Nonlinear theory of parametric excitation of spin waves in antiferromagnets

V. S. L'vov and M. I. Shirokov

Institute of Automation and Electrometry, Siberian Branch, USSR Academy of Sciences
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It is shown that a nonlinear theory of parametric excitation of waves, using the approximation of an interaction Hamiltonian that is diagonal in pairs of waves [11,12] and taking account of the scattering of waves by random inhomogeneities [20] (the *S* theory), gives a good description of the real situation beyond the threshold of the parametric instability of spin waves (SW) in antiferromagnets with anisotropy of the easy-plane type (EP AFM). In pure crystals the anomalously large dipole-dipole contribution to the interaction of the SW leads to sharp anisotropy of their stationary distribution in **k** space. Two-magnon scattering by random defects in the AFM sample makes the SW distribution function isotropic and weakens the pair correlations, and this makes it possible to explain the experimentally observed magnitude temperature dependence of the threshold field. The calculated dependences of the nonlinear susceptibilities χ' and χ'' of the SW system on the supercriticality and on the external field are in qualitative agreement with the experimental dependences from [5,6].

The phenomenon of parametric excitation of spin waves (SW) in antiferromagnets with anisotropy of the easy-plane type (EP AFM) ($CsMnF_3$ [1-3] and $MnCO_3$ [4-7]) has recently been discovered and investigated. The papers [8-10] are devoted to the linear theory of this phenomenon, and the threshold amplitude h_1 of the pumping field was correctly calculated in [8, 9].

A nonlinear theory of parametric excitation of waves, permitting one to describe in detail the phenomena arising in a system of interacting waves when $h > h_1$, was developed in the papers of Zakharov, L'vov and Starobinets [11, 12]. This theory (called the S-theory) was constructed for media that can be described in the framework of the classical Hamiltonian formalism, and is based on simplifying the Hamiltonian of the system of waves to the form

$$\mathcal{H}=\sum_{\mathbf{k}}[\omega_{\mathbf{k}}+\sum_{\mathbf{k}'}T_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{\mathbf{k}'}^{*}]a_{\mathbf{k}}a_{\mathbf{k}}^{*}$$
$$+\frac{1}{2}\sum_{\mathbf{k}}[(h\bar{V}_{\mathbf{k}}+\sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{-\mathbf{k}'})a_{\mathbf{k}}^{*}a_{-\mathbf{k}}^{*}+\text{c.c.}]. \quad (1)$$

Here $\omega_{\mathbf{k}}$ is the dispersion law of the waves, $a_{\mathbf{k}}$ are their complex amplitudes, $\bar{V}_{\mathbf{k}}$ is the effective coefficient of the coupling with the uniform pumping field $h(t) = he^{-i\omega_p t}$, the coefficients $T_{\mathbf{k}\mathbf{k}'}$ describe the nonlinear frequency shift, and the coefficients $S_{\mathbf{k}\mathbf{k}'}$ are the nonlinear parametric interaction of the waves with each other. Such a choice of Hamiltonian corresponds to replacing the exact problem of the interaction of the waves by the problem of the self-consistent interaction of pairs of waves with equal and opposite wave vectors. the justification for this is provided by the "pairing"–the phase correlation of waves in pairs that arises under the action of the pumping.

In this paper we study the concrete problem of the behavior of a system of interacting SW beyond the threshold of their parametric excitation in the EP AFM. In Sec. 1 we discuss the Hamiltonian (2) of this problem and show that in the final analysis it can be brought to the standard form (1) of the S-theory. Here also we give the expressions for the coefficients $\bar{V}_{\mathbf{k}}$ (12), $S_{\mathbf{k}\mathbf{k}'}$ (5) and $T_{\mathbf{k}\mathbf{k}'}$(6) for an EP AFM, which we have obtained taking into account the exchange, dipole-dipole and Zeeman interactions, the Dzyaloshinskiĭ interaction, the field of the uniaxial crystalline anisotropy, and the interaction of the electron spins with the nuclear spins.

In the next section, the stationary amplitudes and phases of the parametrically-excited SW are found. It is shown, in particular, that in the framework of the S-theory the SW should be concentrated on two (or four) lines in **k**-space, lying close to the resonance surface.

Next, in Sec. 3, we investigate the important question of the influence of random magnetic inhomogeneities (impurities, defects) on the threshold of parametric excitation of SW in an EP AFM, and the behavior of the SW beyond the threshold. When there are so many impurities that the damping $\tilde{\gamma}_{\text{imp}}$ of the SW by them exceeds the damping $\gamma_{\mathbf{k}}(T)$ of the SW in the pure AFM, the distribution of the SW over the resonance surface becomes isotropic and the width in $|\mathbf{k}|$ of the excited region is found to be of the order of $\tilde{\gamma}_{\text{imp}}/(\partial\omega_{\mathbf{k}}/\partial k)$. Scattering by random inhomogeneities reduces the phase correlations in the SW pairs; this leads to weakening of their parametric interaction with each other and with the pumping. As a result, the threshold pumping amplitude h_1 and the integral amplitude of the SW are increased by a factor of $[\tilde{\gamma}_{\text{imp}}/\gamma_{\mathbf{k}}(T)]^{1/2}$ by comparison with the pure AFM. In particular, $h_1\bar{V} = \{\langle\gamma_{\mathbf{k}}\rangle[\langle\gamma_{\mathbf{k}}\rangle + \tilde{\gamma}_{\text{imp}}]\}^{1/2}$, where $\langle\gamma_{\mathbf{k}}\rangle$ is the value of the intrinsic damping averaged over the angles.

With the appropriate choice of $\tilde{\gamma}_{\text{imp}}$, this formula correctly describes the absolute value and the temperature dependence of $h_1(T)$ that are observed in experiment [5]. This confirms the theoretically predicted phenomenon of weakening of the phase correlations in a system of parametric waves in a medium with random inhomogeneities.

The results obtained in Secs. 2 and 3 are used in Sec. 4 to calculate the nonlinear susceptibility $\chi = \chi' + i\chi''$ of an EP AFM. It is shown that the theoretical dependences of χ' and χ'' on the supercriticality, the external field and the temperature are in qualitative accord with experiment. However, the absolute values of $(\chi - \chi_0)$ and χ'' in the theory are found to be somewhat too high. Moreover, in experiment the excitation of the SW is observed to be "hard" in character [6]. In [13] we attribute both circumstances to the positive and negative contributions that arise in the damping of parametric SW owing to their interaction with thermal SW.

