Josephson effect in wide superconducting bridges

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The motion of quantum vortices in superconducting bridges of a size greater than the pair size is studied. The volt-ampere characteristic of the bridge is obtained. The amplitude of the harmonics of the alternating voltage that arises at the bridge in the presence of a current exceeding the critical current are calculated. The size and shape of the steps in the volt-ampere characteristic of a bridge placed in an external alternating electromagnetic field are found.

1. INTRODUCTION

Superconducting bridges (superconductors that have a constriction) are the subject of intensive experimental study at the present time. Such bridges possess many of the properties of a Josephson junction—electromagnetic radiation from the bridge is observed when a current greater than the critical current is passed through it, and irradiation of the bridge by an external field leads to the appearance of steps in its volt-ampere characteristic.

There are intrinsically different explanations of the Josephson behavior of superconducting bridges, depending on the value of the constriction size a compared with the pair size ξ. In narrow bridges ($a \ll \xi$), in the presence of a current greater than the critical current a normal component of current appears in the entire region of the bridge[1]. In wide bridges ($a \gg \xi$) the Josephson effects are explained by the periodic motion of quantum vortices at the narrowest point of the bridge[2].

In this paper we find the volt-ampere characteristic of a wide bridge, the amplitude of the harmonics of the alternating voltage, and the size and shape of the steps in the volt-ampere characteristic of a bridge placed in an external electromagnetic field. It has turned out that there are several characteristic ranges of variation of the current, which differ in the number of vortices moving in the bridge.

When the current exceeds the critical value by only a small amount, the time of formation of a vortex at an edge of the bridge is long and is considerably greater than the time of its motion across the bridge. Therefore, for the greater part of the period there are no vortices in the bridge. The dependence of voltage on time has the form of narrow periodic pulses. As the current increases the vortex creation time decreases and becomes less than the time of the vortex motion. The formation of the next vortex is then made difficult because of the forces of repulsion between the vortices. It is created only when the preceding vortex has passed through almost the entire bridge. In this region the mean voltage at the bridge depends weakly on the current, and the amplitude of the first harmonics of the alternating voltage is, in order of magnitude, equal to the value of the mean voltage.

In the region of high currents the force exerted by the current on a vortex increases. The relative magnitude of the repulsive forces between the vortices decreases and vortices are created more frequently. A chain consisting of a large number of vortices moves across the bridge. The number of vortices is proportional to the amount by which the current exceeds the critical value. So long as this excess remains small compared with the critical current, the velocity of the vortices depends weakly on the magnitude of the current. The voltage at the bridge, which is proportional to the number of vortices, depends linearly on the current. The amplitude of the harmonics of the alternating voltage is small in this region.

In the region of current values very much greater than the critical value, the number of vortices and the velocity of their motion are proportional to the size of the current. Therefore, the voltage depends quadratically on the current.

In an alternating external electromagnetic field the current across the bridge has an alternating component, varying with the frequency ω of the external field. If the amplitude of the alternating current is small, an appreciable change in the volt-ampere characteristic of the bridge occurs near the resonance at which the frequency of the external field coincides with that of the characteristic radiation of the bridge. Then the mean current across the bridge, corresponding to the given voltage, changes in accordance with the phase of the alternating current at the moment of creation of the vortices. On the volt-ampere characteristic there appear constant-voltage portions, the size of which is equal to twice the amplitude of the alternating current.

2. MOTION OF VORTICES IN THE BRIDGE

When the current across the bridge increases and reaches a certain value J_c the current density at the edge of the bridge reaches the critical value. Then the barrier impeding the entry of a vortex into the bridge disappears and the vortex begins to move toward the center of the bridge[3]. We shall consider a bridge in the form of a superconducting film of thickness d, with two cuts along one straight line; the distance between the ends of the cuts is equal to 2a (Fig. 1). As follows from the following account, the principal results do not depend on the shape of the bridge, which only has a strong influence on the magnitude of the critical current J_c.

