Remarks on phase transition in a perfect Bose gas

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Thermodynamical singularities of a perfect Bose gas near the Bose condensation temperature T_0 are considered in the presence of a generalized field H that serves as a source of condensate particles. The dependence of the generalized angular momentum (the square of which gives the number of particles in the condensate) on the field H and on the temperature T near T_0 is quite peculiar and its form is identical to the corresponding equation of state for the two-dimensional Slater model with mobile impurities. The influence on the transition of randomly distributed pinned impurities is also investigated. Their presence results in the smoothing of the singularities in the thermodynamical quantities, and Bose condensation becomes possible in the one- and two-dimensional cases as well. However, since the Bose condensation in this case corresponds to localization of a macroscopically large number of particles in a microscopic region, the inclusion of the effect of finite compressibility is expected to lead to an essential alteration of the results and in particular to disappearance of the transition in the one-dimensional case.

1. INTRODUCTION

Singularities in the thermodynamic quantities at second-order phase transitions are widely discussed in literature. Particular interest has been attracted by the form of the equation of state in the critical region, i.e., by the dependence in this region of the order parameter on the temperature and on the generalized field (the latter induces that phase which is realized below T_0 in the absence of the field). The perfect Bose gas is the only exactly soluble three-dimensional model with a second-order phase transition (except for the somewhat non-physical spherical model [1]). It seems, therefore, interesting to discuss the form of the corresponding equation of state near the Bose-condensation point.

True, the above-mentioned generalized field has no direct physical meaning for a Bose gas, and corresponds to a certain source producing the Bose condensate particles. [2] However, as has been discussed in a number of works, e.g., [2,3] introduction of such a field makes the thermodynamical problem analogous to those of magnetic, ferroelectric, and the like transitions in the presence of a field. Therefore, the results can provide some information about such transitions. It will be shown that the appropriate equation of state is quite peculiar and is of the same form as the analogous equation in the ferroelectric two-dimensional Slater model with mobile impurities. [4]

We shall moreover consider the effect which randomly distributed pinned impurities may have on the character of transition. This question has also been discussed in literature. Thus, it has been shown in [5] that in various spin systems the presence of such impurities gives rise to "diffuse" transitions and smooths out all the thermodynamical singularities. It was pointed out in [6] that the presence of pinned impurities makes Bose condensation possible even in the one-dimensional case, despite the fact that no phase transition occurs in the pure one-dimensional Bose gas.

Following the arguments of [5] it will be shown that in the field of impurities the Bose condensation is "diffuse" as well, i.e., both the specific heat and the generalized susceptibility become less singular. As to the transition in the one-dimensional case, we shall note that Bose condensation in the field of impurities in this case actually corresponds to a state of infinite density, i.e., to a macroscopically large number of particles in a microregion. Therefore, any inclusion of the effect of finite compressibility should probably eliminate the possibility of such transitions. This is illustrated in the instance of a Bose-gas model with strong repulsion, considered in [3].

2. THERMODYNAMICS OF THE PERFECT BOSE GAS IN THE PRESENCE OF A GENERALIZED FIELD

The Hamiltonian for the perfect Bose gas in the presence of a generalized field H, creating condensate particles, can be written in the form: [2]

$$\mathcal{H}=\sum_\nu \varepsilon_\nu a_\nu{}^+a_\nu-\frac{1}{2}H\int d\mathbf{r}[\psi^+(\mathbf{r})+\psi(\mathbf{r})],\quad \psi(\mathbf{r})=\sum_\nu a_\nu\varphi_\nu(\mathbf{r}). \tag{1}$$

Here a_ν and a_ν^+ are the Bose operators, and ϵ_ν and $\varphi_\nu(\mathbf{r})$ are, respectively, the single-particle energy levels and wave functions at $H = 0$. If the motion of particles is free, then the functions $\varphi_\nu(\mathbf{r})$ are plane waves $V^{-1/2}\exp(i\mathbf{p}\cdot\mathbf{r})$. But as we are going to apply this to the case when impurities (i.e., scattering centers) are present, we shall use for now the general expression (1).