Thus, the results of the present work show that an

S-theory taking into account the interaction of parametrically excited pairs of SW with each other and their scattering by random inhomogeneities describes the real situation beyond the threshold of the parametric instability of SW in the EP AFM. In our opinion, the S-theory and further improvements of it, associated, e.g., with allowance for the interaction of parametric SW with thermal SW, non-pair interaction of parametric SW, etc., can and should serve as the basis for the interpretation and formulation of new "parametric" experiments in the study of antiferromagnets.

1. THE BASIC EQUATIONS

We shall write out the Hamiltonian of the spin system of an AFM placed in an external magnetic field $\mathbf{h}(t) = \mathbf{h}\cos\omega_p t$ polarized parallel to a constant field H_0 lying in the easy plane:

$$\mathcal{H}=\sum_{\mathbf{k}}(\omega_{\mathbf{k}}a_{\mathbf{k}}a_{\mathbf{k}}^{*}+\Omega_{\mathbf{k}}b_{\mathbf{k}}b_{\mathbf{k}}^{*})+2hU(b_0+b_0^{*})\cos\omega_p t$$
$$+\frac{1}{2}\sum_{\mathbf{k}}[h(V_{\mathbf{k}}e^{-i\omega_p t}a_{\mathbf{k}}^{*}a_{-\mathbf{k}}^{*})+\text{c.c.}]+\mathcal{H}_{int}, \tag{2}$$
$$\mathcal{H}_{int}=\mathcal{H}^{(3)}+\mathcal{H}^{(4)}.$$

Here the canonical variables $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are the complex amplitudes of the traveling SW belonging to the "quasiferromagnetic" and "antiferromagnetic" branches of spectrum, respectively; $\omega_{\mathbf{k}}$ and $\Omega_{\mathbf{k}}$ are the frequencies of these waves. With neglect of the dipole-dipole interaction,

$$\omega_{\mathbf{k}}^2=g^2[H_0(H_0+H_D)+\alpha/T]+(sk)^2,$$
$$\Omega_{\mathbf{k}}^2=g^2[H_D(H_0+H_D)+2H_AH_{ex}]+(sk)^2,$$

where g is the gyromagnetic ratio, H_D is the Dzyaloshinskiĭ field, and H_{ex} and H_A are the exchange field and the uniaxial-anisotropy field. The constant α describes the interaction with the nuclear spins, T is their temperature, and the velocity s characterizes the nonuniform exchange interaction: the characteristic length $l \equiv s/gH_{ex}$ is equal to the lattice constant in order of magnitude.

The most important terms for us in the three-wave interaction Hamiltonian have the form

$$\mathcal{H}^{(3)}=\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3}\left\{\frac{1}{2}V_{1,23}^{(1)}b_1a_2^{*}a_3^{*}+V_{1,23}^{(2)}a_1b_2^{*}a_3^{*}+\text{c.c.}\right\}$$
$$+\frac{1}{2}\sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0}\{U_{123}b_1^{*}a_2^{*}a_3^{*}+\text{c.c.}\}+\ldots. \tag{3}$$

Recognizing that all the parametric SW have similar frequencies, we keep only the following terms in the four-wave Hamiltonian:

$$\mathcal{H}^{(4)}=\frac{1}{2}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4}T_{12,34}a_1^{*}a_2^{*}a_3a_4.$$

It must be said that a contribution to the interaction of four waves with the conservation law

$$\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}=\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4}$$

is made not only by the Hamiltonian $\mathcal{H}^{(4)}$, but also by the Hamiltonian $\mathcal{H}^{(3)}$ in second order of perturbationtheory. It can be shown that the most important contribution in the parameter $(\omega_p/\omega_{ex})^2 \ll 1$ will be that of the processes, written out explicitly in (3), in which a virtual wave from the upper branch participates. Thus, the effective Hamiltonian of the interaction of the parametric waves amongst themselves has the form

$$\mathcal{H}^{(4)}=\frac{1}{2}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4}\tilde{T}_{12,34}a_1^{*}a_2^{*}a_3a_4. \tag{4}$$

We have given complete expressions for the coefficients in the Hamiltonians (2), (3) and (4) in[13]. In the following we shall need only the coefficients of the Hamiltonian (4) that are diagonal in pairs of waves $\pm\mathbf{k}$, viz, $\tilde{T}_{\mathbf{k}-\mathbf{k},\mathbf{k}'-\mathbf{k}'}=S_{\mathbf{k}\mathbf{k}'}$ and $\tilde{T}_{\mathbf{k}\mathbf{k}',\mathbf{k}\mathbf{k}'}=T_{\mathbf{k}\mathbf{k}'}$:

$$S_{\mathbf{k}\mathbf{k}'}=S_0+iS_1(x_{\mathbf{k}}-x_{\mathbf{k}'})+S_2x_{\mathbf{k}}x_{\mathbf{k}'}+S_3x_{\mathbf{k}}x_{\mathbf{k}'}\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2 \tag{5}$$

$$T_{\mathbf{k}\mathbf{k}'}=T_0+T_2\left\{\frac{x_{\mathbf{k}+\mathbf{k}'}^2}{2}+\frac{(x_{\mathbf{k}}+x_{\mathbf{k}'})^2}{8}\right\}, \tag{6}$$

where $x_{\mathbf{k}} = 2k_yk_z/k^2$ in the coordinate frame in which z is the "difficult" axis and x is the direction of $\mathbf{H}_0$;

$$S_0=T_0=-\frac{g^2B}{8\omega_{\mathbf{k}}^2}\left\{\omega_{01}^2+(gH_0)^2\left(\frac{3\Omega_0^2+\omega_p^2}{\Omega_0^2-\omega_p^2}\right)\right\},$$
$$S_1=-g^2B\frac{\pi gM_0}{\omega_{\mathbf{k}}}\frac{\omega_{ex}}{\Omega_0}\frac{gH_0}{2\omega_{\mathbf{k}}},$$
$$S_2=T_2=-g^2B\left(\frac{\pi gM_0}{\omega_{\mathbf{k}}}\right)^2\left(\frac{\omega_{ex}}{\Omega_0}\right)^2. \tag{7}$$
$$S_3=4S_2(\omega_{ex}lk/\Omega_0)^4,\ B\equiv H_{ex}/M_0,$$

where M_0 is the sublattice magnetization. We note that if the dipole-dipole interaction is disregarded, $T_{\mathbf{k}\mathbf{k}'} = S_{\mathbf{k}\mathbf{k}'} = S_0$. Incidentally, $S_2 > S_1 > S_0$. We now write the canonical equations of motion for $a_{\mathbf{k}}$ with allowance for the intrinsic damping:

$$\frac{da_{\mathbf{k}}}{dt}+\gamma_{\mathbf{k}}a_{\mathbf{k}}=-i\frac{\delta\mathcal{H}}{\delta a_{\mathbf{k}}^{*}}.$$