For the superconducting current density j in that region of the bridge in which it is less than the critical value, we can make use of the following expression:

$$\mathbf{j}=\frac{c^2\hbar}{8\pi e\lambda^2}\left(\nabla\varphi-\frac{2e}{c\hbar}\mathbf{A}\right), \tag{1}$$

where φ is the phase of the order parameter $\mathbf{A}$ is the vector potential and λ is the depth of penetration of the

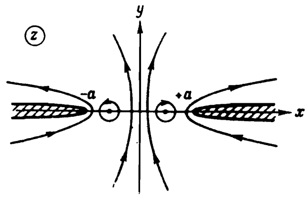

FIG. 1. Superconducting bridge with two quantum vortices.

magnetic field. Usually, the size a of the bridge is less than the effective penetration depth $\lambda_{eff} = \lambda^2/d$ in the film. Then the effect of the intrinsic magnetic field on the superconducting current can be neglected and the second term in formula (1) can be disregarded.

The equation for the phase φ of the order parameter is obtained from the condition div j = 0 and has the form

$$\Delta\varphi=0 \tag{2}$$

with the boundary condition $\partial\varphi/\partial\mathbf{n} = 0$ (n is the normal to the surface of the superconductor). In our geometry the solution of this equation, with no singularities in the region of the bridge, is given by the formula

$$\varphi_1(z) = \mathrm{Re}\left[(J/J_0)\ln\left(z/a+\sqrt{z^2/a^2-1}\right)\right], \tag{3}$$

where $z = x + iy$ defines the point with coordinates x and y in the plane of the bridge, the cuts run along the x axis from the point a to $+\infty$ and from the point $-a$ to $-\infty$, J is the total current across the bridge, and $J_0 = c^2\hbar d/8e\lambda^2$.

The current lines are hyperbolas with foci at the points $z = \pm a$. The current density j on the axis in the region of the bridge is equal to

$$j = J/\pi d\sqrt{a^2-x^2} \tag{4}$$

and has a root singularity at the edge of the bridge. Therefore, even if the current density attains the critical value near the edge, when $a - x \sim \xi$, it is still small throughout the rest of the region and formulas (1) and (2) can be used.

The quantum vortices in the bridge correspond to the singular solutions of Eq. (2):

$$\varphi_2(z) = \sum_n q_n \,\mathrm{Im}\left[\ln\left(\sqrt{\frac{a+z}{a-z}}+\sqrt{\frac{a+z_n}{a-z_n}}\right)-\ln\left(\sqrt{\frac{a+z}{a-z}}-\sqrt{\frac{a+z_n}{a-z_n}}\right)\right], \tag{5}$$

where the z_n define the positions of the vortices, the sum is taken over all the vortices, and $q_n = \pm 1$ and is determined for each vortex by the direction of the vortex. The solution (5) is found by means of a conformal mapping of the z-plane into a half-plane.

The energy of the vortices in the bridge is found from the formula

$$E = \int d^3\mathbf{r}\,\frac{2\pi}{c^2}\lambda^2 j^2(\mathbf{r}), \tag{6}$$

where the relation between the current density j and the gradient of the phase φ is given by formula (1), and the phase itself, $\varphi(\mathbf{r}) = \varphi_1(\mathbf{r}) + \varphi_2(\mathbf{r})$, is determined by the expressions (3) and (5). As a result, we obtain

$$E = \frac{\hbar}{2e}J_0\Bigg[\sum_n q_n\frac{J}{J_0}\,\mathrm{Im}\ln\left(\frac{z_n}{a}+\sqrt{\frac{z_n^2}{a^2}-1}\right)-\sum_n \ln\frac{\xi a}{2|a^2-z_n^2|} + \sum_{n,m} q_n q_m\left\{\ln\left|\sqrt{\frac{a+z_n}{a-z_n}}+\sqrt{\frac{a+z_m}{a-z_m}}\right|-\ln\left|\sqrt{\frac{a+z_n}{a-z_n}}-\sqrt{\frac{a+z_m}{a-z_m}}\right|\right\}\Bigg], \tag{7}$$

where an unimportant term representing the energy of the bridge with no vortices has been omitted. The first term of this expression gives the energy of the interaction of the vortices with the current, the second gives the self-energy of a vortex in the bridge, and the third corresponds to the interaction energy between the vortices.