According to (1), the role of the order parameter, similar to the magnetic moment in a ferromagnet, is played by the quantity

$$M=\frac{1}{2}\int d\mathbf{r}\langle\psi^+(\mathbf{r})+\psi(\mathbf{r})\rangle=\frac{1}{2}\sum_\nu\langle a_\nu{}^+\bar\varphi_\nu{}^*+a_\nu\bar\varphi_\nu\rangle,\quad \bar\varphi_\nu=\int d\mathbf{r}\varphi_\nu(\mathbf{r}), \tag{2}$$

where the angle brackets denote Gibbs averaging. In what follows, this quantity will be referred to as the moment. Following the analogy, noted in [3], between the phase transitions in a Bose gas and in a lattice formed by two-dimensional dipoles, we can say that M and H precisely correspond to the moment of the lattice and the field acting on an individual dipole.

The thermodynamic potential Ω is connected with the Hamiltonian (1) by the usual relation

$$\Omega=-T\ln\operatorname{Sp}e^{-\beta(\hat{\mathcal{H}}-\mu\hat{\mathbf{N}})},\quad \hat N=\sum_\nu a_\nu{}^+a_\nu. \tag{3}$$

Here $\beta = 1/T$, $\hat{N}$ is the particle number operator, and the chemical potential μ is related to the number of particles in the system by $N = -\partial\Omega/\partial\mu$.

Formally, the Hamiltonian (1) does not conserve the total number of particles. However, as can be readily seen, the time derivative of the particle number density $n = N/V$ behaves like $\dot{n} \propto (a_0^+ - a_0)/\sqrt{V}$. Even though the quantities a_0^+ and a_0 are themselves of the order of $\sqrt{V}$, their difference is of the order of unity, and hence the particle number density is conserved in the statistical limit as $N \to \infty$ and $V \to \infty$.

The operator $\hat{\mathscr{H}} - \mu\hat{N}$, with $\hat{\mathscr{H}}$ given by (1), can be diagonalized by a canonical transformation $\hat{a}_\nu = \alpha_\nu + f_\nu$ where the constant f_ν is determined by the condition that in $\hat{\mathscr{H}} - \mu\hat{N}$ have no terms linear in α_ν and α_ν^+. As a result, we find from (3) that

$$\Omega = -\frac{1}{4}H^2 \sum_\nu \frac{|\overline{\varphi}_\nu|^2}{\varepsilon_\nu - \mu} + T \sum_\nu \ln[1 - e^{\beta(\mu - \varepsilon_\nu)}]. \tag{4}$$

For the case of free particles $\overline{\varphi}_\nu$, given by (2), equals $V^{1/2}\delta_{\mathbf{p}0}$, and (4) assumes the form

$$\frac{\Omega}{V} = \frac{H^2}{4\mu} + \frac{Tg}{(2\pi\hbar)^3}\int d\mathbf{p} \ln[1 - e^{\beta(\mu - \varepsilon_\mathbf{p})}], \tag{5}$$

where $\epsilon_\mathbf{p} = p^2/2m$, $g = 2S + 1$, and S is the spin of the particle. Differentiating (5) with respect to μ, we find

$$n = \frac{N}{V} = \frac{H^2}{4\mu^2} + \frac{g}{(2\pi\hbar)^3}\int \frac{d\mathbf{p}}{\exp[\beta(\varepsilon_\mathbf{p} - \mu)] - 1}. \tag{6}$$

Expressions (5) and (6) determine the thermodynamics of the system. In particular, the generalized moment M and associated susceptibility χ are given by

$$\frac{M}{V} = -\frac{1}{V}\frac{\partial\Omega}{\partial H} = -\frac{H}{2\mu}, \quad \chi = \frac{1}{V}\frac{\partial M}{\partial H} = -\frac{1}{2\mu}. \tag{7}$$

It will be recalled that for the perfect Bose gas $\mu < 0$[7], and hence both M and χ are positive.

We shall now consider the thermodynamics in the vicinity of the Bose condensation point T_0 in the absence of the field, i.e., near the temperature at which the second term in (6) equals n at $\mu = 0$. The explicit expression for T_0 is given, e.g., in [7]. We then obtain from (6) for small H and $\tau = (T - T_0)/T_0 \ll 1$ the following expression

$$\gamma\tau + \frac{H^2}{4\mu^2} = \sqrt{\frac{2|\mu|}{\lambda}}, \tag{8}$$

where the constants γ and λ are given by

$$\gamma = {}^3/_2 n, \quad \lambda = T_0\zeta^2({}^3/_2)(2\pi n^2)^{-1}, \quad \zeta({}^3/_2) = 2.612. \tag{9}$$