Using the Hamiltonian (2) and keeping, in accordance with the S-theory, the terms diagonal in the pairs in $\mathcal{H}_{\text{int}}^{(4)}$ (4), we obtain the equation

$$\left[\frac{d}{dt}+\gamma_{\mathbf{k}}+i\tilde{\omega}_{\mathbf{k}}\right]a_{\mathbf{k}}+iP_{\mathbf{k}}a_{-\mathbf{k}}^{*}=0, \tag{8}$$

where

$$\tilde{\omega}_{\mathbf{k}}=\omega_{\mathbf{k}}+2\sum_{\mathbf{k}'}T_{\mathbf{k}\mathbf{k}'}|a_{\mathbf{k}'}|^2, \tag{9}$$

$$P_{\mathbf{k}}=hV_{\mathbf{k}}e^{-i\omega_p t}+[V_{0\,\mathbf{k}-\mathbf{k}}^{(1)}b_0+U_{0\,\mathbf{k}-\mathbf{k}}b_0^{*}]+\sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{-\mathbf{k}'}. \tag{10}$$

The fact that in a typical experimental situation[1-7] the oscillations of the uniform precession (UP) occur at a frequency ω_p far from the resonance frequency Ω_0 is of fundamental importance. Therefore, the amplitude b_0 is small and this enables us to confine ourselves in the canonical equations of motion

$$db_0/dt=-i\delta\mathcal{H}/\delta b_0^{*}$$

to the linear approximation in b_0, and, having solved them, to eliminate b_0 from (10). As a result we obtain

$$P_{\mathbf{k}}=h\tilde{V}_{\mathbf{k}}e^{-i\omega_p t}+\sum_{\mathbf{k}'}\tilde{S}_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{-\mathbf{k}'}, \tag{11}$$

where

$$\tilde{V}_{\mathbf{k}}=V_{\mathbf{k}}-\frac{UV_{0\,\mathbf{k}-\mathbf{k}}^{(1)}}{\Omega_0-\omega_p}-\frac{UU_{0\,\mathbf{k}-\mathbf{k}}}{\Omega_0+\omega_p}=\frac{g^2}{4\omega_{\mathbf{k}}}\left[H_D+2H\frac{\Omega_0^2}{\Omega_0^2-\omega_p^2}\right];$$
$$\tilde{S}_{\mathbf{k}\mathbf{k}'}=S_{\mathbf{k}\mathbf{k}'}-\frac{1}{2}\left\{\frac{V_{0\,\mathbf{k}-\mathbf{k}}^{(1)}V_{0\,\mathbf{k}'-\mathbf{k}'}^{*(1)}}{\Omega_0-\omega_p}+\frac{U_{0\,\mathbf{k}-\mathbf{k}}U_{0\,\mathbf{k}'-\mathbf{k}'}^{*}}{\Omega_0+\omega_p}\right\}. \tag{12}$$

We see that the uniform precession in the upper branch plays a double role: first, it leads to an additional "indirect" coupling of the SW with the external field, i.e., to the renormalization (12) of the coefficient $V_{\mathbf{k}}$, and, secondly, it leads to an additional interaction between the SW. It is not difficult to see that this interaction arises because of the forced motion of the UP (not at resonance!) under the influence of the SW. A similar interaction also arises via other nonresonance beats (with $k \neq 0$). We have already taken all these contributions to the interaction of the SW into account in formu-

las (5)–(7). Naturally, we must not take the same effect into account twice, and therefore in the formula for P_k the coefficient $S_{kk'}$ must be left unremornalized:

$$P_k=h\bar{V}_k+\sum_{k'}S_{kk'}a_{k'}a_{-k'}. \quad (13)$$

Thus, we have obtained closed equations of motion ((8), (9) and (13)) for the parametric SW in the lower branch in an EP AFM. These equations no longer contain antiferromagnetic characteristics, and are universal in the sense that they describe the nonlinear stage of the parametric instability of waves of any type, e.g., SW in ferromagnets[14], Langmuir waves in a nonisothermal plasma (in the "undissociated" part of the spectrum)[15], etc. The equations obtained correspond to the S-theory Hamiltonian (1). The general properties of their solutions have been studied in detail in[12, 16-18]. Below we shall write out the stationary solutions of these equations, using explicit "antiferromagnetic" expressions for the coefficients $\bar{V}_k$, γ_k and $S_{kk'}$.

2. DISTRIBUTION AND LEVEL OF CONTAINMENT OF PARAMETRICALLY EXCITED SW

We shall study the stationary solutions of the equations of motion (8), (9) and (13) for parametric SW. It is known from the S-theory[12] that such a distribution is singular: $a_k \neq 0$ only on the resonance surface

$$2\bar{\omega}_k=\omega_p. \quad (14)$$

The distribution N_Ω of waves over this surface is normalized such that

$$N=\sum_k|a_k|^2=\int N_\Omega\, d\Omega,$$

and is determined most conveniently using a geometrical interpretation of the external-stability condition[12]–the self-consistent pumping surface $|P_\Omega|$ should lie entirely inside the surface γ_Ω: $|P_\Omega| \leq \gamma_\Omega$, the pair-amplitudes N_Ω being nonzero only in those directions Ω in which these surfaces touch[1)].