Differentiating the energy with respect to the coordinates of a vortex, we can obtain the force acting on the vortex in the bridge. When the vortex lies on the x axis, at the edge of the bridge at a point with coordinate x_n close to a, the force equals

$$F_n = -\frac{\hbar}{2e}J_0\left(\frac{1}{a-x_n}-q_n\frac{J}{J_0}\sqrt{\frac{2}{a}}\frac{1}{\sqrt{a-x_n}} + \sqrt{\frac{2}{a}}\frac{1}{\sqrt{a-x_n}}\sum_{m\neq n} q_n q_m\sqrt{\frac{a+x_m}{a-x_m}}\right) \tag{8}$$

where x_m are the coordinates of the remaining vortices, which we also assume to be positioned on the x axis. It follows from this expression that the force attracting the vortex toward the edge of the bridge (the first term) increases more rapidly than the force from the interaction with the current (the second term) as the edge is approached. Therefore, for low currents there is a barrier impeding the entry of vortices into the bridge. The barrier for formation of the first vortex disappears at a critical current J_c determined from the condition that these forces be equal at $x - a \sim \xi$, where ξ is the size of the vortex core: $J_c \sim J_0\sqrt{a/\xi}$. The current density at the edge of the bridge then reaches the critical value.

If there are already vortices in the bridge, the barrier for the formation of a new vortex is changed. From formula (8) we find that, in this case, the barrier disappears at a current $J = J_c + \Delta J$, and ΔJ equals

$$\Delta J = J_0\sum_m q_m\sqrt{\frac{a+x_m}{a-x_m}}. \tag{9}$$

We note that although the value of the critical current has been found only in order of magnitude, the formula (9) for ΔJ is exact. This follows from the fact that the moment of creation of a vortex is determined by the current density near the edge of the bridge, and does not depend on whether this density is created by a transport current or by other vortices.

A vortex that has been formed at the edge will move toward the center of the bridge. Then, as can be seen from formula (8), at distances greater than ξ from the edge the force of attraction toward the edge becomes smaller than the force from the interaction with the current. The ratio of the force exerted on a vortex by the other vortices to the force from its interaction with the current has the order of magnitude of $\Delta J/J$. In the following we shall be interested in the region of currents $\Delta J \ll J$, in which this ratio is small. Then the motion of the vortices occurs principally under the influence of the force from the interaction of each vortex with the current.

For a vortex positioned in the x axis at an arbitrary distance x_n from the center of the bridge, the force from its interaction with the current equals

$$F_n = -\frac{\hbar J}{e}\frac{q_n}{\sqrt{a^2-x_n^2}}. \tag{10}$$

Comparing this formula with formula (4), it is easy to see that the force is proportional to the current density at the position of the vortex.

The equation of the viscous motion of the vortex has the form

$$\dot{x}_n = F_n/\eta, \tag{11}$$

where η is the viscosity coefficient, for which there is an expression in terms of the microscopic parameters of the superconductor[4]. Solving Eq. (11) with the force given by formula (10), we obtain the following implicit time dependence of the vortex coordinate:

$$t = \frac{\eta e a^2}{2\hbar J}\left(\frac{\pi}{2} - \arcsin\frac{x}{a} - \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right), \tag{12}$$

where the time origin corresponds to the moment of creation of the vortex. Since it has been assumed in the derivation of formula (12) that $\Delta J \ll J$, in it we must replace J by J_c.

3. VOLT-AMPERE CHARACTERISTIC OF THE BRIDGE

During the motion of the vortices the phase φ of the order parameter varies with time. Far from the bridge, the phase in the superconductor does not depend on the coordinates and the rate of change of the phase difference between the two sides of the bridge is proportional to the voltage at the bridge:

$$V(t) = \frac{\hbar}{2e}(\dot{\varphi}_{+\infty} - \dot{\varphi}_{-\infty}). \tag{13}$$

By means of the formulas (5) we find

$$V(t) = \frac{\hbar}{e}\sum_m \frac{q_m \dot{x}_m(t)}{\sqrt{a^2 - x_m^2(t)}} \tag{14}$$

For a given total current J across the bridge the motion of vortices has a periodic character. Each succeeding vortex repeats the motion of the previous one, shifted in time by the period T. As a result, the change of the phase over the period is the same as in the passage of one vortex across the whole of the bridge, and is equal to 2π. Then from formula (13) the Josephson relation

$$\hbar\omega = 2\pi\hbar/T = 2e\bar{V}, \tag{15}$$

follows, where $\bar{V}$ is the mean voltage at the bridge. Of course, this same relation also follows from formula (14), if the voltage V(t) is averaged over the period T.