Going over from μ to M according to Eq. (7), and replacing the magnetization M and field H by dimensionless quantities m and h defined by the relations $M = mV\gamma^{1/2}$ and $H = h\gamma^{5/2}\lambda$, we obtain the equation of state $h(m, \tau)$ in the following form

$$h = m(m^2 + \tau)^2. \tag{10}$$

Equation (10) is identical in form to the analogous equation for the two-dimensional Slater model with mobile impurities near the upper transition temperature T_2.[4] In that model the specific heat C_V in the absence of the field has a kink at T and its derivative suffers a discontinuity. This again coincides with the behavior of C_V and dC_V/dT in the three-dimensional perfect Bose gas under consideration. Thus, these two models have the same thermodynamical singularities at the phase transitions.

As was pointed out in [4], Eq. (10) suggests rather peculiar magnetic properties. Let us give some asymptotic relations:

$$\chi = 1/\tau^2 \quad \text{for} \quad \tau > 0, \quad h \ll \tau^{5/2}; \tag{11a}$$

$$\chi = 1/5h^{4/5} \quad \text{for} \quad h \gg |\tau|^{5/2}; \tag{11b}$$

$$\chi = 1/4|\tau|^{3/4}h^{1/2}, \quad m = \sqrt{|\tau|} \quad \text{for} \quad \tau < 0, \quad h \ll |\tau|^{5/2}. \tag{11c}$$

Thus, in a weak field and at $T \to T_0 + 0$ the susceptibility increases as τ^{-2}, whereas for all $T < T_0$ and small h it rises as $h^{-1/2}$. The magnetization m increases proportionally to the square root of $|\tau|$, as it does in the phenomenological theory of phase transitions.[7] The square of the spontaneous moment M^2 represents the condensate particle density $N_0 = {}^3/_2 N\tau$, which is an expansion near T_0 of the exact expression[7] $N_0 = N[1 - (T/T_0)^{3/2}]$.

In contrast to C_V, the specific heat at constant pressure C_p has a singularity at $T \to T_0$. Expressing C_p through C_V[7] and using the thermodynamic identity $(\partial P/\partial V)_T = -n(\partial\mu/\partial N)_T$, we obtain the following asymptotic forms for the singular part of C_p in the regions considered in (11):

$$\text{a)}\ C_p \sim \frac{1}{\tau}; \quad \text{b)}\ C_p \sim \frac{1}{h^{2/5}}; \quad \text{c)}\ C_p \sim \frac{|\tau|^{3/2}}{h}. \tag{12}$$

It should be noted that the singularity in C_p at $h = 0$ and $\tau \to +0$ is the same as in the phenomenological theory of the critical point ([7], Sec. 84). We also note that the particle correlation radius r_c ([7], Sec. 120) increases on approach to T_0 as well: $r_c \sim |\mu|^{-1/2}$. In particular, at $h = 0$, $\tau \to +0$ this gives $r_c \sim 1/\tau$ which is markedly different from the Ornstein and Zernike result $r_c \sim \tau^{-1/2}$ ([7], Sec. 119). For $h \gg \tau^{5/2}$ the radius r_c increases as $h^{-2/5}$.

It can be shown that the critical exponents for C_V, χ, m, and r_c satisfy all the relations of the scaling hypothesis which is usually employed to describe the singularities at second-order phase transitions (see, e.g., [8]). If we write the singular parts of these quantities at $h = 0$, $\tau \ll 1$ and at $\tau = 0$, $h \ll 1$, respectively, in the form

$$C_V \sim [\tau^{-\alpha}, h^{-\varepsilon}], \quad \chi \sim [\tau^{-\gamma}, h^{-\lambda}], \quad m \sim [(-\tau)^\beta, h^{1/\delta}], \quad r_c \sim [\tau^{-\mu}, h^{-\rho}],$$

and the correlation function $K(\mathbf{r})$ at $h = 0$, $\tau = 0$ in the form $r^{-\nu}$, then in our model we have for the critical exponents

$$\alpha = -1, \quad \varepsilon = -{}^2/_5; \quad \gamma = 2, \quad \lambda = {}^4/_5;$$
$$\beta = {}^1/_2, \quad \delta = 5; \quad \mu = 1, \quad \rho = {}^2/_5; \quad \nu = 1.$$

Note that negative values for the exponents α and ϵ, which characterize the singularity in the specific heat, correspond to finiteness of C_V at the transition point.

