Theory of a two-dimensional Wigner crystal of surface electrons in helium

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(Submitted October 17, 1974)
Zh. Eksp. Teor. Fiz. 68, 1423–1433 (April 1975)

The effect of a static deformation of the helium surface on the localization energy of surface electrons forming a two-dimensional Wigner crystal is investigated. It is shown that the action of an alternating electric field, normal to the helium surface, on the periodic electron lattice leads to the excitation of standing capillary waves. For certain frequencies, uniquely related to the lattice constant, this excitation has a resonance character. The shape of the resonance lines is calculated in different temperature intervals. It is noted that mechanical action on the helium surface can lead to resonant excitation of standing waves of surface second sound.

PACS numbers: 73.20.C

INTRODUCTION

One of the interesting phenomena that may occur when free electrons are localized on the surface of helium is the formation by the electrons of a periodic structure, known, in problems with Coulomb interaction, as a Wigner crystal. The situation with surface electrons is not entirely standard, since we are concerned with a system of particles of like charge, whereas, in its usual formulation, the problem of the formation of a Wigner crystal requires the presence of a compensating background. The way out of this situation was found by Crandall and Williams[1], who suggested using metallic plates positioned close to the charged surface of the helium and parallel to it, in order to neutralize the gas of surface electrons, i.e., to obtain as a result a uniform electron density over the helium surface. The uniformity of the mean electron density over the helium surface in this case is a consequence of the uniformity of the fields and surface charges in systems of the plane-parallel capacitor type. The corresponding experiments[1] confirm the possibility of producing a uniformly charged helium surface, and this is a reliable base for further theoretical work on the problem of the crystallization of the surface electrons.

By now, a number of concrete results have been obtained in the theory of two-dimensional electron crystals on a helium surface: the "melting" temperature has been estimated and under certain conditions, by the estimates of Crandall and Williams[2], reaches values $T_m \sim 1$ K; the collective modes of the Wigner crystal and the magnitude of the mean-square displacements have been calculated (Crandall[3]), and different aspects of the problem of the stability of a charged surface have been investigated (Gor'kov and Chernikova[4]).

It must be noted that the possibility of a deformation of the helium surface in the vicinity of a localized electron has not been taken into account in any of the papers mentioned. Moreover, such a deformation should occur, and in sufficiently strong clamping fields $E_\perp$ and for not very high concentrations of surface electrons, its presence substantially affects the localization energy of the electrons in the lattice. In the limit in which the electron concentration tends to zero the deformation mechanism of electron localization, as shown in a paper by the authors[5], becomes the main mechanism. In this case, the total energy W associated with the formation of the complex (electron plus deformation of the helium surface) and the electron localization length L are determined by the following relations:

$$W = -\frac{(eE_\perp)^2}{4\pi\alpha}\left(\ln\frac{\sqrt{2}\,\gamma^{-1/2}}{\varkappa L} - 1\right),$$
$$L^2 = 2\pi\alpha\hbar^2/m(eE_\perp)^2,\quad \varkappa^2 = \rho g/\alpha,\quad \gamma \approx 1.78, \tag{1}$$

where κ is the capillary constant of helium, ρ and α are the density and surface tension of helium, and g is the acceleration due to gravity. According to (1), the formation of bound complexes is energetically favorable under the conditions $\kappa L \ll 1$. The capillary length κ^{-1} appears in (1) because of the fact that, in the one-particle problem, the perturbation of the shape of the helium surface induced by the pressure of a localized electron on the surface vanishes only at distances greater than the capillary length κ^{-1}.

We shall assume now that the density n_s of surface electrons is finite, and $r_s^2 = n_s^{-1} \ll \kappa^{-2}$. In these conditions it is natural to expect that the role of the cutoff factor at large distances in the problem of the deformation of the helium surface by a localized electron will be played by the length r_s. The gain in energy as a result of localization of an electron in the region of the self-consistent deformation of the helium surface should then be of the order of

$$W \approx \frac{(eE_\perp)^2}{4\pi\alpha}\ln\frac{r_s}{L}. \tag{1a}$$

It is worth stressing that the condition

$$r_s/L \gg 1, \tag{2}$$

ensuring the efficacy of the deformation mechanism of localization of the surface electrons, coincides with one of the conditions necessary for the existence of a Wigner crystal in general. For this reason, allowance for the deformation of the helium surface is evidently an integral part of the theory of a Wigner crystal of surface electrons in helium.

In the present work calculations are performed of the deformational localization of surface electrons forming a two-dimensional periodic lattice, in an external electric field $E_\perp$. The basic purpose of these calculations is to generalize the results of[5] to the case of a finite density of surface electrons. In addition, the question of the resonant excitation of capillary waves in liquid helium by the periodic lattice of surface electrons is investigated in detail in the paper. The discovery of such resonances, the existence of which was predicted by one

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of the authors[6] and can occur only in the presence of well-defined periodicity in the distribution of the electrons over the surface, will serve as a direct proof of the onset of crystallization of the surface-electron gas.