If the current J across the bridge is not too close to the critical current J_c, the time for the vortex to break away is short compared with the period T, and in describing the motion of the vortices we can disregard it. Then, at the moment of creation of a new vortex, the coordinates of the vortices already moving in the bridge are $x_m = x(mT)$, where x(t) is given by formula (12). Substituting these coordinates into formula (9) and taking the relation (15) into account, we obtain the volt-ampere characteristic

$$\Delta J = J_0 \sum_m q_m \left(\frac{a + x(m\pi\hbar/e\bar{V})}{a - x(m\pi\hbar/e\bar{V})}\right)^{1/2}. \tag{16}$$

The details of the volt-ampere characteristic depend on the shape of the bridge. In a strictly symmetric bridge two vortices of opposite orientations are created simultaneously at the edges, move toward the center of the bridge and disappear there when they meet. We shall study formula (16) for this case, in different regions of the current ΔJ.

In the initial region of currents no more than two pairs of vortices can be formed in the bridge (by the time a third pair is created at the edge, the first has already disappeared at the center). Therefore, at the moment of creation of a new pair there is only one pair in the bridge, and in formula (16) we must retain only two terms, corresponding to this pair of vortices. Using formula (12), for the volt-ampere characteristic we obtain

$$\bar{V} = \frac{\pi}{2}V_0\left(\frac{\pi}{2} - \frac{2J_0\Delta J}{4J_0^2 + (\Delta J)^2} - \arcsin\frac{\Delta J}{(4J_0^2 + (\Delta J)^2)^{1/2}}\right)^{-1}. \tag{17}$$

Here $V_0 = \hbar\pi/et_m = 4\hbar^2 J_c/\eta e^2 a^2$, where t_m is the total time of the motion of the pair across the bridge. Formula (17) is valid for $\bar{V} < 2V_0$. The limits of the range of applicability are determined from the condition for which t_m is equal to twice the period T.

For $\Delta J \ll J_0$ the mean voltage depends weakly on the current and there is a plateau $\bar{V} = V_0$ in the volt-ampere characteristic (Fig. 2). In this region the amount by which the current exceeds the critical value is small and a second pair can be created only when the first has almost reached the center of the bridge. The period coincides with the total time of the motion of the pair across the bridge, and does not depend on ΔJ.

In the general case, the ratio $\bar{V}/V_0$ equals the mean number of pairs in the bridge. In the region $NV_0 < \bar{V} < (N+1)V_0$, where N is an integer, the number of pairs in the bridge is equal to either N or N + 1. When $\bar{V}_N = NV_0$ the number of pairs of vortices in the bridge is always equal to N.

When the voltage passes through the values NV_0, the number of terms in formula (16) for the volt-ampere characteristic changes. As a result, sharp bends appear in the volt-ampere characteristic (Fig. 2), and the discontinuities in the derivative $dJ/d\bar{V}$ at the positions $\bar{V}_N = NV_0$ of the bends are determined by the formula

$$\Delta(dJ/d\bar{V}) = \pi J_0/2NV_0. \tag{18}$$

We shall now determine the volt-ampere characteristic in the region $\bar{V} \gg V_0$, when the number of pairs of vortices in the bridge $N \gg 1$. In this case, the sum over m in formula (16) can be replaced by an integral over x, by introducing the density of vortices $\rho(x) = 1/T\dot{x}$. Determining $\rho(x)$ by means of formulas (10) and (11), from formula (16) we then obtain

$$\bar{V} = \pi V_0 \Delta J/4J_0. \tag{19}$$

Thus, in the region $\Delta J \gg J_0$ the volt-ampere characteristic of the bridge is linear. Here, formula (19) is valid for currents $\Delta J \ll J_c$.

In the region $\Delta J \gtrsim J_c$ the force exerted on a vortex by the other vortices is of the same order as the force from its interaction with the current, and formula (12) cannot be used for the time dependence of the vortex coordinate. For such current values the Josephson effects are small, and we shall confine ourselves to estimating only the dependence of the mean voltage on the current in the region $\Delta J \gg J_c$.

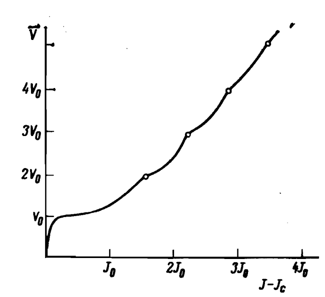

FIG. 2. Volt-ampere characteristic of the bridge.