A specific feature of the antiferromagnetic situation is the fact that the kernel $S_{\Omega\Omega'}$ is degenerate (cf. (5)), and therefore the character of the dependence $|P(\Omega)|^2$ is completely determined for any distribution N_Ω. For not too large k ($sk < \omega_0$) we can neglect the term proportional to S_3[2)], and then $|P_\Omega|^2$ depends only on $x = 2k_yk_z/k^2$:

$$|P_\Omega|^2=\left|h\bar{V}+\sum_{x_i}S_{xx_i}N_ie^{-i\Phi_i}\right|^2=a+bx+cx^2, \quad a>0, \quad c>0. \quad (15)$$

On the other hand, as we have shown in[13], the damping constant γ_k of the SW (for fixed $|k|$) also depends, to good accuracy, only on x:

$$\gamma_\Omega=\gamma_0+\gamma_1x^2. \quad (16)$$

The surfaces (15) and (16) can touch only at one or two (for b = 0) points, Therefore, for any supercriticality (in the framework of the S-theory!), a regime should be realized in which N_Ω is concentrated on two or on four lines ($x = x_0$ or $x = \pm x_0$ respectively), belonging to the resonance surface. The theory starting from the dependences (15) and (16) determines only the integral intensity N_Ω on each line; the detailed distribution N_Ω along the lines is determined by the deviation of the dependences γ^2_Ω and $|P_\Omega|^2$ from (15) and (16).

Clearly, for small values of the supercriticality, when $N \to 0$ and $|P_\Omega| \to h\bar{V}$ = const, the surfaces will touch near the line x = 0, where γ_Ω is a minimum. We represent Eqs. (8) for the integral amplitude N_1 on the line $x = x_0$ and for the phase $\Phi_1 = (\varphi_k + \varphi_{-k})|_{x=x_0}$ in the form

$$P_{x_0}=h\bar{V}+(S_0+S_2x_0^2)N_1e^{-i\Phi_1}=i\gamma_{x_0}e^{-i\Phi_1}.$$

Hence,

$$h\bar{V}\sin\Phi_1=\gamma_{x_0}, \qquad (S_0+S_2x_0^2)N_1^2=(h\bar{V})^2-\gamma_{x_0}^2. \quad (17)$$

It remains to determine the location of the line $x = x_0$. For this we shall write the expression (15) for the self-consistent pumping $|P_x|^2$ on a certain line x under the condition that N_Ω is concentrated on the line $x = x_0$ and is determined from (17). The condition for touching of the surfaces $|P_x|^2$ and γ_x^2 at the point $x = x_0$ gives (for more detail, see in[13]):

$$2\gamma_1x_0=S_1N_1. \quad (18)$$

Thus, at the threshold pumping amplitude, when $N_1 = 0$, SW are excited on the line x = 0 corresponding to the minimum damping, and the line is then displaced as the supercriticality increases. This shift, as can be seen from (17), leads to further limitation of the amplitude, firstly, because of the growth of the interaction coefficient $S_{x_0x_0} = S_0 + S_2x_0^2$ and, secondly, because of the increase of γ_{x_0}.

The quantities x_0 and N_1 must be determined from Eqs. (17) and (18). These can be simplified using the inequalities $S_2^2 > S_1^2 > S_0^2$. Then,

$$\frac{2\gamma_1}{\gamma_0}x_0^2\left[1+\frac{2\gamma_1}{\gamma_0}\left(\frac{S_2}{S_1}\right)^2x_0^4\right]=\xi, \qquad N_1=\left|\frac{2\gamma_1x_0}{S_1}\right|, \quad (19)$$

where $\xi = [(h\bar{V})^2/\gamma_0^2 - 1]$ is the supercriticality the pumping power in excess of the critical value. It can be seen from this that for small supercriticalities the containment mechanism associated with the increase of the damping is dominant.

The displacement of the line and the growth of N_1 will continue in accordance with (19) until the surfaces $|P_x|^2$ and γ_x^2 touch at the second point, along the line $x = -x_0$. It is obvious that this occurs for that amplitude $N_1 \equiv N_c$ for which the function $|P_x|^2$ becomes even: $|P_x| = |P_{-x}|$. Then the location of the first line $x = x_0 = x_c$ at the point at which SW are excited on the second ($x = -x_c$) is determined from a relation following from the condition b = 0 in (15):

$$x_c^2=\frac{\gamma_{x_c}^2S_1^2}{[S_1^2+S_2^2x_c^2]^2}\frac{1}{N_c^2}.$$

Using (18), we obtain from this a biquadratic equation for x_c:

$$2\gamma_1x_c^2=S_1^2(\gamma_0+\gamma_1x_c^2)/(S_1^2+S_2^2x_c^2). \quad (20)$$

Taking into account that $\gamma_0 \sim 2\gamma_1$[13], we obtain from (19) and (20) an estimate for the supercriticality ξ_2 at which pair creation occurs on the second line:

$$\xi_2=2\left|\frac{S_1}{S_2}\right|\sqrt{\frac{2\gamma_1}{\gamma_0}}\leqslant\frac{2}{3}. \quad (21)$$

Assuming that $\xi > \xi_2$, we shall study the stationary state in which two groups of SW pairs, arranged symmetrically on the lines $x = \pm x_0$, are excited. We represent Eqs. (8), (13) and (14) for the amplitudes N_1, N_2 and phases Φ_1, Φ_2 of these groups in the form

$$h\bar{V}+S_{\alpha1}N_1e^{-i\Phi_1}+S_{\alpha2}N_2e^{-i\Phi_2}=i\gamma e^{-i\Phi_\alpha},$$

where

$$\alpha=1,\ 2;\ S_{11}=S_{22}=S_0+S_2x_0^2;$$
$$S_{12}=S_{21}^*=(S_0-S_2x_0^2)+2iS_1x_0$$

and $\gamma \equiv \gamma_{x_0}$. From these equations it is comparatively easy to find the amplitude difference and phase difference:

$$N_1-N_2=\frac{\gamma S_1}{x_0[S_1^2+S_2^2x_0^2]},\quad \sin\left(\frac{\Phi_1-\Phi_2}{2}\right)=\frac{-S_1}{\sqrt{S_1^2+S_2^2x_0^2}}. \tag{22}$$

Of course, for $\xi = \xi_2$, $N_2 \to 0$, formula (22) also determines the critical amplitude $N_1 = N_c$, which coincides with the value of N_c that follows from (12), since for $\xi = \xi_2$ it is natural to assume that $x_0 = x_c$. Eliminating $(N_1 - N_2)$ and $(\Phi_1 - \Phi_2)$ from half the sum of the equations for N_1 and N_2, we obtain with the aid of (22) a simple equation for $N = N_1 + N_2$ $\Phi = 1/2(\Phi_1 + \Phi_2)$:

$$h\nabla e^{i\Phi}+\tilde{S}N=i\tilde{\gamma}, \tag{23}$$

where

$$\tilde{\gamma}=\gamma_{x_0}\frac{S_2x_0^2(S_1^2+S_2^2x_0^2)+S_1^2(S_0+S_2x_0^2)}{|x_0|(S_1^2+S_2^2x_0^2)^{3/2}}\approx\gamma_{x_0}\left[1+\frac{S_1^2}{2S_2^2x_0^2}\right],$$
$$\tilde{S}=S_0\cos\left(\frac{\Phi_1-\Phi_2}{2}\right),\qquad S_0=S_0\left[1-\frac{S_1^2}{S_0S_2}\right]. \tag{24}$$