SELF-CONSISTENT CALCULATION OF THE LOCALIZATION ENERGY OF SURFACE ELECTRONS IN A PERIODIC LATTICE

The surface electrons, being in a strong clamping electric field $E_\perp$, exert a pressure $p_{el}(\mathbf{r}) = eE_\perp n(\mathbf{r})$ on the helium surface. Here, e is the electron charge and $n(\mathbf{r})$ is the surface density of the electrons, which is a function of the two-dimensional position vector $\mathbf{r}$. Therefore, the deviation $\xi(\mathbf{r})$ of the shape of the surface from that of a plane will be determined by an equilibrium equation of the following form:

$$\alpha\Delta\xi = eE_\perp[n(\mathbf{r}) - n_s], \qquad (3)$$

where n_s is the mean surface density of the electrons.

Let the surface electrons form a two-dimensional periodic lattice. In this case, their density $n(\mathbf{r})$ can be expanded in a series in the reciprocal-lattice vectors:

$$n(\mathbf{r}) = \sum_{\mathbf{q}} n_{\mathbf{q}} e^{i\mathbf{q}\mathbf{r}}, \quad n_{\mathbf{q}} = \frac{1}{S}\int n(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}} d^2\mathbf{r}. \qquad (4)$$

Here $\mathbf{q}$ are the vectors of the two-dimensional reciprocal lattice and S is the area of the helium surface occupied by the electrons. When (4) is taken into account, the solution of Eq. (3) is written as:

$$\xi(\mathbf{r}) = -\frac{eE_\perp}{\alpha}\sum_{\mathbf{q}\neq 0}\frac{n_{\mathbf{q}}}{q^2}e^{i\mathbf{q}\mathbf{r}}. \qquad (3a)$$

If the electron density is chosen in the form $n(\mathbf{r}) = \Sigma_{\mathbf{l}}\delta(\mathbf{r} - \mathbf{l})$, where $\mathbf{l}$ is the position vector of a lattice site, then $n_{\mathbf{q}} = S_0^{-1}$ (S_0 is the area of a unit cell) and, consequently, the series (3a) will diverge logarithmically at $\mathbf{r} = \mathbf{l}$. This divergence is easily removed if we take into account that an electron at a lattice site is slightly smeared-out. Regarding an individual electron at a site as a two-dimensional oscillator, we can write the density of electrons in the crystal in the following form:

$$n(\mathbf{r}) = \frac{1}{\pi L^2}\sum_{\mathbf{l}}\exp\left(-\frac{|\mathbf{l}-\mathbf{r}|^2}{L^2}\right), \qquad (5)$$

where L is the characteristic electron localization length, which will be determined self-consistently below. It is legitimate to use the approximation (5) so long as the condition $L \ll a$ is fulfilled (a is the lattice constant).

On the basis of (5) we find an expression for $n_{\mathbf{q}}$:

$$n_{\mathbf{q}} = S_0^{-1}\exp(-q^2L^2/4), \qquad (5a)$$

which must be substituted into (3a). The sums over $\mathbf{q}$ appearing in (3a) will be calculated for a square lattice, since in this case the calculations are considerably simplified. It is clear, however, that the specific choice of the type of lattice cannot have a strong effect on the character of the solution of the electron part of the problem near the sites.

Proceeding to the concrete calculations, we shall find the displacement of the helium surface near a certain site $\mathbf{l}$. To this end, putting $\mathbf{r} = \mathbf{l} + \mathbf{u}$ and $u \ll a$, we expand the exponential in the right-hand side of the expression (3a) in a series in u to order u^2. As a result, with $n_{\mathbf{q}}$ from (5a) taken into account, the solution $\xi(\mathbf{r})$ takes the form

$$\xi(\mathbf{l}+\mathbf{u}) \approx \xi(\mathbf{l}) + \frac{eE_\perp}{\alpha S_0}u^2\left[\left(\sum_{n=1}^{\infty}\exp\{-\delta^2n^2\}\right)^2 + \sum_{n=1}^{\infty}\exp\{-\delta^2n^2\}\right],$$

$$\xi(\mathbf{l}) = -\frac{eE_\perp}{\alpha(2\pi)^2S_0}\sum_{\substack{nm \\ n^2+m^2\neq 0}}\frac{1}{m^2+n^2}\exp\{-\delta^2(m^2+n^2)\}, \qquad (6)$$

where $\delta = \pi L/a \ll 1$, and n and m run over integer values from $-\infty$ to ∞. Calculating the resulting sums over n and m by means of Poisson's formula, we can bring the expression (6) to the following form:

$$\xi(\mathbf{l}+\mathbf{u}) \approx \xi(\mathbf{l}) + eE_\perp u^2/4\pi\alpha L^2,$$

$$\xi(\mathbf{l}) \approx -\frac{eE_\perp}{2\pi\alpha}\left\{\int_\delta^\infty[1-\Phi(x)]\frac{dx}{x} + \frac{2\pi+7}{12}\right\} \qquad (6a)$$

where $\Phi(x)$ is the error function. Taking into account that as $x \to 0$ the function $\Phi(x) \to 0$, we have, with logarithmic accuracy,

$$\int_\delta^\infty[1-\Phi(x)]\frac{dx}{x} \approx \ln\frac{1}{\delta}$$

for $\delta \ll 1$. Therefore, the maximum depression of the helium surface for $a \gg L$ is approximately equal to

$$\xi(\mathbf{l}) \approx -\frac{eE_\perp}{2\pi\alpha}\ln\frac{a}{L}. \qquad (6b)$$