As is well-known, in the one- or two-dimensional cases there is no phase transition in a perfect Bose gas. Still, at low temperatures both the correlation radius and the susceptibility χ increase sharply. For the two-dimensional case (6) is replaced by

$$n = \frac{H^2}{4\mu^2} - \varkappa T \ln(1 - e^{\beta\mu}), \quad \varkappa = \frac{mg}{2\pi\hbar^2}. \tag{13}$$

(Here and below m is the particle mass.) Whence we obtain for χ at $H = 0$

$$\chi = \frac{1}{2|\mu|} = -\frac{1}{2T}\left\{\ln\left[1 - \exp\left(-\frac{n}{\varkappa T}\right)\right]\right\}^{-1}. \tag{14}$$

Thus, in the two-dimensional perfect Bose gas at low T the quantities χ and r_c grow exponentially, as in the one-dimensional Ising model. In the one-dimensional case we find for small T

$$-\mu = {}^1/_2 m(gT/n\hbar)^2, \tag{15}$$

which gives $\chi \sim 1/T^2$ and $r_c \sim 1/T$.

3. PERFECT BOSE GAS IN A RANDOM EXTERNAL FIELD

We shall now consider Bose condensation in the presence of randomly distributed pinned impurities. The effect of each impurity will be described as an external repulsive field acting on the system. The Hamiltonian will still be given by (1), but ϵ_ν and $\varphi_\nu(\mathbf{r})$ will now represent the particle energies and wave functions in the random field of impurities at H = 0. The thermodynamic potential is given by Eq. (4) which can be written in the form

$$\frac{\Omega}{V}=-\frac{H^2}{4}\int_0^\infty\frac{\Phi(\varepsilon)\,d\varepsilon}{\varepsilon-\mu}+T\int_0^\infty\ln[1-e^{\beta(\mu-\varepsilon)}]\rho(\varepsilon)\,d\varepsilon, \tag{16}$$

where

$$\rho(\varepsilon)=\frac{1}{V}\sum_\nu\delta(\varepsilon-\varepsilon_\nu),\quad \Phi(\varepsilon)=\frac{1}{V}\sum_\nu|\varphi_\nu|^2\delta(\varepsilon-\varepsilon_\nu). \tag{17}$$

$\rho(\epsilon)$ is interpreted as the density of states for a particle in the field of impurities. This function, as well as $\Phi(\epsilon)$, can be expressed in terms of the retarded Green's function G_R of a particle in this field:[9]

$$\rho(\varepsilon)=\frac{1}{V}\int d\mathbf{r}\,\frac{1}{\pi}\operatorname{Im}G_R(\mathbf{r},\mathbf{r}),\quad \Phi(\varepsilon)=\frac{1}{V}\int d\mathbf{r}\,d\mathbf{r}'\,\frac{1}{\pi}\operatorname{Im}G_R(\mathbf{r},\mathbf{r}'). \tag{18}$$

The exact expressions for the spectrum and the wavefunctions of a particle in a random field are unknown for the three-dimensional case. However, only small values of ϵ in (16) will matter in describing the transition, specifically, $\epsilon \ll U$, $\hbar^2/ma^2$, where U is the height of the potential due to a single impurity, and a is its range. Then, for the case of low impurity concentrations c (to which, for simplicity, we shall confine ourselves in the sequel), just from dimensionality considerations, similar to those used in[10,9], it is natural to assume that for small ϵ the functions $\rho(\epsilon)$ and $\Phi(\epsilon)$ are of the form

$$\rho(\varepsilon,c)=\rho_0(\varepsilon)f\left(\frac{\varepsilon}{c^\alpha}\right),\quad \Phi(\varepsilon,c)=\frac{1}{c^\alpha}g\left(\frac{\varepsilon}{c^\alpha}\right). \tag{19}$$

Here $\alpha = 2/d$, d being the space dimensionality, and $\rho_0(\epsilon)$ is the density of free particle states: $\rho_0(\epsilon) = \mathrm{const} \times \epsilon^{d/2-1}$. At $x \to \infty$ the function f(x) tends to unity, and at $x \ll 1$ both f(x) and g(x) exponentially vanish: f, g $\sim \exp(-x^{d/2})$. The factor $c^{-\alpha}$ in front of g in (19) is determined by the condition

$$\int_0^\infty\Phi\,d\varepsilon=1. \tag{20}$$

which must be fulfilled in order that Eq. (16) should go over into (5) as $c \to 0$.