The location of the lines $\pm x_0$ must be determined, as usual, from the condition for touching of the surfaces $|P_x|^2$ and γ_x^2. Calculating $|P_x|^2$ from formula (15) by means of the relations (22) and (23), and then equating the derivatives $d|P_x|^2/dx$ and $d\gamma_x^2/dx$, we finally obtain an unexpectedly simple and interesting result: the location of the lines $\pm x_0$ does not depend on the supercriticality, and, naturally, x_0 coincides with the original (for $\xi = \xi_2$) "coordinate" x_c determined by formula (20).

Thus, except in a not very wide region near the threshold ($\xi < \xi_2$—cf. (21)), two groups of spin-wave pairs, differing little from each other, are excited, and their resultant amplitude $N = N_1 + N_2$ and mean phase $\Phi = 1/2(\Phi_1 + \Phi_2)$ are determined by the simple relations (23).

To obtain concrete results it is necessary to calculate the value of x_0 appearing in the expressions (24) for $\tilde{\gamma}$ and $\tilde{S}$. From Eqs. (20) we have

$$x_0^2=\frac{|S_1|}{2S_2}\left[\sqrt{\left(\frac{S_1}{2S_2}\right)^2+\frac{2\gamma_0}{\gamma_1}}-\frac{S_1}{2S_2}\right]\approx\frac{S_1}{S_2}\sqrt{\frac{\gamma_0}{2\gamma_1}}-\frac{S_1^2}{4S_2^2}. \tag{25}$$

If $(S_1/S_2)\sqrt{\gamma_1/2\gamma_0} \ll 1$, we obtain for $\tilde{\gamma}$ and $\tilde{S}$, approximately,

$$\tilde{\gamma}=\gamma_0+\frac{S_1^2}{S_2}\sqrt{2\gamma_1\gamma_0},\quad \tilde{S}=\left(1-\frac{S_1}{S_2}\sqrt{\frac{\gamma_1}{2\gamma_0}}\right)S_0. \tag{26}$$

It is necessary to note that the ratio S_1^2/S_2S_0, as can be seen from (7), does not contain a small parameter. As a result, $\tilde{S}_0$ differs substantially from S_0:

$$\tilde{S}_0=S_0-\frac{S_1^2}{S_2}=-\frac{g^2B}{8\omega_k^2}\{\omega_{10}^2+(gH_0)^2\}\neq S_0. \tag{27}$$

It is interesting to note that in this expression the contribution to the interaction of the parametric SW via virtual SW with $k \neq 0$ has been cancelled. We add also that the ratio $S_1/S_2 \sim 0.3$, and therefore the difference between $\tilde{\gamma}$ and γ_0 can also be considerable.

3. EFFECT OF RANDOM MAGNETIC INHOMOGENEITIES

3.1 The Interaction Hamiltonian

In real crystals of antiferromagnetic substances, there are always inhomogeneities violating the translational symmetry of the problem. The presence of inhomogeneities leads to the result that the quadratic part of the Hamiltonian acquires a correction that is off-diagonal in $\mathbf{k}$:

$$\mathcal{H}_{imp}=\frac{1}{N}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}g_{\mathbf{kk}'}a_{\mathbf{k}}a_{\mathbf{k}'}^*b_{\mathbf{q}}^*\Delta(\mathbf{k}-\mathbf{k}'-\mathbf{q}). \tag{28}$$

Here N is the number of magnetic atoms in the crystal and the amplitude $b_{\mathbf{q}}$ is a Fourier component of the static random field of the inhomogeneities, the scattering properties of which are characterized by the matrix $g_{\mathbf{kk}'}$. The inhomogeneities can be widely different in character—impurities, point defects, surface roughness, and so on.

For point inhomogeneities $b_{\mathbf{q}}$ is conveniently chosen in the form

$$b_{\mathbf{q}}=\sum_n e^{i\mathbf{q}\mathbf{r}_n},$$

where $\mathbf{r}_n$ are the random coordinates of the defects.

The matrix elements $g_{\mathbf{kk}'}$ can be calculated by taking any simple model of the defects. If they are interstitial impurities that do not deform the lattice around themselves and differ from the atoms of the matrix only in their exchange integral (B + B′ in place of B), then

$$g_{\mathbf{kk}'}=g_0+g_1\frac{\mathbf{kk}'}{kk'},$$
$$g_0=\frac{B'}{B}\frac{\omega_k^2+g^2(H_0+H_D)^2}{\omega_k}\qquad g_1=\frac{B'}{B}\frac{(sk)^2}{\omega_k}. \tag{29}$$

We note that the matrix $g_{\mathbf{kk}'}$ in this model was calculated in the work of Bar'yakhtar and Sanina[19]; the expression they obtained differs somewhat from (29).

3.2 Basic Equations

The effect of inhomogeneities on the parametric excitation of SW was studied in the framework of the Hamiltonian (28) in the[20] by Zakharov and L'vov. It was shown by means of a diagram technique that in the stationary state the correlation functions $n_{\mathbf{k}} = \overline{a_{\mathbf{k}}a_{\mathbf{k}}^*}$, $\sigma_{\mathbf{k}} = \overline{a_{\mathbf{k}}a_{-\mathbf{k}}}e^{i\omega pt}$ (the bar denotes averaging over the arrangement of the impurities) characterizing the system of parametric SW are concentrated in a narrow layer near the resonance surface (14). For the integral quantities

$$N_\Omega=\frac{k_\Omega^2}{(2\pi)^3}\int n_{\mathbf{k}}\,d\mathbf{k},\quad \Sigma_\Omega=\frac{k_\Omega^2}{(2\pi)^3}\int\sigma_{\mathbf{k}}\,d\mathbf{k}$$