The potential energy of an electron above the deformed surface is determined by the equality[5]

$$U_\xi(\mathbf{r}) = eE_\perp\xi(\mathbf{r}).$$

Therefore, when we take into account (6a), (6b) and the expression obtained in[2] for the potential energy U_e of one electron in the field of the others, the total potential energy of the electron near the lattice-site $\mathbf{l}$ will look like:

$$U(\mathbf{l}+\mathbf{u}) = U_\xi(\mathbf{l}+\mathbf{u}) + U_e(\mathbf{l}+\mathbf{u}) \approx eE_\perp\xi(\mathbf{l}) + U_e(\mathbf{l}) + u^2\left[\frac{(eE_\perp)^2}{4\pi\alpha L^2} + 2.7\frac{e^2}{a^3}\right]. \qquad (7)$$

with $\xi(\mathbf{l})$ given by (6b).

According to (7), the wave equation for an electron at a given lattice site has, in the region of small quantum numbers, an oscillatory form, and this justifies the choice of the expression (5) for the electron density. The quantity L appearing in the definition (5) of $n(\mathbf{r})$ should be found from the solution of an oscillatory equation with a potential energy (7) that depends, in turn, on L. As a result, there arises a quadratic equation for L^{-2}, of which the solution is

$$L^{-2} = {}^1/_2L_1^{-2} + ({}^1/_4L_1^{-4} + L_2^{-4})^{1/2}. \qquad (8)$$

where

$$L_1^2 = 2\pi\hbar^2\alpha/m(eE_\perp)^2, \quad L_2^4 = \hbar^2a^3/5.4me^2,$$

and m is the free-electron mass. We note that L_1 coincides with the characteristic surface-anion localization length obtained in[5]. The quantity L_2 is the characteristic length of the region in which the wavefunction of an electron at a lattice site is smeared out in the absence of depressions in the helium surface. Estimates show that for $E_\perp \approx 10$ esu and $a \approx 10^{-4}$ cm, the two lengths L_1 and L_2 are of the same order. In stronger fields $E_\perp$ or for large a, the characteristic electron localization length L will coincide with L_1. For smaller $E_\perp$ or a, the quantity $L \approx L_2$.

As noted in the Introduction, the presence of depressions in the surface leads to a certain gain in energy. In fact, the difference between the total energies of a system of localized electrons above a deformed surface and above a planar surface is equal to

$$\Delta W = eE_\perp \int \xi(\mathbf{r}) n(\mathbf{r}) d^2\mathbf{r} + \frac{\alpha}{2}\int (\nabla\xi)^2 d^2\mathbf{r} + N\frac{\hbar^2}{m}(L^{-2} - L_2^{-2}), \qquad (9)$$

where N is the total number of electrons. Substituting n(**r**) from (5) and ξ(**r**) from (6) into (9), we find, after calculations analogous to (6) and (6a),

$$\Delta W = -\frac{(eE_\perp)^2}{4\pi\alpha} N\left\{\int_{\delta'}^{\infty}[1-\Phi(x)]\frac{dx}{x} + \frac{2\pi+7}{12}\right\} + N\frac{\hbar^2}{m}(L^{-2} - L_2^{-2}). \qquad (10)$$

Here $\delta' = \sqrt{2}\pi L/a$ and L is given by (8). In the case $\delta' \ll 1$, the same arguments as in the analysis of (6a) enable us to write, with logarithmic accuracy,

$$\Delta W \approx -N\frac{(eE_\perp)^2}{4\pi\alpha}\ln\frac{a}{L}. \qquad (11)$$

In writing the formula (11) we have neglected the difference $N\hbar^2(L^{-2} - L_2^{-2})/m$, inasmuch as it is at least logarithmically small compared with the term retained.

The result (10), (11) quantitatively characterizes the energy gain ΔW due to the self-consistent surface-deformation accompanying the crystallization of the electrons. The fact that this result is obtained confirms the correctness of the estimate (1a) cited in the Introduction.

