Inverse problem of scattering theory in complex λ plane in the presence of a Coulomb field with allowance for absorption[1)]

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An algebraic approach to the solution of the inverse problem of scattering theory for charged particles in the plane of the complex orbital angular momentum l at a fixed physical value of the energy is examined. Formulas are obtained for the nuclear interaction potential in which absorption is taken into account and the "nuclear" phase shift $\delta(\lambda, k)$ is assumed to be of the same form as in Eqs. (3) and (4).

1. It is known that the inverse problem of scattering theory has a solution in the following cases: 1) if the phase shifts at all values of the energy, the normalization constant, and the bound-state energy are specified for a fixed value of the orbital angular momentum; 2) if the phase shifts at all values of the orbital angular momentum are specified for a fixed value of the energy. In the first formulation, this problem was solved by Gel'fand and Levitan [1], Kreĭn [2], Marchenko [3], and others. The solution for the case of charged particles was obtained by Gugushvili and Mentokovskiĭ [4]. In the second formulation, the solution of the problem can be found in the papers of Burdet, Giffon, and Predazzi [5] (for uncharged particles) and Poplavskiĭ [6] (with allowance for the Coulomb interaction). In all cases, the authors started from considerations of the general theory of boundary-value problems. The algebraic approach to the solution of the inverse problem in the first formulation was considered by Theis [7].

In this paper we propose an algebraic method of solving the problem by using phase shifts, specified at a fixed energy, at different values of the orbital angular momentum.

2. We consider the radial Schrödinger equation for the partial wave

$$y''(r)+\left(k^2-\frac{\lambda^2-{}^1/_4}{r^2}-\frac{a}{r}-V(r)\right)y(r)=0, \tag{1}$$

where $k = (2ME)^{1/2}/\hbar$, $a = 2MZze^2/\hbar^2$, $\lambda = l + {}^1/_2$. In this equation $\hbar^2(\lambda^2 - {}^1/_4)/2Mr^2$ and $\hbar^2 a/2Mr$ are respectively the centrifugal and Coulomb potentials, and $\hbar^2V(r)/2M$ is an arbitrary potential (for example, this can be the nuclear-interaction potential). The quantity k assumes physical values, while λ is complex.

We shall show below how to obtain the potential V(r) if the scattering function S(λ, k) corresponding to the sum of the three aforementioned potentials can be represented in the form

$$S(\lambda, k)=\exp\{2i[\eta(\lambda, k)+\delta(\lambda, k)]\}, \tag{2}$$

$$\exp[2i\delta(\lambda, k)]=R(\lambda, k)/R(\lambda, -k), \tag{3}$$

$$R(\lambda,k)=\prod_1^n[\lambda-\alpha_\mu(k)]\Big/\prod_1^n[\lambda-\beta_\nu(k)], \tag{4}$$

where $\eta(\lambda, k)$ are the phase shifts for the Coulomb and centrifugal potentials; $\delta(\lambda, k)$ is the additional shift due to the presence of the potential V(r); $\alpha_\mu(k)$ and $\beta_\nu(k)$ are complex constants which are different for all μ and ν, with Re $\alpha_\mu(k) > 0$, Re $\beta_\nu(k) > 0$.

We present first the known results for the case V(r) = 0. We rewrite (1) in the form

$$u''(r)+\left(k^2-\frac{\lambda^2-{}^1/_4}{r^2}-\frac{a}{r}\right)u(r)=0. \tag{5}$$

We shall henceforth be interested in the regular solution $\varphi(\lambda, k, r)$ and the Jost solution $f(\lambda, \pm k, r)$ of Eq. (5), which satisfy the boundary conditions

$$\lim_{r\to0}\varphi(\lambda,k,r)r^{-(\lambda+{}^1/_2)}=1, \tag{6}$$

$$\lim_{r\to\infty}f(\lambda,\pm k,r)\exp\left[\pm i\left(kr-\frac{a}{2k}\ln 2kr\right)\right]=1. \tag{7}$$