(k_Ω is the radius of the surface (14)), a system of integral equations was obtained:

$$\Gamma_\Omega(N_\Omega-\tilde{N}_\Omega)+\mathrm{Im}\{\Pi_\Omega^*(\Sigma_\Omega-\tilde{\Sigma}_\Omega)\}=0,$$
$$\Gamma_\Omega\Sigma_\Omega+i\Pi_\Omega N_\Omega=0. \tag{30}$$

Here Γ_Ω and Π_Ω are the damping and pumping, renormalized on account of the inhomogeneities. They are determined by Dyson's integral equations:

$$\Gamma_\Omega=\gamma_\Omega+\frac{\pi ck_\Omega^2a^3}{(2\pi)^3v_\Omega}\int|g_{\Omega\Omega'}|^2\frac{\Gamma_{\Omega'}}{v_{\Omega'}}d\Omega',$$
$$\Pi_\Omega=P_\Omega+\frac{\pi ck_\Omega^2a^3}{(2\pi)^3v_\Omega}\int g_{\Omega\Omega'}g_{-\Omega,-\Omega'}\frac{\Pi_{\Omega'}}{v_{\Omega'}}d\Omega', \tag{31}$$

in which

$$v_\Omega^2=\Gamma_\Omega^2-|\Pi_\Omega|^2, \tag{32}$$

c is the concentration of inhomogeneities, v_Ω is the projection of the group velocity along the normal to this surface, and a^3 is the volume of the unit cell.

The quantities $\tilde{N}_\Omega$ and $\tilde{\Sigma}_\Omega$ in Eq. (30) are defined by the relations

$$\tilde{N}_\Omega = \frac{\pi c k_\Omega^2 a^3}{(2\pi)^3 v_\Omega v_\Omega} \int |g_{\Omega\Omega'}|^2 N_{\Omega'}\, d\Omega',$$
$$\tilde{\Sigma}_\Omega = \frac{\pi c k_\Omega^2 a^3}{(2\pi)^3 v_\Omega v_\Omega} \int g_{\Omega\Omega'} g_{-\Omega,-\Omega'} \Sigma_{\Omega'}\, d\Omega', \tag{33}$$

In[20] these equations were analyzed in the case when the problem has axial symmetry, which is realized in the parametric excitation of SW in an isotropic ferromagnet, and with the use of the model assumption $g_{\mathbf{kk}'}$ = const. The assumptions are not valid for an AFM: the coefficients γ_Ω and $S_{\Omega\Omega'}$, gemerally speaking, do not have any symmetry; the coefficients $g_{\mathbf{kk}'}$ depend differently on $\mathbf{k}$ and $\mathbf{k}'$ for different types of inhomogeneities. Therefore, we shall analyze Eqs. (30)–(33) without making assumptions about the form of its coefficients.

3.3 General Analysis of the Equations

We integrate the first of Eqs. (30) over Ω and, using Eq. (31), obtain the integral relation

$$\int \gamma_\Omega N_\Omega\, d\Omega = \operatorname{Im}\left\{\int P_\Omega^* \Sigma_\Omega\, d\Omega\right\}, \tag{34}$$

which has the meaning of an energy-balance equation. The integral on the left describes the flow of energy out of the system of parametric waves as a result of the characteristic relaxation mechanisms that occur in a homogeneous crystal. Two-magnon scattering occurs with conservation of frequency; it does not carry energy out of the system of parametric waves and for this reason cannot appear in the relation (34). The right-hand side of this relation can be transformed identically to the form

$$\operatorname{Im}\int h\bar{V}_\Omega^* \Sigma_\Omega\, d\Omega,$$

and describes the influx of energy from the external pumping.

The influence of inhomogeneities, as we now show, leads to two effects: the obvious one of making the distribution N_Ω isotropic, and the less obvious one of violating the phase correlations in the pairs, leading to a decrease of $|\Sigma_\Omega|$ in comparison with N_Ω. We shall study this phenomenon in the most interesting limiting case, when the concentration of inhomogeneities is large. We shall characterize the inhomogeneities of the AFM by the parameter $\gamma_{imp} = cg^2k^2a^3/2\pi v$, where $g^2 \langle\langle |g_{\Omega\Omega'}|^2\rangle_\Omega\rangle_{\Omega'}$; the symbol $\langle\rangle$ denotes averaging over the resonance surface:

$$\langle f\rangle_\Omega = \frac{1}{4\pi}\int f\, d\Omega.$$

In the limit $\gamma_{imp} \gg \gamma$, the initial integral equations are considerably simplified. With one exception, with which we shall be concerned below, they have a solution for which $N_\Omega \gg |\Sigma_\Omega|$ and $\Gamma_\Omega \gg |\Pi_\Omega|$. Using this inequalities, we obtain from (30)–(33) an equation determining the distribution N_Ω:

$$\int |g_{\Omega\Omega'}|^2 (N_\Omega - N_{\Omega'})\, d\Omega' = 0. \tag{35}$$

The only reasonable solution of this equation is the isotropic distribution: $N_\Omega = N/4\pi$. To calculate Σ_Ω, as can be seen from (30), it is necessary to know Γ_Ω and Π_Ω. Taking into account that we seek a solution for which $\Gamma \gg \gamma$ and $\Gamma \approx \nu$, we obtain from (31), firstly

$$\Gamma_\Omega \approx \nu_\Omega \approx \gamma_{imp} \left\langle \frac{|g_{\Omega\Omega'}|^2}{g^2} \right\rangle_{\Omega'}, \tag{36}$$

and secondly, a Fredholm integral equation of the second kind for Π_Ω:

$$\Pi_\Omega = P_\Omega + \langle K_{\Omega\Omega'} \Pi_{\Omega'}\rangle_{\Omega'}, \tag{37}$$

in which

$$K_{\Omega\Omega'} = g_{\Omega\Omega'} g_{-\Omega,-\Omega'} / \langle |g_{\Omega,\Omega''}|^2\rangle_{\Omega''}. \tag{38}$$

In a number of cases, e.g., when $g_{\Omega\Omega'}$ = const, Eq. (37) has no solution. This implies that our assumption $\Gamma_\Omega \gg |\Pi_\Omega|$ is not valid. It can be shown that in this case the initial equations have another solution, in which $\Gamma \approx |\Pi| \approx \gamma_{imp}(\gamma_{imp}/\gamma)^{1/3} \gg \nu \approx \gamma_{imp}$. For the present, we shall not be interested in this degenerate case, and we shall assume that the equation for Π_Ω has a single solution with $|\Pi|$ of the order of $|P|$.