Differentiating (16) with respect to μ we obtain an equation for $\mu(T, H, c)$:

$$n=\frac{H^2}{4}\int_0^\infty\frac{\Phi(\varepsilon)\,d\varepsilon}{(\varepsilon-\mu)^2}+\int_0^\infty\frac{\rho(\varepsilon)\,d\varepsilon}{\exp[\beta(\varepsilon-\mu)]-1} \tag{21}$$

The transition temperature $T_0(c)$ is determined from Eq. (21) at H = 0 and $\mu = 0$. At the considered small c, the shift in the transition point $\delta T_0 = T_0(c) - T_0(0)$, according to (19), is proportional to $c^{1/3}$ in the three-dimensional case. Moreover, Eq. (21) suggests that $\delta T_0 > 0$, since in the important region of small ϵ the level density decreases, $f(x) - f(\infty) < 0$.

The above-mentioned exponential decrease in $\rho(\epsilon)$ and $\Phi(\epsilon)$ tends to smooth out the singularities in all the thermodynamic quantities near T_0. For example, the susceptibility χ given by the coefficient at $H^2/2$ in (16) becomes finite at all temperatures. In particular, in the three-dimensional case at H = 0 and $\tau \ll c^{1/3}$ we find from (19) and (16) that $\chi \sim c^{-2/3}$, and at $\tau \gg c^{1/3}$ we have $\chi \sim \tau^{-2}$, as before. The specific heat C_V is infinitely differentiable as $T \to T_0 \pm 0$ even though it suffers a jump at the point of transition: $\delta C_V = C_V(T_0 - 0) - C_V(T_0 + 0) \sim c^{1/3}$. The specific heat C_p remains finite at $T \to T_0$: $C_p(T_0) \sim c^{-1/3}$. At $T < T_0$ and in the presence of field H the chemical potential is nonzero only if the field exceeds a certain critical value $H_c(T)$. Near T_c, in particular, we find from (21) that $|\mu| = a\tau + bH^2$, where the constants a, b > 0 so that for small τ we have $H_c(T) \sim \sqrt{T_0 - T}$.

As in the impurity-free case, the condensate particle density near T_0 depends on τ linearly: $N_0 \sim N(T_0 - T)$. However, the spontaneous magnetization $M/V = -1/V(\partial\Omega/\partial H)_{H=0}$ is absent below T_0. Indeed, let us denote by ν_0 the lowest energy state at which the condensation occurs. The quantity $\overline{\varphi}_{\nu_0}$ which, according to (2), determines the moment M, will be a finite microscopic quantity in the presence of impurities, in contrast to the impurity-free case when $\overline{\varphi}_{\nu_0} = \sqrt{V}$. The distinction is associated with the fact that, as pointed out in[10,11], in the presence of impurities the lowest energy states are always localized. Therefore, in the statistical limit $V \to \infty$, $N/V = \mathrm{const}$ the quantity $\overline{\varphi}_{\nu_0}N_0^{1/2}/V$ vanishes, so that in the absence of the field the moment is zero at all T.

All these results remain also valid for the one-dimensional (and two-dimensional) case, discussed in[6], if we replace the concentration C in the above relations by c^3 (or, respectively, by c^2). In the one-dimensional case the functions $\rho(\epsilon)$ and $\Phi(\epsilon)$ in Eqs. (16) and (19) can be written down explicitly (for small ϵ we are interested in). The function $\rho(\epsilon)$ was found in[9], and $\Phi(\epsilon)$ enters also in the solution of a random-walk problem considered in[12]. Therefore, if we write the result of[12] for the probability $\overline{W}(t)$ in the one-dimensional case in the form

$$\overline{W}(t)=\int_0^\infty\Phi(\varepsilon)e^{-\varepsilon t}d\varepsilon, \tag{22}$$

then, $\Phi(\epsilon)$ can be found by comparing (22) with Eq. (12) from[12]:

$$\rho(\varepsilon)=\frac{\pi c^2}{4k^3}\operatorname{sh}^{-2}\frac{c\pi}{2k},\qquad \Phi(\varepsilon)=\frac{4c^2}{k^4}\operatorname{sh}^{-1}\frac{c\pi}{k}, \tag{23}$$

where $k = (2m\epsilon/\hbar^2)^{1/2}$. Formulae (23) illustrate the general properties of the functions ρ and Φ in (19) which have been discussed above.