Using the notation

$$[u;\ v]=u(r)v'(r)-u'(r)v(r),$$

we introduce the Jost functions

$$f(\lambda, \pm k)=[f(\lambda, \pm k, r);\ \varphi(\lambda, k, r)]. \tag{8}$$

Recognizing that $[f(\lambda, k, r); f(\lambda, -k, r)] = 2ik$, we can express $\varphi(\lambda, k, r)$ in terms of $f(\lambda, \pm k, r)$:

$$\varphi(\lambda,k,r)=-\frac{1}{2ik}[f(\lambda,-k)f(\lambda,k,r)-f(\lambda,k)f(\lambda,-k,r)]. \tag{9}$$

The solution (9) has as $r \to \infty$ the asymptotic form

$$\varphi(\lambda,k,r)\sim e^{i\eta(\lambda,k)}\frac{f(\lambda,-k)}{k}\sin\left[kr-\frac{a}{2k}\ln 2kr+\eta(\lambda,k)\right], \tag{10}$$

where

$$f(\lambda, k)/f(\lambda, -k)=\exp[2i\eta(\lambda, k)] \tag{11}$$

is the scattering function.

We introduce the notation

$$f(\lambda, -k, r)=f_\lambda,\ f(\alpha_\mu(k), -k, r)=f_\mu,\ \alpha_\mu(k)=\alpha_\mu, \tag{12}$$

$$\varphi(\lambda, k, r)=\varphi_\lambda,\ \varphi(\beta_\nu(k), k, r)=\varphi_\nu,\ \beta_\nu(k)=\beta_\nu,$$

$$a_{\nu\lambda}=\frac{[\varphi_\nu; f_\lambda]}{\beta_\nu^2-\lambda^2},\quad a_{\nu\mu}=\frac{[\varphi_\nu; f_\mu]}{\beta_\nu^2-\alpha_\mu^2}. \tag{13}$$

On the basis of (13) and of the relation

$$\varphi_\nu f_\lambda''-\varphi_\nu''f_\lambda=\frac{\lambda^2-\beta_\nu^2}{r^2}\varphi_\nu f_\lambda,$$

which follows from (5), we obtain

$$a_{\nu\lambda}'=-\varphi_\nu f_\lambda/r^2. \tag{14}$$

3. We shall show that the scattering function (2)–(4) can be obtained by choosing the potential V(r) in (1) to be the function

$$V(r)=-2\left[\left(\frac{A_\nu\varphi_\nu}{r^2}\right)'+\frac{A_\nu\varphi_\nu}{r^3}\right], \tag{15}$$

where A should satisfy the system of equations

$$A_\nu a_{\nu\mu}=-f_\mu. \tag{16}$$

(summation is carried out over the repeated indices ν and μ). We first verify that

$$y_\lambda = f_\lambda + A_\nu a_{\nu\lambda} \quad (17)$$

is the solution of Eq. (1). To this end we substitute (17) in (1). Taking (13)–(15) into account, we obtain

$$C_\nu a_{\nu\lambda} = 0, \quad (18)$$

where

$$C_\nu = A_\nu'' + \left\{k^2 - \frac{\beta_\nu^2 - 1/4}{r^2} - \frac{a}{r} + 2\left[\left(\frac{A_\nu \varphi_\nu}{r^2}\right)' + \frac{A_\nu \varphi_\nu}{r^3}\right]\right\} A_\nu. \quad (19)$$

At $\lambda = \alpha_\mu$ we have, according to (16), $y_\mu = 0$. In this case the system of homogeneous equations $C_\nu a_{\nu\mu} = 0$ relative to C_ν, which is obtained from (18), contains only the trivial solution $C_\nu = 0$, inasmuch as $|a_{\nu\mu}| \not\equiv 0$ if α_μ and β_ν are different. Consequently, A_ν is a solution of (1). Therefore y_λ is also a solution of this equation.