Using the relations (31) and (32) we transform the equality (34) to the form

$$\gamma_{imp}\langle\gamma_\Omega\rangle = \operatorname{Re}\left\langle \frac{P_\Omega^* \Pi_\Omega}{\langle |g_{\Omega\Omega'}|^2\rangle_{\Omega'}} \right\rangle_\Omega g^2. \tag{39}$$

From this it can be seen that $|P|^2 \approx \gamma_{imp}\langle\gamma_\Omega\rangle$, and it then follows quickly from (30) that $|\Sigma_\Omega| \approx \sqrt{\gamma/\gamma_{imp}}N \ll N$. This circumstance– violation of the phase correlations– leads to an increase of the threshold. In place of the estimate $h_c\tilde{V} \approx \langle\gamma\rangle$, which would be obtained in the case $|\Sigma| \approx N$, it follows from (34) that

$$h_c\tilde{V} \approx (\gamma_{imp}\langle\gamma\rangle)^{1/2}.$$

3.4 Case of Point Defects

We shall study the behavior of the SW in the framework of the equations that we have obtained, using the explicit form (29) for the functions $g_{\mathbf{kk}'}$, calculated for a simple model of substitutional impurities. For definiteness, we shall assume that $S_{\Omega\Omega'} = S_0 + iS_1(x - x') + S_2 xx'$). This is valid for not too large k. Taking into account that in this case P_Ω depends only on x:

$$P_\Omega = P_0 + xP_1,$$
$$P_0 = h\bar{V} + 4\pi[S_0\langle\Sigma_\Omega\rangle - iS_1\langle x\Sigma_\Omega\rangle],$$
$$P_1 = 4\pi[S_2\langle x\Sigma_\Omega\rangle + iS_1\langle\Sigma_\Omega\rangle], \tag{40}$$

and the kernel $K_{\Omega\Omega'}$ depends only on $\mathbf{k}\cdot\mathbf{k}'$ (which can be seen by substituting (29) into (38)), it is easy to obtain the solution of the integral equation (37):

$$\Pi_\Omega = \Pi_0 + x\Pi_1,$$
$$\Pi_0 = P_0/d,\ \Pi_1 = P_1/b,$$
$$d = 2g_1^2/(3g_0^2 + g_1^2),\ b = 1 + d/5. \tag{41}$$

Remembering that in our case it follows from (36) that $\Gamma_\Omega = \gamma_{imp}$, we obtain from (30) and (41) the dependence

$$\Sigma_\Omega = \Sigma_0 + x\Sigma_1,$$
$$\Sigma_0 = -i\frac{N}{4\pi}\frac{P_0}{\bar{\gamma}_{imp}},\quad \Sigma_1 = -\frac{iN}{4\pi}\frac{d}{b}\frac{P_1}{\bar{\gamma}_{imp}},$$
$$\bar{\gamma}_{imp} = d\gamma_{imp}. \tag{42}$$

Substituting (42) into (40), we obtain a system of linear equations for P_0 and P_1, having the solution

$$P_0 = h\bar{V}\frac{\bar{\gamma}_{imp} + i\varepsilon S_2 N}{\bar{\gamma}_{imp}\Delta},\quad P_1 = h\bar{V}\frac{S_1 N_1}{\bar{\gamma}_{imp}\Delta},$$
$$\Delta = 1 + \frac{(S_1^2 - S_0 S_2)}{\bar{\gamma}_{imp}^2}N^2 + i\frac{(S_0 + \varepsilon S_2)N}{\bar{\gamma}_{imp}},$$
$$\varepsilon = \frac{4}{15}\frac{d}{b} \approx \frac{8}{45}\left(\frac{g_1}{g_0}\right)^2 \text{ for } g_1 \ll g_0. \tag{43}$$

To determine the dependence of N on the pumping

amplitude we make use of the energy-balance equation (39), which takes the form

$$\tilde{\gamma}_{imp}\langle\gamma_0\rangle = d\,\mathrm{Re}\,\langle P_0^{*}\Pi_0\rangle = |P_0|^2+\varepsilon|P_1|^2.$$

From the last two relations we obtain a biquadratic equation for the dimensionless amplitude $m \equiv NS_0/\tilde{\gamma}_{imp}$:

$$A^2m^4+2m^2[B-D\xi]+\xi=0, \qquad (44)$$

in which

$$A=\varepsilon\frac{S_0S_2}{S_0^2}, \quad 2B=1+\varepsilon\frac{S_1^2}{S_0^2} \qquad 2D=\varepsilon^2\frac{S_2^2}{S_0^2}+\varepsilon\frac{S_1^2}{S_0^2},$$

and $\xi = (h^2/h_1^2 - 1)$ is the pumping power in excess of the critical value. The critical amplitude h_1 is determined by the relation

$$h_1\tilde{V}=(\langle\gamma\rangle\tilde{\gamma}_{imp})^{1/2}, \qquad (45)$$

in which

$$\tilde{\gamma}_{imp}=d\gamma_{imp}=\omega_k\frac{c}{3\pi}\left(\frac{B'}{B}\right)^2\left(\frac{a}{l}\right)^3\left(\frac{\omega_{ex}lk}{\omega_k}\right)^5\left(\frac{\omega_k}{\omega_{ex}}\right)^3.$$

Using the estimate $B'/B \sim 10$ (following the authors of[19]) and taking $sk/\omega_k \sim 1/2$, we obtain for $MnCO_3$ the estimate $(\tilde{\gamma}_{imp}/\omega_k) \approx 10^{-5}c$. This means that only in the low-temperature region and in sufficiently impure crystals ($c \gtrsim 10^{-2}$) can point defects lead to the situation $\tilde{\gamma}_{imp} > \gamma$ in which the asymptotic formulas (44) and (45) are valid. Taking into account that γ_{imp} is greater than $\tilde{\gamma}_{imp}$ (by a factor of $1/d = 3g_0^2/2g_1^2 = 3(\omega_k/sk)^4(1 + H_D/2H_0)$), it is understandable that the situation in which $\gamma_{imp} > \gamma$ and the relationship between $\tilde{\gamma}_{imp}$ and γ is arbitrary is much more often realized. Comparing the results we have obtained with the solution of the initial equations that can be obtained in the case $\tilde{\gamma}_{imp} \ll \gamma$, $\gamma_{imp} \gg \gamma$, it is easy to convince oneself that the formulas (42)–(45) will describe both limiting cases $\tilde{\gamma}_{imp} < \gamma$ and $\tilde{\gamma}_{imp} > \gamma$ (for $\gamma_i > \gamma$) correctly if in them we replace $\tilde{\gamma}_{imp}$ by $(\tilde{\gamma}_{imp} + \langle\gamma\rangle)$. In particular, in place of (45) we obtain the interpolation threshold formula

$$h_1\tilde{V}=[(\langle\gamma\rangle+\tilde{\gamma}_{imp})\langle\gamma\rangle]^{1/2}. \qquad (46)$$