For the one-dimensional case Eqs. (16), (21), and (23) yield $T_0(c) \sim nc^2$, and, in view of the above, $\chi(T_0) \sim 1/c^2$, and $N_0 \sim T_0 - T$. The specific heat $C_V(T_0 \pm 0)$ is infinitely differentiable and $\delta C_V \sim c$. For the two-dimensional case $T_0(c) \sim n(\ln(n/c))^{-1}$, $\chi(T_0) \sim 1/c$, $\delta C_V \sim \sqrt{c}$, and so on.

For the most part, the results obtained agree with those of[5], especially in the region $T > T_0$: all the thermodynamic quantities become finite and infinitely differentiable at the transition point. On the face of it, this seems to corroborate the point made in[5] that the presence of pinned impurities results in "diffuseness" of the thermodynamical singularities at a transition.

However, certain features of the model considered demand great caution if we want to compare its results with more realistic models. The trouble is that the lowest energy level, at which condensate particles are to be found, would correspond to an impurity-free region of maximum possible size V_0. This arises from the fact that the impurities surrounding this region create a potential barrier on its boundary, so that the particles "locked up" in the region have the lowest energy $\epsilon_0 \sim \hbar^2/mV_0^{2/3}$[9-11]. But the probability to find such a region for a Poisson distribution of impurities is equal to $c \exp(-cV_0)$. Therefore, as was mentioned above, in the statistical limit, $N, V \rightarrow \infty$, $N/V = \text{const}$ we must assume that $V_0/V \rightarrow 0$, or else $\exp(-cV_0) \equiv 0$. We thus come to the conclusion that in such a system all the condensate particles fall into a region of a microscopic size, i.e., their density in space is infinite: $N_0/V_0 \rightarrow \infty$ as $N \rightarrow \infty$. It can be easily seen that these results will still be valid if we allow for the possibility of condensation into a group of lowest impurity levels and that the present argument pertains to the case of arbitrary (one, two, or three) dimensions.

The fact that Bose condensation makes no physical sense under such circumstances is also apparent in the case of impurities with an attractive potential. One such impurity with an attraction sufficient to create a bound state of a single particle is enough for the whole perfect Bose gas to condense into this level, i.e., a macroscopically large number of particles $N_0 \sim N$ would appear in a micro-region around that impurity.

Incidentally, similar arguments seem to be appropriate in considering Bose condensation in a gravitational field.[13] In this case, like in the presence of impurities, the singularity in the specific heat C_V again changes from a kink, a discontinuity in dC_V/dT, to a jump δC_V. However, an examination of the state of particles in the condensate shows that the Bose condensation in this case corresponds to localization of particles with microscopic height near the container's bottom.

Thus, any inclusion of the effect of finite compressibility (repulsion at small separations) should probably banish some of the above features of Bose condensation in the field of impurities, notably the one-dimensional Bose condensation.

To illustrate this assertion we shall consider a model of the Bose gas with strong repulsion, proposed in[3]. According to that work, the singular part of the Bose-gas partition function can be reduced to the partition function for a lattice of planar dipoles with a nearest-neighbor interaction. In doing so, the interaction constant J is expressed through the interaction parameters and the density of the Bose gas. In the one-dimensional case this model corresponds to a chain of planar dipoles. The presence of the above pinned impurities of concentration c can then be described by introducing randomly distributed impurity bonds $J' \neq J$ of a relative concentration $c_2 \sim c/n$. The resulting partition function is analogous to that discussed in[14] for the one-dimensional Ising model with impurities, and is of the form

$$Z \sim [I_0(\beta J))^{Nc_1}[I_0(\beta J')]^{Nc_2}. \tag{24}$$

Here $I_0(x)$ is the Bessel function of an imaginary argument and $c_1 = 1 - c_2$ is the concentration of "normal" bonds J. Other ways of simulating the effect of impurities in the chain also yield similar results (see, for example, [14]).

Expression (24) has no singularities at any temperature, which indicates that no phase transition occurs. It is, therefore, believed that allowance for repulsion indeed leads to the disappearance of the phase transition in a one-dimensional Bose gas with impurities.

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