Three types of solutions of the Einstein–Maxwell equations

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Three types of exact solutions of the Cauchy problem for the gravitation equations are obtained for centrally symmetric nonstationary worlds consisting of dust matter and of an electromagnetic field of charges. These solutions depend on two arbitrary functions of the radial coordinate and on a constant. An expression is derived for the maximum energy of particles accelerated by an electric field and scattered in the field of isotropic radiation in such worlds.

I. We consider 4-systems (worlds) of dustlike matter with energy density ϵ_s and an electromagnetic field with energy density ϵ_f, produced by charges of density ϵ_f, with the metric

$$ds^2=e^{\nu}d\tau^2-e^{\lambda}dr^2-R^2(d\theta^2+\sin^2\theta d\varphi^2)$$

in co-moving reference frames in gravitation theory [1]. If $R^{\cdot}\neq 0$ and $R'\neq 0$ (the dot and the prime denote the differentiation with respect to τ and r) everywhere except in a finite number of hypersurfaces, the problem reduces to the solution of the equations

$$\mathscr{E}=4\pi R(1+e^{-\nu}R^{\cdot 2}-e^{-\lambda}R'^2-\Lambda R^2/3)/\varkappa, \tag{1}$$

$$\mathscr{E}^{\cdot}=4\pi R^2R^{\cdot}\varepsilon_f, \tag{2}$$

$$\mathscr{E}'=4\pi R^2R'(\varepsilon_f+\varepsilon_s), \tag{3}$$

$$\varepsilon_f^{\cdot}+4\varepsilon_fR^{\cdot}/R=0, \tag{4}$$

$$\varepsilon_f'+4\varepsilon_fR'/R-(8\pi\varepsilon_f)^{1/2}\rho_fe^{\lambda/2}=0, \tag{5}$$

$$\rho_f^{\cdot}+\rho_f(\lambda^{\cdot}+4R^{\cdot}/R)/2=0, \tag{6}$$

$$\varepsilon_s^{\cdot}+\varepsilon_s(\lambda^{\cdot}+4R^{\cdot}/R)/2=0, \tag{7}$$

$$\varepsilon_f'+4\varepsilon_fR'/R-\varepsilon_s\nu'/2=0, \tag{8}$$

where $\kappa = 8\pi k/c^4$ and Λ is the cosmological term.

The integrals that determine the initial state of the worlds were obtained by Markov and Frolov [2]. Let us find the possible exact solutions of the Cauchy equation for (1)—(8). Integration of Eqs. (2), (4), (6), and (7) with respect to τ with allowance for (3), (5), and (8) leads to the appearance of three physically different functions of r:

$$\varepsilon_f=Q^2(r)/8\pi R^4,\quad \rho_f=\Phi(r)e^{-\lambda/2}/R^2,$$
$$\varepsilon_s=F(r)e^{-\lambda/2}/R^2,\quad \mathscr{E}=\mathscr{E}_r(r)-Q^2/2R,$$
$$Q'=4\pi\Phi,\ \mathscr{E}_r'=4\pi Ff(r),$$
$$f=R'e^{-\lambda/2}+QQ'/4\pi RF.$$

It remains to integrate (8) with respect to r and (1) with respect to τ, which are now transformed in the following manner:

$$\nu'=2QQ'e^{\lambda/2}f/\mathscr{E}_r'R^2, \tag{9}$$

$$e^{-\nu}R^{\cdot 2}=\varkappa(\mathscr{E}_r-Q^2/2R)/4\pi R+f^2(1-QQ'/\mathscr{E}_r'R)^2+\Lambda R^2/3-1. \tag{10}$$

