamma}\to0 \quad (\gamma\neq2); \tag{A.1}$$

2) $\eta = -\gamma$, $1/\mu = 0$ (saddle);

3) $1/\eta = 0$, $1/\mu = 0$:

$$\mu\approx(1+\eta)^2+\mathrm{const}\cdot(1+\eta)^{(10-9\gamma)/(4-3\gamma)} \text{ at } 1\leq\gamma<10/9,$$
$$\mu\approx(1+\eta)^2+\mathrm{const} \text{ at } 10/9\leq\gamma<4/3$$

correspond to asymptotic isotropization with

$$c\varkappa\eta\approx(4-3\gamma)\xi, \quad \varkappa\xi\to\pm\infty, \quad \xi\sim(x^0)^{4-3\gamma},$$
$$R_1\sim R_2\sim x^0, \quad v/c\sim(x^0)^{3\gamma-4}, \quad |x^0|\to\infty; \tag{A.2}$$

at $\gamma = 4/3$ we have

$$\mu\approx(1+\eta)^2+\mathrm{const}\cdot\eta\exp(3\eta), \quad \eta\to-\infty,$$
$$\xi\approx(-3c\varkappa/16)\eta^2, \quad \varkappa\xi\to-\infty;$$

at $4/3 < \gamma < 2$ there enter in the singular point (3.10) and the solution $1/\mu = 0$.

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1)The considered situation is analogous to the presence of a horizon for the Schwarzschild metric and for the vacuum Taub-NUT (Newman-Unti-Tamburino) metric [16].

2)The methods of the qualitative theory of differential equations as applied to cosmological models were developed in general form by S. P. Novikov and Bogoyavlenskiĭ [19].

3)The sign of $\kappa = K/c$ characterizes, in accordance with (2.2), (2.3a), and (1.2), the direction of the velocity $v^1 = cu^1/u^0 = c\tanh\varphi/R_1$. The behavior of the solution as $\kappa\xi \to -\infty$ and $\kappa\xi \to +\infty$ is different in accordance with Figs. 1–3. According to (2.8) and (2.3a) we have $d\xi/dx^0 > 0$.

4)The vanishing of the determinant $(-g)$ at these points in the system (1.1) (or its becoming infinite at $e < 0$) at finite x^0 and e is due to the incompleteness of (1.1).

5)It is important to note the presence of an isotropic vacuum solution in the type V model, in contrast to the vacuum solutions in the models of type I and IX.

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