Formula (16) is valid in this region too, and it follows from it that the ratio $\Delta J/J_0$, which in this region is almost equal to J/J_0, coincides in order of magnitude with the total number of vortices in the bridge. The mean voltage at the bridge is proportional to the total number of vortices and inversely proportional to the total time of the motion of a vortex in the bridge. Since the interaction of a vortex with the other vortices has the same order of magnitude as its interaction with the current, to estimate the time of motion we can take only the latter interaction into account. Therefore, the time of motion is inversely proportional to the magnitude of the current J. As a result, in the region $J \gg J_c$ we obtain $\overline{V} \sim V_0 J^2/J_0 J_c$. This law is valid until the superconductivity is destroyed in the entire region of the bridge.

We shall now estimate the dependence of the mean voltage on the current at currents very close to the critical value, when the time of formation of a vortex exceeds the time of its motion. For the greater part of the period there are then no vortices in the bridge, and only at the edge of the bridge, in a region of size $\sim\xi$, does a weak normal current of the order of ΔJ exist. This region can be regarded as a bridge with dimensions of the order of ξ, and for the estimates we can make use of the formulas obtained for a point bridge. The time for a vortex to break away in such a bridge is of the order of the period of the Josephson oscillations, which is inversely proportional to $\sqrt{\Delta J}$[1].

The qualitative picture of the creation of a vortex has the following form. At some moment of time, a superconducting current, equal to its maximum critical value, and a normal current, equal to ΔJ, flow through the bridge. The electric field at the edge of the bridge leads to a change $\Delta\varphi$ in the phase of the order parameter and to a decrease of the superconducting current by an amount $\sim(\Delta\varphi)^2$. The normal current, proportional to $\dot{\varphi}$, increases by the same amount. The time over which the normal current undergoes a serveralfold increase, i.e., by an amount $\sim\Delta J$, is found from the condition $\Delta\varphi/t \sim \Delta J \sim (\Delta\varphi)^2$, whence $t \sim (\Delta J)^{-1/2}$. Then the change in the phase occurs very rapidly and a vortex is formed. Thus, the vortex formation time $t_0 \sim (\Delta J)^{-1/2}$. This time does not depend on the size of the bridge.

For the ratio of the time of formation of the vortex to its time of motion we have $t_0/t_m \sim \sqrt{J_0/\Delta J}(\xi/a)^{3/2}$, since the time of motion $t_m \sim a^{3/2}$. For $\Delta J \ll J_0(\xi/a)^3$, the period, which is inversely proportional to the mean voltage, is equal to the vortex formation time; the volt-ampere characteristic is the same as for a point bridge: $\overline{V} \sim \sqrt{\Delta J}$. For $\Delta J \gg J_0(\xi/a)^3$ formula (17) is valid.

4. ALTERNATING ELECTROMAGNETIC FIELD IN THE BRIDGE

When a direct current is passed through the bridge, an alternating voltage, determined by formula (14), appears at the bridge. This voltage has a periodic dependence on the time, with period T, and it can be expanded in a Fourier series:

$$V(t)=\sum_{k=-\infty}^{+\infty} v_k e^{-2\pi ikt/T},$$

$$v_k=\frac{\hbar}{eT}\int_0^T \sum_m \frac{q_m \dot{x}_m(t)}{\sqrt{a^2-x_m^2(t)}} e^{2\pi ikt/T}\,dt. \qquad (20)$$

For vortices situated to one side of the center of the bridge, $x_m(t) = x(t + mT)$, where $x(t)$ is defined by formula (12). Taking this relation into account and using the expression for $\dot{x}_m(t)$ from formulas (10) and (11), we obtain for v_k:

$$v_k=\frac{2\hbar^2 J}{\eta e^2 T}\int_0^{t_m} \frac{1}{a^2-x^2(t)} e^{2\pi ikt/T}\,dt, \qquad (21)$$

where t_m is the total time of the motion of the vortex across the bridge.

To find the asymptotic expression for v_k when $t_m k/T \gg 1$ we note that the important contribution to the integral in this case is given by the region of short times t, when x is close to a. As a result, we have

$$v_k=\Gamma(1/3)\, e^{i\pi/6} (V_0/3\pi^2 k\overline{V})^{1/3}\overline{V}, \qquad (22)$$

where $\Gamma(z)$ is the gamma-function. Thus, the amplitude of the harmonics decreases slowly as the label k increases.