There are at least three types of worlds for which the variables in (9) and (10) separate:

$$1.\ Q=Q_0,\qquad 2.\ R_g=R_{g0},\qquad 3.\ R_f=R_{f0},$$

where $R_g = \kappa\mathscr{E}_r/4\pi$, $R_f = Q^2/2\mathscr{E}_r$, the zero subscript labels the constants. For these we have

$$e^{\nu}=e^{\nu_\tau}\begin{cases}1\\ \mathrm{const}^2/R^2,\\ (1-R_{f0}/R)^2\end{cases}\qquad e^{\lambda}=\begin{cases}R'^2/f^2, & Q=Q_0\\ R'^2R^2/Q^2q^2, & R_g=R_{g0}.\\ R'^2/f^2(1-R_{f0}/R)^2, & R_f=R_{f0}\end{cases}$$

Here $\nu_\tau(\tau)$ is an arbitrary function of τ because of the admissibility of a transformation $\tau = \tau(\tau_1)$ [1], $q = \Phi/F$.

The solution of (10) can now be represented in quadratures:

$$\pm\int e^{\nu_\tau/2}d\tau=\int e^{-\nu_R/2}dR[\varkappa(\mathscr{E}_r-Q^2/2R)/4\pi R+(f-qQ/R)^2+\Lambda R^2/3-1]^{-1/2} \tag{11}$$

where $\nu_R(R) = \nu - \nu_\tau$. The integrals in (11) can be evaluated. At $\Lambda = 0$ they are expressed in terms of elementary functions, and yield the solutions for various nonstationary worlds. For lack of space, we cannot write them all out. We present one solution for $Q = Q_0$ and $f^2 < 1$:

$$R=R_g\left[1-\left(1-4R_f\frac{1-f^2}{R}\right)^{1/2}\cos\eta\right]\Big/2(1-f^2),$$
$$\pm\int e^{\nu_\tau/2}d\tau=R_g\left[\eta-\left(1-4R_f\frac{1-f^2}{R_g}\right)^{1/2}\sin\eta\right]\Big/2(1-f^2)^{3/2}. \tag{12}$$

The existence of the worlds (11) should be ensured by physically admissible initial conditions, which we obtain by integrating (1) with respect to r at $\tau = 0$:

$$\pm R'(0)e^{\lambda(0)/2}=[1-\Lambda R^2(0)/3-\varkappa(\mathscr{E}_r-Q^2/2R(0))/4\pi R(0)-e^{-\nu(0)}R^{\cdot 2}(0)]^{1/2}, \tag{13}$$

where the zero in the brackets indicates the initial values of the functions, and $\lambda(0)$ is an arbitrary function of r because of the admissibility of the transformation $r = r(r_1)$ [1].

The solution of (13) reduces to quadratures at least in the following cases:

1) $Q=Q_0,\quad R^{\cdot}(0)=0,\quad f=\pm\cos\left(\int e^{\lambda(0)/2}dr/2a_0\right);$

2) $Q=Q_0,\quad R^{\cdot}(0)=\pm 2^{1/2}e^{\nu(0)/2}\,\mathrm{sh}\left(\int e^{\lambda(0)/2}dr/2a_0\right),$
$f=\pm\mathrm{ch}\left(\int e^{\lambda(0)/2}dr/2a_0\right);$

3) $R_g=R_{g0},\quad R^{\cdot}(0)=0,\quad \varepsilon_f(0)+\varepsilon_s(0)=\varepsilon_0;$

4) $R_g=R_{g0},\quad R^{\cdot}(0)=0,\quad q=q_0;$

5) $R_g=R_{g0},\quad q=q_0,\quad \varepsilon_f(0)+\varepsilon_s(0)=\varepsilon_0;$

6) $R_f=R_{f0},\quad R^{\cdot}(0)=0,\quad \varepsilon_f(0)+\varepsilon_s(0)=\varepsilon_0;$

7) $R_f=R_{f0},\quad f=f_0,\quad \varepsilon_f(0)+\varepsilon_s(0)=\varepsilon_0;$

8) $R^{\cdot}(0)=0,\quad q=q_0,\quad \varepsilon_f(0)+\varepsilon_s(0)=\varepsilon_0.$

For example, for the world (12) the initial conditions 1) yield

$$R(0)=2a_0\sin\chi,$$
$$\pm\int e^{\lambda(0)/2}\,dr=2a_0\chi, \tag{14}$$
where $2a_0=(3/\kappa\epsilon_0)^{1/2}$.