To construct the Jost solutions of Eq. (1) with the potential (15), we investigate the asymptotic form of the solution (17) as $r \to \infty$. To this end we obtain A_ν from the system (16)

$$A_\nu = -f_\mu \bar{a}_{\nu\mu} / |a_{\nu\mu}|, \quad (20)$$

where $\bar{a}_{\nu\mu}$ is the cofactor of $a_{\nu\mu}$. Substituting (20) in (17), we obtain

$$y_\lambda = \frac{f_\lambda}{|a_{\nu\mu}|} \left| a_{\nu\mu} - \frac{f_\mu a_{\nu\lambda}}{f_\lambda} \right|.$$

We consider the element

$$a_{\nu\mu} - \frac{f_\mu a_{\nu\lambda}}{f_\lambda} = a_{\nu\mu}\left[1 - \frac{f_\mu[\varphi_\nu; f_\lambda](\beta_\nu^2 - \alpha_\mu^2)}{f_\lambda[\varphi_\nu; f_\mu](\beta_\nu^2 - \lambda^2)}\right].$$

According to (7) and (10), $f_\mu[\varphi_\nu; f_\lambda]/f_\lambda[\varphi_\nu; f_\mu] \to 1$ as $r \to \infty$. Consequently,

$$\lim_{r\to\infty}\left(a_{\nu\mu} - \frac{f_\mu a_{\nu\lambda}}{f_\lambda}\right) = \frac{\lambda^2 - \alpha_\mu^2}{\lambda^2 - \beta_\nu^2} \lim_{r\to\infty} a_{\nu\mu},$$

therefore

$$\lim_{r\to\infty} y_\lambda = \left\{\prod_1^n (\lambda^2 - \alpha_\mu^2) \Big/ \prod_1^n (\lambda^2 - \beta_\nu^2)\right\} \lim f_\lambda,$$

and the function

$$F_\lambda = y_\lambda \prod_1^n (\lambda^2 - \beta_\nu^2) \Big/ \prod_1^n (\lambda^2 - \alpha_\mu^2)$$

is a solution of Eq. (1) and has as $r \to \infty$ the asymptotic form (7), i.e., F_λ is the Jost solution.

According to (4) and (12), F_λ can be represented in the form

$$F(\lambda, -k, r) = \frac{y(\lambda, -k, r)}{R(\lambda, k) R(-\lambda, k)}. \quad (21)$$

To find the regular solution of (1), we construct the solution $\tilde{\Phi}_\lambda$ of this equation in analogy with (9) in the form

$$\tilde{\Phi}_\lambda = -\frac{1}{2ik}[f(\lambda, -k) y(\lambda, k, r) - f(\lambda, k) y(\lambda, -k, r)] \quad (22)$$

and determine its asymptotic form as $r \to 0$. Substituting (17) in (22) (recognizing at the same time that $\varphi(\beta_\nu(-k), -k, r) = \varphi(\beta_\nu(k), k, r)$), we obtain

$$\tilde{\Phi}_\lambda = \varphi_\lambda + A_\nu b_{\nu\lambda} = \frac{\varphi_\lambda}{|a_{\nu\mu}|}\left| a_{\nu\mu} - \frac{f_\mu b_{\nu\lambda}}{\varphi_\lambda}\right|,$$

$$b_{\nu\lambda} = [\varphi_\nu; \varphi_\lambda] / (\beta_\nu^2 - \lambda^2).$$

We consider the element

$$a_{\nu\mu} - \frac{f_\mu b_{\nu\lambda}}{\varphi_\lambda} = a_{\nu\mu}\left[1 - \frac{f_\mu[\varphi_\nu; \varphi_\lambda](\beta_\nu^2 - \alpha_\mu^2)}{\varphi_\lambda[\varphi_\nu; f_\mu](\beta_\nu^2 - \lambda^2)}\right].$$

To establish the asymptotic form of this expression as $r \to 0$, we used the following definition of the Jost function [8]

$$f(\lambda, \pm k) = \lim_{r\to 0}[2\lambda r^{\lambda - 1/2} f(\lambda, \pm k, r)]. \quad (23)$$