In the region of currents $\Delta J \gg J_0$ and, correspondingly, $V_0/\overline{V} = T/t_m \ll 1$, formula (22) is applicable for all harmonics. The amplitude of the harmonics in this case is small compared with the mean voltage. For $\Delta J \lesssim J_0$ formula (22) is valid only for $k \gg 1$. In this region the amplitude of the first harmonics has the order of magnitude of the mean voltage.

When the bridge is placed in an external alternating electromagnetic field with frequency ω the total current across the bridge becomes an alternating current: $J(t) = \overline{J} + i_1 \sin(\omega t + \delta)$. In this case the condition for the creation of each new vortex is given, as before, by formula (9). However, we must substitute $J(t_n) - J_c$ into the left-hand side of this formula, where $J(t_n)$ is the total current at the moment of creation of the n-th vortex.

When the frequency ω of the external field is a multiple of the frequency of the motion of the vortices, the vortices are created, as before, after equal time intervals $T = 2\pi k/\omega$. The mean voltage $\overline{V}$ is connected with the frequency ω of the external field by the Josephson relation: $\overline{V} = \hbar\omega/2ek$. Equation (16), relating the mean current to the mean voltage, takes the form

$$\overline{J}+i_1 \sin\delta=J(\hbar\omega/2ek), \qquad (23)$$

where the value of the current in the right-hand side is that corresponding to the voltage $\hbar\omega/2ek$ in the absence of the alternating field.

As can be seen, the mean current across the bridge takes different values for the same voltage, depending on the phase δ of the current at the moment of creation of the vortex. This means that steps of width $2i_1$ appear in the volt-ampere characteristic of the bridge at the voltages $\hbar\omega/2ek$.

We shall study now the shape of the volt-ampere characteristic near a step, under the assumption that the steps are small: $i_1 \ll \Delta J$. If the mean voltage at the bridge is not equal to $\hbar\omega/2ek$, the phases of the current are different at the moments of creation of the vortices. Near resonance, however, the phase varies slowly: the phase δ_n of the current at the moment of creation of the n-th vortex depends weakly on the label n. The time of the motion of the n-th vortex before the creation of the next vortex changes by an amount

$$\tau_n=\frac{\delta_n-\delta_{n-1}}{\omega}=\frac{1}{\omega}\frac{\partial\delta_n}{\partial n}, \qquad (24)$$

which we shall assume to be the same for all vortices situated in the bridge at a given time. Correspondingly, the coordinates of the vortices will be the same as for periodic motion with period $T = 2\pi k/\omega + \tau_n$. Expanding the right-hand side of Eq. (9) in the small parameter τ_n, we obtain, in place of Eq. (23),

$$i+i_1 \sin \delta_n = -\frac{\bar{V}}{2\pi k} \frac{\partial J}{\partial \bar{V}} \frac{\partial \delta_n}{\partial n}, \tag{25}$$

where $i = \bar{J} - J(\hbar\omega/2ek)$, and the dependence of the current J on the voltage $\bar{V}$ in the absence of the alternating field is given by formula (16).

For $|i| < i_1$ Eq. (25) describes the transitional process establishing the periodic motion of the vortices with period $T = 2\pi/k\omega$. Here δ_n tends to the constant value $\arcsin(i/i_1)$. For $|i| > i_1$ the solution of Eq. (25) is given by the formula

$$\tau_n = \frac{1}{\omega} \frac{\partial \delta_n}{\partial n}$$

$$= -\frac{2\pi k}{\omega \bar{V} \partial J/\partial \bar{V}} (i^2 - i_1^2) \left[i + i_1 - 2i_1 \sin^2 \left(\frac{\pi k \sqrt{i^2 - i_1^2}}{\bar{V} \partial J/\partial \bar{V}} n \right) \right]^{-1}. \tag{26}$$

The motion of the vortices in this case is frequency-modulated. The correction τ_n to the period is a periodic function of the vortex label n or of the time $t = nT$. As follows from formula (26), close to a step, for $i \ll \Delta J/k$, the frequency of modulation of the period is small, and τ_n, as we assumed, is a slow function of the label n. Near the edge of a step the time dependence of τ_n has the form of sharp pulses: for most of the time the frequency at which the vortices break away does not change, and only occasionally do the vortices begin to break away more (or less) frequently.