Let us analyze the results.

1. The total charge of the system Q and the "free energy" of matter $\mathscr{E}_S^0$

$$Q=4\pi\int\rho_f R^2 e^{\lambda/2}\,dr=4\pi\int\Phi\,dr,$$
$$\mathscr{E}_s^0=4\pi\int\varepsilon_s R^2 e^{\lambda/2}\,dr=4\pi\int F\,dr,$$

do not depend on the time τ, and the functions Φ and F have the meaning of the linear densities of the charge and of the free energy of matter.

2. The total energy $\mathscr{E}_r$ of the system does not depend on the time and is equal to the sum of the total energies of substance $\mathscr{E}_S$ and of the field $\mathscr{E}_f$:

$$\mathscr{E}_s=4\pi\int\varepsilon_s R^2R'\,dr,$$
$$\mathscr{E}_f=4\pi\int\varepsilon_f R^2R'\,dr+Q^2/2R.$$

For example, in the world (12) with initial state (14) we have

$$\mathscr{E}_r=8\pi a_0\sin^3\chi/\varkappa+Q_0^2/4a_0\sin\chi.$$

The free energy of the substance in this world is

$$\mathscr{E}_s^0=12\pi a_0(\chi-{}^1/_2\sin 2\chi)/\varkappa+Q_0^2/4a_0\operatorname{tg}\chi$$

and is larger than its total energy by an amount equal to the gravitational mass defect.

3. The world 1 with constant charge Q_0 is a single charge placed in uncharged matter ($\Phi=0$). If there is no matter ($F=0$), then the static Reissner-Nordstrom world is obtained [2], in which

$$R\equiv r,\quad R^{\cdot}\equiv 0,\quad e^{\nu}=e^{-\lambda}=1-\Lambda r^2-\varkappa(\mathscr{E}_{r0}-Q_0^2/2r)/4\pi r.$$

4. In world 2 with constant gravitational radius R_{g0} (constant total energy $\mathscr{E}_{r0}$) and in world 3 with constant "classical radius" R_{f0}, the charge Q is distributed over the system. For example, the realization of the initial conditions 4) yields

$$R(0)=R_{g0}(1+\sin\chi)/2,\qquad \pm\int e^{\lambda(0)/2}\,dr=R_{g0}(\chi-\cos\chi)(\varkappa/8\pi q_0^2-1)^{1/2}/2;$$
$$Q=\pm R_{g0}\cos\chi/2q_0(\varkappa/8\pi q_0^2-1)^{1/2}.$$

We note that the free energy of the substance in the given world is always larger than the critical value $ec^2/k^{1/2}$, i.e., than the energy of the system whose gravitational radius is equal to the classical radius.

5. The worlds 1 and 3 go over at $Q=0$ into the Tolman-Friedmann worlds [1], while the world 2 is possible only in the presence of a charge, since constancy of the total energy in this world is ensured by cancellation of the gravitational attraction by the electrostatic repulsion of the charges.

II. Among the solutions (11) and (13) there are worlds in which the electric field is small during the period of maximum expansion, but reaches an appreciable value on definite sections of the earlier stage of their development.