On the basis of (6) and (23) we have

$$\frac{f_\mu[\varphi_\nu; \varphi_\lambda]}{\varphi_\lambda[\varphi_\nu; f_\mu]} \to \frac{\beta_\nu - \lambda}{\beta_\nu + \alpha_\mu} \text{ as } r \to 0.$$

Consequently

$$\lim_{r\to 0}\left(a_{\nu\mu} - \frac{f_\mu b_{\nu\lambda}}{\varphi_\lambda}\right) = \frac{\lambda + \alpha_\mu}{\lambda + \beta_\nu} \lim_{r\to 0} a_{\nu\mu},$$

therefore

$$\lim_{r\to 0} \tilde{\Phi}_\lambda = \left\{\prod_1 (\lambda + \alpha_\mu) \Big/ \prod_1 (\lambda + \beta_\nu)\right\} \lim_{r\to 0} \varphi_\lambda,$$

and the function

$$\Phi_\lambda = \tilde{\Phi}_\lambda \prod_1^n (\lambda + \beta_\nu) \Big/ \prod_1^n (\lambda + \alpha_\mu)$$

is a solution of Eq. (1) that has the asymptotic form (6) as $r \to 0$, i.e., Φ_λ is a regular solution.

According to (4) and (12) we can represent Φ_λ in the form

$$\Phi(\lambda, k, r) = \tilde{\Phi}(\lambda, k, r) / R(-\lambda, k). \quad (24)$$

Substituting (21) and (24) in (22) and letting r go to infinity, we obtain

$$\Phi(\lambda, k, r) \sim e^{i[\eta(\lambda, k) + \delta(\lambda, k)]} \frac{f(\lambda, -k) R(\lambda, -k)}{k}$$

$$\times \sin\left[kr - \frac{a}{2k}\ln 2kr + \eta(\lambda, k) + \delta(\lambda, k)\right],$$

where

$$\frac{f(\lambda, k)}{f(\lambda, -k)} \frac{R(\lambda, k)}{R(\lambda, -k)} = \exp\{2i[\eta(\lambda, k) + \delta(\lambda, k)]\}$$

is the scattering function (2)–(4).

On the basis of (14), (15), and (20), the potential $V(r)$ is equal to

$$V(r) = -2\left[\left(-\frac{f_\mu \bar{a}_{\nu\mu}\varphi_\nu}{|a_{\nu\mu}| r^2}\right)' - \frac{f_\mu \bar{a}_{\nu\mu}\varphi_\nu}{|a_{\nu\mu}| r^3}\right] \quad (25)$$

$$= -2\left[\left(\frac{\bar{a}_{\nu\mu} a_{\nu\mu}'}{|a_{\nu\mu}|}\right)' + \frac{1}{r}\frac{\bar{a}_{\nu\mu} a_{\nu\mu}'}{|a_{\nu\mu}|}\right] = -2\left[\left(\frac{|a_{\nu\mu}|'}{|a_{\nu\mu}|}\right)' + \frac{1}{r}\frac{|a_{\nu\mu}|'}{|a_{\nu\mu}|}\right] = -2\Delta_r \ln|a_{\nu\mu}|,$$

where

$$\Delta_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right).$$

If the potential $V(r)$ is real at $r > 0$, then it follows from (1) and (7) that

$$f(\lambda, -k) = f^*(\lambda^*, k), \quad R(\lambda, -k) = R^*(\lambda^*, k).$$

Then the S function (2)–(4) will satisfy the unitarity condition, i.e., $S^{-1}(\lambda, k) = S^*(\lambda^*, k)$.

The results can be used to find the nuclear potential $V(r)$ from the "nuclear" phase shift $\delta(\lambda_j, k)$ that are known from the phase-shift analysis at different values of the orbital angular momentum ($\lambda_j = 1/2, 3/2, 5/2, \ldots$). These shifts depend not only on $V(r)$, but also on the Coulomb and on the centrifugal potentials. If they can be approximated in an acceptable manner by the function (3), then (25) can be used to calculate the pure nuclear potential.

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