To end the discussion of the conditions facilitating localization of the electrons, it must be noted that it is possible to use a magnetic field H normal to the helium surface for this purpose. In conditions when the magnetic length $L_H = (2\hbar c/eH)^{1/2}$ of the electron becomes comparable with L from (8), the magnetic field begins to effectively increase the ratio $a/L \gg 1$, and this favors localization of the electrons. Quantitatively, the presence of a magnetic field only changes the definition of the parameter L:

$$L^{-2} = {}^1/_2 L_1^{-2} + ({}^1/_4 L_1^{-4} + L_2^{-4} + L_H^{-4})^{1/2}.$$

RESONANT EXCITATION OF STANDING SURFACE WAVES BY A PERIODIC SYSTEM OF CHARGES ON A HELIUM SURFACE

It was shown earlier[6] that the existence of a periodic lattice of electrons on a free helium surface can be used for the resonant excitation of capillary waves of comparatively short wavelength. Below we carry out a detailed calculation of the shape of the resonance lines in two limiting cases. In the first of these we shall assume that the temperature is not too low, in order that we can make use of volume two-fluid hydrodynamics. In the second case the temperature is assumed to be so low that the dissipative phenomena describable by the surface hydrodynamics of Andreev and Kompaneets[7] begin to play the main role in the estimate of the width of the resonance lines.

At the end of this section, estimates of the effect of the thermal motion of the localized electrons on the lineshapes are given.

A. Case of Volume Hydrodynamics

The linearized equations of two-fluid volume hydrodynamics with neglect of the compressibility have the following form[8]:

$$\begin{gathered}\Delta\varphi = 0, \quad \operatorname{div}\mathbf{v}_n = 0, \\ \rho_n\frac{\partial\mathbf{v}_n}{\partial t} + \nabla p' = \eta\Delta\mathbf{v}_n, \quad p' = p + \rho_s\,\partial\varphi/\partial t,\end{gathered} \qquad (12)$$

where φ is the hydrodynamic potential of the superfluid flow of the liquid; $\mathbf{v}_n$ is the velocity of the normal flow; ρ_s and ρ_n are the superfluid and normal helium densities; p is the pressure of the liquid; η is the helium viscosity coefficient.

Let an alternating electric field $\tilde{E}_\perp e^{-i\omega t}$, normal to the helium surface, be switched on in addition to the constant clamping field $E_\perp$. In the presence of this perturbation the boundary conditions at the liquid surface will have the following form:

$$\begin{gathered}v_{sz} = v_{nz} = \partial\xi/\partial t, \\ \sigma_{zy} \equiv \eta\left(\frac{\partial v_{nz}}{\partial y} + \frac{\partial v_{ny}}{\partial z}\right) = 0, \quad \sigma_{zx} \equiv \eta\left(\frac{\partial v_{nz}}{\partial x} + \frac{\partial v_{nx}}{\partial z}\right) = 0, \\ \sigma_{zz} \equiv -p + 2\eta\frac{\partial v_{nz}}{\partial z} = \alpha\Delta\xi + e\tilde{E}_\perp e^{-i\omega t}(n - n_s),\end{gathered} \qquad (13)$$

where σ_{ik} are the components of the viscous-stress tensor, ξ is the deviation of the shape of the surface from the equilibrium shape, and the z axis is directed perpendicular to the liquid boundary, such that $z \le 0$ corresponds to the liquid phase and **r** = (x, y).

Taking into account the explicit form of n(**r**) from (4), we shall seek the solution of the problem under consideration in the form

$$\begin{gathered}\xi(\mathbf{r}) = \sum_{\mathbf{q}} \xi_{\mathbf{q}} e^{i(\mathbf{qr} - \omega t)}, \quad \varphi(\mathbf{r}, z) = \sum_{\mathbf{q}} \varphi_{\mathbf{q}} e^{qz + i(\mathbf{qr} - \omega t)}, \\ p'(\mathbf{r}, z) = \sum_{\mathbf{q}} p_{\mathbf{q}}' e^{qz + i(\mathbf{qr} - \omega t)}, \\ \mathbf{v}_n(\mathbf{r}, z) = \sum_{\mathbf{q}} [i\mathbf{q} f_{\mathbf{q}}(z) + \mathbf{k} w_{\mathbf{q}}(z)] e^{i(\mathbf{qr} - \omega t)},\end{gathered} \qquad (14)$$

where **k** is the unit vector along the z axis. Substituting formulas (14) into Eq. (12), solving the resulting system of ordinary differential equations and satisfying the boundary conditions (13) gives the following results:

$$\begin{gathered}\varphi_{\mathbf{q}} = -\frac{e\tilde{E}_\perp}{\rho}\frac{i\omega n_{\mathbf{q}}}{\Delta_{\mathbf{q}}}, \quad \xi_{\mathbf{q}} = -\frac{q}{i\omega}\varphi_{\mathbf{q}}, \\ p_{\mathbf{q}}' = -\eta(q^2 + \lambda_{\mathbf{q}}^2)\varphi_{\mathbf{q}}, \quad \lambda_{\mathbf{q}} = \lambda_{\mathbf{q}}^{(0)} - i\beta_{\mathbf{q}}, \\ \lambda_{\mathbf{q}}^{(0)} = \left\{\frac{q^2}{2} + \left[\frac{q^4}{4} + \left(\frac{\omega\rho_n}{2\eta}\right)^2\right]^{1/2}\right\}^{1/2}, \quad \beta_{\mathbf{q}} = \frac{\omega\rho_n}{2\lambda_{\mathbf{q}}^{(0)}\eta} \\ \Delta_{\mathbf{q}} = \omega^2 - \omega_{\mathbf{q}}^2 + 4\omega\frac{\eta}{\rho}q^2\frac{q\beta_{\mathbf{q}}}{(\lambda_{\mathbf{q}}^{(0)} + q)^2 + \beta_{\mathbf{q}}^2} + 2i\omega\gamma_{\mathbf{q}}, \\ \omega_{\mathbf{q}}^2 = \frac{\alpha}{\rho}q^3, \quad \gamma_{\mathbf{q}} = 2\frac{\eta}{\rho}q^2\frac{\lambda_{\mathbf{q}}^{(0)}(\lambda_{\mathbf{q}} + q) + \beta_{\mathbf{q}}^2}{(\lambda_{\mathbf{q}}^{(0)} + q)^2 + \beta_{\mathbf{q}}^2}, \\ w_{\mathbf{q}}(z) = \frac{qp_{\mathbf{q}}'}{2\eta\lambda_{\mathbf{q}}}\left(\frac{e^{qz} - e^{\lambda_{\mathbf{q}}z}}{q - \lambda_{\mathbf{q}}} - \frac{e^{qz}}{q + \lambda_{\mathbf{q}}} + c_{\mathbf{q}}e^{\lambda_{\mathbf{q}}z}\right), \\ f_{\mathbf{q}}(z) = \frac{p_{\mathbf{q}}'}{2\eta q\lambda_{\mathbf{q}}}\left(\frac{qe^{qz} - \lambda_{\mathbf{q}}e^{\lambda_{\mathbf{q}}z}}{q - \lambda_{\mathbf{q}}} - \frac{qe^{qz}}{q + \lambda_{\mathbf{q}}} + c_{\mathbf{q}}\lambda_{\mathbf{q}}e^{\lambda_{\mathbf{q}}z}\right), \\ c_{\mathbf{q}} = \frac{2q^2}{(q + \lambda_{\mathbf{q}})(q^2 + \lambda_{\mathbf{q}}^2)} - \frac{q + \lambda_{\mathbf{q}}}{q^2 + \lambda_{\mathbf{q}}^2} \\ \rho = \rho_s + \rho_n, \quad q = |\mathbf{q}|.\end{gathered} \qquad (15)$$

Having the solution (14), (15) at our disposal, we shall compute the amount of energy dissipated per unit time; for this, we make use of the well-known definition[9]:

$$\frac{d\mathscr{E}}{dt} = \eta\left\{\int \boldsymbol{\omega}_n^2\,dV - \int \frac{d\mathbf{v}_n^2}{\partial n}dS + 2\int [\mathbf{v}_n \times \boldsymbol{\omega}_n]\mathbf{n}\,dS\right\}. \qquad (16)$$

Here $\boldsymbol{\omega}_n = \operatorname{curl}\mathbf{v}_n$, and **n** is the unit vector of the normal to the helium surface, directed into the liquid.

Taking formulas (14) and (15) into account, we can obtain for the components of $\boldsymbol{\omega}_n$ the following expressions:

$$\omega_{nx}=\sum_{\mathbf{q}} iq_y\Omega_{\mathbf{q}}(z)e^{i(\mathbf{qr}-\omega t)},\qquad \omega_{ny}=-\sum_{\mathbf{q}} iq_x\Omega_{\mathbf{q}}(z)e^{i(\mathbf{qr}-\omega t)}$$

$$\omega_{nz}=0,\quad \Omega_{\mathbf{q}}(z)=2q\varphi_{\mathbf{q}}\exp(\lambda_{\mathbf{q}}z).$$

Calculating $d\mathscr{E}/dt$ (16) with allowance for the concrete form of the expressions for $\mathbf{v}_n$ and $\boldsymbol{\omega}_n$ obtained above, and averaging the result over the time, we obtain

$$\overline{\frac{d\mathscr{E}}{dt}}=2\eta S\left(\frac{eE_\perp}{\rho}\right)^2\omega^2\sum_{\mathbf{q}} q^3\left|\frac{n_{\mathbf{q}}}{\Delta_{\mathbf{q}}}\right|^2\left[\frac{(\lambda_{\mathbf{q}}^{(0)})^2-q^2+\beta_{\mathbf{q}}^2}{(\lambda_{\mathbf{q}}^{(0)}+q)^2+\beta_{\mathbf{q}}^2}+\frac{q}{2\lambda_{\mathbf{q}}^{(0)}}\right].\qquad(17)$$

It follows from formulas (15) and (17) that $\varphi_{\mathbf{q}}$ and $\overline{d\mathscr{E}/dt}$ contain a resonance denominator $\Delta_{\mathbf{q}}$. For $\omega\to\omega_{\mathbf{q}}$, where $\mathbf{q}$ is one of the reciprocal-lattice vectors, the amplitude of the surface vibrations and the mean energy dissipated per unit time become resonantly large, if $\gamma_{\mathbf{q}}\ll\omega_{\mathbf{q}}$.