To find the mean voltage of the bridge we shall average the correction τ_n to the period over a large number of vortices n:

$$\bar{V} = \frac{\pi\hbar}{e} \frac{1}{T + \bar{\tau}_n} = \frac{\pi\hbar\omega}{ek} + \frac{\partial V}{\partial J} i \sqrt{1 - i_1^2/i^2}. \tag{27}$$

This formula determines the volt-ampere characteristic of the bridge near a step, for $|i| > i_1$. It can be seen that the mean voltage varies hyperbolically with the current.

DISCUSSION OF THE RESULTS

The picture obtained for the motion of vortices in a superconducting bridge assumes that the bridge is sufficiently wide: $a \gg \xi$. The shape of the bridge does not affect the main results and only has a substantial influence on the magnitude of the critical current.

The critical current is determined from the condition for which the current density at the edge of the bridge is equal to the critical value. For a bridge in the shape of a hyperboloid with "aperture" angle 2α, the critical current is, as before, of the order of $J_0\sqrt{a/\xi}$, if $\alpha \ll \sqrt{\xi/a}$. In the opposite case $\alpha \gg \sqrt{\xi/a}$ the critical current increases: $J_c \sim J_0(a/\xi)\sin\alpha$. For a bridge in the shape of an acute angle, $J_c \sim J_0(a/\xi)^{\pi/2(\pi-\alpha)}$ for any aperture 2α.

The Josephson properties of a bridge are determined by the motion of the quantum vortices in it. This motion is described by formula (12), for any bridge with a small aperture angle. Therefore, all the derived dependences on the quantity $\Delta J = J - J_c$ will remain unchanged for any acute-angled symmetric bridge.

However, even slight asymmetry of the bridge is sufficient to change the pattern of motion of the vortices: vortices will be created at only one edge of the bridge, pass through the entire region of the bridge and disappear at the opposite edge. For a bridge in the form of a half-plane with a cut, the appropriate formulas are obtained by simply replacing ΔJ by twice the amount by which the current exceeds the critical value.

In the general case, asymmetry of the bridge leads to changes of the numerical coefficient in the dependences found. For example, if both angles in an asymmetric bridge are acute, the total time of motion of a vortex across the bridge is twice as long as in the symmetric case. Correspondingly, the values of the voltages at which the plateau and kinks are observed in the volt-ampere characteristic are decreased by a factor of two. The kinks in this case are sharper.

In the case when the bridge is formed not by a superconducting film but by a bulk superconductor (e.g., a clamped bridge), the pattern described for the motion of the vortices is conserved. For a symmetric bridge the vortices have the shape of rings that form at the edge of the bridge and link up at the center under the influence of the current. The motion of such vortices in a bridge has been studied experimentally[5].

We shall also discuss the difference in the properties of wide and narrow superconducting bridges. First of all, we compare the volt-ampere characteristics of the bridges. In a narrow bridge there are two characteristic regions in the volt-ampere characteristic. For $\Delta J = J - J_c \lesssim J_c$ the mean voltage is proportional to $\sqrt{\Delta J}$. In the region $\Delta J \gg J_c$ it is mainly normal current that flows across the bridge and the mean voltage is proportional to the magnitude of the current. The Josephson effects in this region are small. In a wide bridge the dependence $\bar{V} \sim \sqrt{\Delta J}$ is valid only in a narrow region:

$$\Delta J < (\xi/a)^4 J_c^2/J_0,$$

where $J_0 = c^2\hbar d/8e\lambda^2$ has the order of magnitude of the critical current of a bridge of size $\sim\xi$. The most interesting region in which Josephson effects are manifested is $\Delta J \lesssim J_0 \ll J_c$. The volt-ampere characteristic in this region is determined by the formula (17) and has a plateau for $\Delta J \ll J_0$: $\bar{V}_0 = \pi\hbar/et_m$, where t_m is the total time of the motion of the vortex across the bridge. There are kinks in the volt-ampere characteristic at voltages equal to multiples of V_0.

In the region $J_0 < \Delta J < J_c$ the volt-ampere characteristic is linear. Its slope, however, is considerably smaller than the resistance of the bridge in the normal state, since the destruction of the superconductivity in the entire region of the bridge sets in at considerably higher currents. In this region the number N of vortices in the bridge is large and the amplitudes of the Josephson harmonics, which are inversely proportional to $N^{1/3}$, are already small.

Thus, as the width of the bridge increases its critical current increases, and the region of currents ΔJ in which Josephson effects are important does not change.

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