We consider the motion of a particle with charge e and rest energy $\mathscr{E}_0$ in such systems. For the sake of generality, we take into account the energy loss by scattering in the field of isotropic radiation with energy density ϵ_W [1]:

$$\mathscr{E}_0\left(\frac{du^i}{ds}+\Gamma_{kl}{}^i u^k u^l\right)=eF^{ik}u_k+\sigma(T^{ik}u_k-u^iu_ku_lT^{kl}), \tag{15}$$

where $\sigma=8\pi(e^2/\mathscr{E}_0)^2/3$, and the nonzero components of the tensors F_{ik} and T^{ik} are

$$F_{01}=(8\pi\varepsilon_f)^{1/2}\exp[-(\nu+\lambda)/2],$$
$$T^{00}=\varepsilon_W e^{-\nu},\quad T^{11}=T^{22}=T^{33}=\varepsilon_W e^{-\lambda}/3.$$

For radial motion, the interval between events is equal to

$$ds=e^{\nu/2}(1-\beta^2e^{-\nu})^{1/2}d\tau,$$

where $\beta=e^{\lambda/2}dr/d\tau$. Let the Lorentz factor $\gamma=\mathscr{E}/\mathscr{E}_0$ be u_0:

$$\gamma=e^{\nu/2}(1-\beta^2e^{-\nu})^{-1/2}.$$

We can then rewrite (15) in the energy-dependent form

$$d\gamma/d\tau=-4\sigma\varepsilon_W(1-e^{\nu}/\gamma^2)\gamma^2/3\mathscr{E}_0+[\nu^{\cdot}-\lambda^{\cdot}(1-e^{\nu}/\gamma^2)]\gamma/2+e(8\pi\varepsilon_f)^{1/2}(1-e^{\nu}/\gamma^2)^{1/2}e^{\nu}/\mathscr{E}_0. \tag{16}$$

In the case of greatest interest, that of strongly curved space or a strongly relativistic particle, when $\gamma e^{-\nu/2}\gg 1$, Eq. (16) simplifies and takes the form of a Riccati equation

$$d\gamma/d\tau=-4\sigma\varepsilon_W\gamma^2/3\mathscr{E}_0+(\nu-\lambda)^{\cdot}\gamma/2+e(8\pi\varepsilon_f)^{1/2}e^{\nu}/\mathscr{E}_0. \tag{17}$$

Equation (17) can be solved rigorously if $\epsilon_W=0$. In the absence of radiation losses, the solution is

$$\gamma=\exp\left(\int(\nu-\lambda)^{\cdot}d\tau/2\right)\left[C_0+e\int(8\pi\varepsilon_f)^{1/2}\exp\left(\nu-\int(\nu-\lambda)^{\cdot}d\tau/2\right)d\tau\right],$$

where C_0 is a constant. In a neutral world ($\epsilon_f=0$), the particle energy decreases when the system expands ($\nu\to 0$, $\lambda\to 0$). This agrees with the conclusions [1] concerning the motion of particles in a Friedmann universe. In a charged world, it becomes possible for the energy of the arriving particles to increase as a result of their acceleration in the electric field.

It is impossible to integrate (17) for a system with radiation. But if the particle is acted upon by the accelerating field during the entire time, then we obtain an expression for the limiting energy

$$\gamma_m=3\mathscr{E}_0(\nu-\lambda)^{\cdot}\{1\pm[1+64\sigma e(8\pi\varepsilon_f)^{1/2}e^{\nu}/3\mathscr{E}_0^2(\nu-\lambda)^{\cdot 2}]^{1/2}\}/16\sigma\varepsilon_W, \tag{18}$$

where ϵ_f, ϵ_W, ν, and λ are taken at the point at which $d\gamma/d\tau=0$.

We consider the case when the limiting energy is reached at the instant of maximum expansion, which we shall call initial [2]. We choose as the initial condition $R^{\cdot}(0)=0$. As $R^{\cdot}\to 0$ we have $(\nu-\lambda)^{\cdot}\to 0$, and (18) simplifies to

$$\gamma_m=[3e(8\pi\varepsilon_f)^{1/2}e^{\nu}/4\sigma\varepsilon_W]^{1/2}.$$

The limiting energy of the accelerated particles is determined by the initial values of the energy densities of the field and of the isotropic radiation, and depends little on the electric field at the instant of maximum expansion. The use of the representation of the nonstationary charged world in cosmology (e.g., [3] entails the solution of a number of problems [4] that are beyond the scope of this paper.

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