Thus, the position of the resonance frequencies is determined by the condition

$$\omega=\omega_{\mathbf{q}}.\qquad(18)$$

For a square lattice this condition can be written out in the form[6]

$$\omega^2=\frac{\alpha}{\rho}\left(\frac{2\pi}{a}\right)^3(n^2+m^2)^{3/2},\qquad(18a)$$

where n and m are integers. If we are dealing with a triangular lattice, then

$$\omega^2=\frac{\alpha}{\rho}\left(\frac{4\pi}{a\sqrt{3}}\right)^3(n^2+m^2+nm)^{3/2}.\qquad(18b)$$

The smearing-out of the resonances is proportional to $\gamma_{\mathbf{q}}$. In the limit $\rho_n>2\eta q^2/\omega$,

$$\gamma_{\mathbf{q}}\approx2\eta q^2/\rho.\qquad(18c)$$

In the opposite limiting case $\rho_n<2\eta q^2/\omega$, we have

$$\gamma_{\mathbf{q}}\approx\eta q^2/\rho.\qquad(18d)$$

Thus, with increasing $|\mathbf{q}|$ the linewidth grows like q^2.

B. Case of Surface Hydrodynamics

According to Andreev and Kompaneets[7], at sufficiently low temperatures the influence of the "volume" normal component on the surface phenomena can be neglected and we can assume that there is only a surface normal density, due to the capillary waves and surface impurity excitations. The boundary conditions for the equation

$$\Delta\varphi=0$$

(φ is the hydrodynamic potential of the superfluid flow) are, in the case under consideration, a system of surface hydrodynamic equations[7]. After linearization, these equations can be written as:

$$\nu_n\frac{\partial}{\partial t}(v_{n\beta}-v_{s\beta})-\frac{\partial\alpha}{\partial x_\beta}-\eta_s\frac{\partial^2v_{n\beta}}{\partial x_\gamma\partial x_\gamma}-\zeta_s\frac{\partial^2v_{n\gamma}}{\partial x_\beta\partial x_\gamma}=0,$$

$$\nu_n\frac{\partial}{\partial x_\beta}(v_{n\beta}-v_{s\beta})=\rho v_{sz}-\rho\frac{\partial\xi}{\partial t},\qquad \alpha\Delta\xi+p=e\tilde{E}_\perp e^{-i\omega t}(n-n_s),\qquad(19)$$

$$\frac{\partial\sigma}{\partial t}+\sigma\frac{\partial v_{n\beta}}{\partial x_\beta}=0,\qquad \frac{\partial\nu}{\partial t}+\nu\frac{\partial v_{n\beta}}{\partial x_\beta}=0,$$

where $\mathbf{v}_n$ is the velocity of the normal surface flow; $\mathbf{v}_s=\nabla\varphi$; ν_n is the normal surface density; η_s and ζ_s are the first and second surface-viscosity coefficients; ν and σ are the impurity mass and entropy per unit area of the surface; the indices β, γ take the values x and y; summation over repeated indices is assumed. A term associated with the electron pressure on the helium surface has been added to the third equation in (19).

By analogy with case A, we shall seek the solution of (19) in the following form:

$$v_{n\beta}=\sum_{\mathbf{q}} iq_\beta f_{\mathbf{q}}e^{i(\mathbf{qr}-\omega t)}.$$

The functions φ and ξ have the same structure as in (14). Substituting these expressions into (19) and eliminating the nonequilibrium corrections to σ and ν, we obtain

$$f_{\mathbf{q}}\left\{1-\frac{\tilde{u}^2q^2}{\omega^2}+i\frac{2\gamma_2}{\omega}\right\}-\varphi_{\mathbf{q}}=0,$$

$$\varphi_{\mathbf{q}}+i\frac{\omega}{q}\xi_{\mathbf{q}}=-\frac{\nu_nq}{\rho}(f_{\mathbf{q}}-\varphi_{\mathbf{q}}),$$

$$\alpha q^2\xi_{\mathbf{q}}=i\omega\rho\varphi_{\mathbf{q}}-e\tilde{E}_\perp n_{\mathbf{q}},$$

$$\tilde{u}^2=-\frac{1}{\nu_n}\left(\frac{\partial\alpha}{\partial\sigma}\sigma+\frac{\partial\alpha}{\partial\nu}\nu\right),\qquad \gamma_2=(\eta_s+\zeta_s)\frac{q^2}{2\nu_n},$$

where $\tilde{u}$ is the velocity of the surface sound and γ_2 is its attenuation coefficient. It follows from this that $\varphi_{\mathbf{q}}$, $f_{\mathbf{q}}$ and $\xi_{\mathbf{q}}$, expressed in terms of $n_{\mathbf{q}}$, are equal to

$$\varphi_{\mathbf{q}}=-\frac{e\tilde{E}_\perp}{\rho}\frac{i\omega n_{\mathbf{q}}}{\Delta_{\mathbf{q}}^{(1)}},\qquad f_{\mathbf{q}}=-\frac{e\tilde{E}_\perp}{\rho}\omega^3\frac{in_{\mathbf{q}}}{\Delta_{\mathbf{q}}^{(1)}\Delta_{\mathbf{q}}^{(2)}},$$

$$\xi_{\mathbf{q}}=\frac{e\tilde{E}_\perp n_{\mathbf{q}}}{\alpha q^2}\left(\frac{\omega^2}{\Delta_{\mathbf{q}}^{(1)}}-1\right),$$

$$\Delta_{\mathbf{q}}^{(2)}=\omega^2-\tilde{u}^2q^2+2i\omega\gamma_2,\qquad(20)$$

$$\Delta_{\mathbf{q}}^{(1)}=\omega^2-\omega_{\mathbf{q}}^2+\omega_{\mathbf{q}}^2\frac{\nu_nq}{\rho}\left[1-\frac{\omega^2(\omega^2-\tilde{u}^2q^2)}{(\omega^2-\tilde{u}^2q^2)^2+4\omega^2\gamma_2^2}\right]+2i\omega\gamma_1,$$

$$\gamma_1=\gamma_2\frac{\nu_nq}{\rho}\frac{\omega^2\omega_{\mathbf{q}}^2}{(\omega^2-\tilde{u}^2q^2)^2+4\omega^2\gamma_2^2}.$$

In the case under consideration, according to[7] the surface dissipative function R has the following form:

$$R=-\tau_{i\beta}\frac{\partial v_{ni}}{\partial x_\beta};$$

$$\tau_{\alpha\beta}=-\eta_s\left(\frac{\partial v_{n\alpha}}{\partial x_\beta}+\frac{\partial v_{n\beta}}{\partial x_\alpha}-\delta_{\alpha\beta}\frac{\partial v_{n\gamma}}{\partial x_\gamma}\right)-\zeta_s\delta_{\alpha\beta}\frac{\partial v_{n\gamma}}{\partial x_\gamma};\qquad \tau_{z\alpha}=\tau_{\alpha\beta}\frac{\partial\xi}{\partial x_\beta}.$$

Assuming that $\partial\xi/\partial x_\beta\ll1$ (i.e., $\tau_{z\alpha}\ll\tau_{\alpha\beta}$), we can write the expression for the energy dissipated per unit time as:

$$\frac{d\mathscr{E}}{dt}=-2\int\tau_{\alpha\beta}\frac{\partial v_{n\alpha}}{\partial x_\beta}d^2\mathbf{r}.$$

Substituting into this the expression obtained above for $v_{n\alpha}$ and averaging the result over the time, we obtain

$$\overline{\frac{d\mathscr{E}}{dt}}=(\eta_s+\zeta_s)S\left(\frac{e\tilde{E}_\perp}{\rho}\right)^2\omega^6\sum_{\mathbf{q}} q^4\left|\frac{n_{\mathbf{q}}}{\Delta_{\mathbf{q}}^{(1)}\Delta_{\mathbf{q}}^{(2)}}\right|^2.\qquad(21)$$

It follows from formula (21) that the energy dissipated per unit time can increase resonantly for two reasons. First, as $\omega\to\omega_{\mathbf{q}}$ the quantity $\Delta_{\mathbf{q}}^{(1)}$ decreases sharply. This is connected with the fact that, in this case, according to (20), a standing capillary wave with wave-vector equal to a certain reciprocal-lattice vector is resonantly excited. The positions of the resonance frequencies in this case are determined as before by formula (18). As regards the widths of the resonance lines, they have a structure that differs from (18c, d):

$$\gamma_1\approx(\eta_s+\zeta_s)\alpha^2q^5/2\rho^3\tilde{u}^4.$$

Secondly, as $\omega\to q\tilde{u}$ the quantity $\Delta_{\mathbf{q}}^{(2)}$ decreases, and this corresponds to resonant excitation of a standing wave of surface second sound. In this case the energy dissipated (21) increases by a factor of $(\tilde{u}q/\gamma_2)^2$, and the width of the resonance is proportional to γ_2. It is necessary, however, to note that the areas of the maxima in the curve of $\overline{d\mathscr{E}(\omega)/dt}$ at $\omega\approx\tilde{u}q$ are smaller by a factor of ν_nq/ρ than at $\omega\approx\omega_{\mathbf{q}}$.

For the existence of resonances it is necessary that the sum, over the reciprocal-lattice vectors, of the nonresonance terms in the right-hand side of the expression (21) be much smaller than the resonance term. When $\omega = \omega_{\mathbf{q}}$ this question does not arise, since the series (21) converges well even when $n_{\mathbf{q}} = S_0^{-1} = \text{const}$. But in the case $\omega = \tilde{u}q \gg \omega_{\mathbf{q}}$ the series in the right-hand side of the expression (21) is equivalent to the series $\Sigma_{\mathbf{q}} |n_{\mathbf{q}}|^2$, which diverges for $n_{\mathbf{q}} = \text{const}$. If we take into account that $n_{\mathbf{q}}$ has the form (5), the sum of this series will be proportional to the square of the parameter $\delta' = \sqrt{2}\pi L/a < 1$. Therefore, for a resonance to exist it is necessary that the condition

$$(2\gamma_2/\tilde{u}q)^2 \ll 2\pi^2 L^2/a^2. \qquad (22)$$

be fulfilled. For $a/L \approx 10$, the right-hand side of the inequality (22) is found to be of the order of 10^{-1}. Therefore, the condition (22) will be well-fulfilled even for $2\gamma_2/\tilde{u}q \approx 10^{-1}$.

C. Allowance for Vibrations of the Electrons in the Lattice

In the derivation of the formulas for the energy dissipated per unit time, the vibrations of the electrons in the lattice were not taken into account. To take these into account, we proceed as follows. Let $\mathbf{u}_{\mathbf{l}}$ be the displacement of an electron from its equilibrium position at the lattice site $\mathbf{l}$. In the case under consideration, the electron density can be expanded as before in the series (2), with $n_{\mathbf{q}}$ having the following form:

$$n_{\mathbf{q}} = \frac{1}{S} \sum_{\mathbf{l}} \exp\{-i\mathbf{q}(\mathbf{l}+\mathbf{u}_{\mathbf{l}})\}. \qquad (23)$$

Here the set of vectors $\mathbf{q}$ is determined not by the structure of the lattice, as was the case in (2), but by the area of the surface occupied by electrons. The value of $|n_{\mathbf{q}}|^2$ appearing in the expression for $\overline{d\mathscr{E}/dt}$, with $n_{\mathbf{q}}$ from (23), must be averaged over all $\mathbf{u}$. The corresponding calculation are analogous to those encountered in the theory of neutron scattering and lead to the result that, in formulas (17) and (21), in which, as before, the summation is taken over the vectors of the reciprocal lattice, there appears a factor

$$\chi_{\mathbf{q}} = \exp(-{}^1\!/_2 q^2 \langle u^2 \rangle).$$

multiplying $|n_{\mathbf{q}}|^2$, with the $n_{\mathbf{q}}$ for an ideal lattice. Here $\langle \mathbf{u}^2 \rangle$ is the mean-square displacement of the electron as a result of thermal and zero-point vibrations. The factor $\chi_{\mathbf{q}}$ is essentially the Debye-Waller factor for a two-dimensional crystal.

The result obtained shows that the presence of displacements $\mathbf{u}_{\mathbf{l}}$ does not affect the width of the absorption maxima, but only reduces their height. For the first few resonances, the factor $\chi_{\mathbf{q}} \approx 1$, since $\langle \mathbf{u}^2 \rangle \ll a^2$. The conditions for which the latter inequality is fulfilled were investigated in detail in[3].

CONCLUSION

We shall do some summing-up. In the paper we have investigated the group of phenomena due to the deformation of the free surface of liquid helium under the action of a system of surface electrons arranged in a two-dimensional lattice and "clamped" to the surface by an external electric field.

In the static variant of the problem the self-consistent deformation of the helium surface in the vicinity of each localized electron leads to a certain additional (compared with the analogous problem on a planar helium surface) localization of the electron, and this, in the final analysis, should increase the stability of the crystalline phase of the surface electrons. The nature of the deformational localization of the electrons and the scale of the expected effects were discussed in detail in the Introduction. The subsequent calculations carried out in the first part of the article completely confirm the validity of the preliminary qualitative estimates.

The introduction into the problem of an additional alternating electric field normal to the helium surface leads to the excitation, by the system of surface electrons, of standing capillary waves in the liquid. Owing to the assumed periodicity of the surface charge density, under certain conditions this excitation has a resonance character. The main part of the paper is devoted to a detailed description of the structure of these resonances.

In the region of not very low temperatures ($T \lesssim 1$ K) the second viscosity of helium exerts the decisive influence on the shape of the resonance lines. The position of the resonances in this case is determined by the formulas (18a), (18b), and the width of these lines follows from (15), (18c) and (18d). The main practical interest here is in the very fact of the existence of resonance absorption of an alternating field normal to the helium surface, and the detection of this will be a convincing proof of the presence of periodicity in the distribution of electrons.

Much more interesting is the situation in the region of low temperatures, when the hydrodynamic part of the problem begins to be described by the Andreev-Kompaneets equations. In the framework of these equations, mechanical excitation of surface vibrations leads to resonance phenomena not only at the frequencies (18a) and (18b) corresponding to ordinary capillary waves, but also at frequencies $\omega = \tilde{u}q$ as a result of the appearance of standing waves of surface second sound. The results of subsection B have a fairly general character and show that any forced vibrations of the helium surface with an arbitrary given wave-vector $\mathbf{q}$ resonantly excite surface second-sound waves as $\omega \to \tilde{u}q$, and this leads to a sharp increase in the energy losses in the system supporting these vibrations. In addition, the width of the resonances caused by the excitation both of capillary waves and of surface second-sound waves are proportional, in the given case, to the surface dissipative coefficients η_S and ζ_S, on which there is practically no experimental information at the present time.

The authors are grateful to A. F. Andreev for discussing the results of the paper.

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Translated by P. J. Shepherd
155