3.5 Discussion of the Results

In the preceding subsection we have obtained formulas determining the parametric-excitation threshold (46) and describing the behavior beyond the threshold in an EP AFM with point defects ((42), (43) and (44)). Comparison of these formulas with the results of the general analysis of the initial equations that was carried out in Sec. 3.3 gives us every reason to assume that they are also at leads qualitatively valid when we go outside the framework of the simple model that we have considered for the impurities. It is necessary only to assume that $\tilde{\gamma}_{imp}$ and ϵ, appearing in these formulas, are phenomenological parameters describing the impurities (and independent, naturally, of the temperature). The parameter $\tilde{\gamma}_{imp}$, which is proportional to the concentration of defects, characterizes the SW relaxation frequency in the absence of pumping, and ϵ characterizes the anisotropy of the scattering, or, more accurately, the deviation of $g_{kk'}$ from a constant:

$$\varepsilon \approx \frac{\langle[g_{kk'}-\langle g_{kk'}\rangle]^2\rangle}{\langle|g_{kk'}|^2\rangle}.$$

We shall show that with the aid of the parameter $\tilde{\gamma}_{imp}$ it is possible to explain naturally the experimentally observed temperature dependence of the threshold field $h_1(T)$ in the framework of formula (46). In the temperature range 1.2–2.2 K of interest to the experimenters the strongest SW relaxation mechanism in an EP AFM is the three-magnon coalescence process[21, 13]

$$\omega_k+\omega_{k'}=\Omega_{k+k'}. \qquad (47)$$

The corresponding damping constant γ_k, which we calculated in[13] with allowance for the dipole-dipole interaction, is determined by the formula

$$\frac{\langle\gamma_k\rangle}{\omega_k} = \frac{v_0}{16\pi Sl^3}\frac{T}{sk}e^{-\hbar\omega_-/T}\,[e^{\hbar\omega_k/T}-1] \times\left[\left(\frac{H_0}{H_{ex}}\right)^2\omega_-^2+\frac{4}{15}(\pi gM_0)^2\right]\frac{1}{\omega_k^2}, \qquad (48)$$

which is valid for $sk > T\Omega_0^2/\omega_0^2$. Here v_0 is the volume of the unit cell (two formula units of $MnCO_3$ or $CsMnF_3$), S is the spin, $l \equiv s/gH_{ex}$ (2.5 Å for $MnCO_3$[4]), and

$$\omega_-=sk+\Omega_0^2/2[\omega_k+sk].$$

Figure 1 shows the dependence (48) of $\log(\langle\gamma_k\rangle/\omega_k)$ on the temperature for $MnCO_3$ for H_0 = 2.5 kOe, 3.3 kOe for 4.0 kOe. Figure 2 shows the quantities γ_{exp} determined by Kotyuzhanskiĭ and Prozorova[5] from the experimental values of the threshold field $h_1(T)$ by means of the relation $h_1\tilde{V} \equiv \gamma_{exp}(T)$ (for $\tilde{V}$, see formula (12)) obtained by Ozhogin. A considerable discrepancy can be seen, especially at low temperatures. Here also are shown the theoretical values for the quantity $h_1\tilde{V}$, calculated from (46) for our chosen values of $\tilde{\gamma}_{imp}(H_0)$: $\tilde{\gamma}_{imp}/\omega_k = 3 \times 10^{-5}$ (H_0 = 2.5 kOe), 4×10^{-5} (H_0 = 3.3 kOe) and 6.5×10^{-5} (H_0 = 4.0 kOe). The quantitative agreement of the theory with experiment for $h_1(T)$ indicates, firstly, the fundamentally important role of random inhomogen-

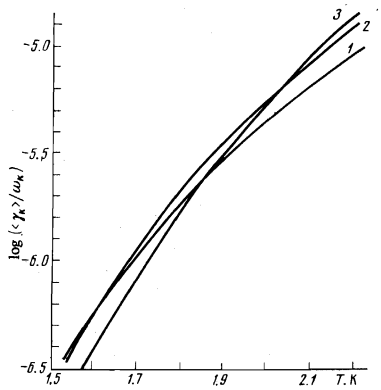

FIG. 1. Temperature dependence of the mean damping $\log(\langle\gamma_k\rangle/\omega_k)$ for $MnCO_3$ for fields (1) H_0 = 2.5 kOe, (2) H_0 = 3.3 kOe, (3) H_0 = 4.0 kOe.

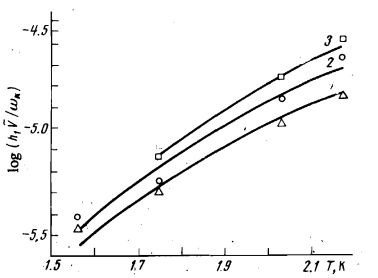

FIG. 2. Temperature dependence of the threshold field $h_1\tilde{V}$ for $MnCO_3$ in fields (1) H_0 = 2.2 kOe, (2) H_0 = 3.3 kOe, (3) H_0 = 4.0 kOe. The curves 1, 2 and 3 are the theoretical dependences (46); the points ○, △, and □ are the corresponding experimental values.

eities in the experiment of[4-6], and, secondly, the fact that the relaxation mechanism (47) does indeed make the main contribution to $\gamma(T)$ for $T > 1.2$–$1.5°$. At lower temperatures, as shown in[13], four-magnon processes must also be taken into account.

We note that the point-defect model we have studied leads to a different dependence $\tilde{\gamma}_{imp}(H_0)$ and cannot explain the observed value of $\tilde{\gamma}_{imp}$. It may be thought that the main contribution (which does not become small as k does) to $\tilde{\gamma}_{imp}$ is made by scattering at extended (linear and surface) defects, including scattering of SW at the faces of the crystal.

4. NONLINEAR SUSCEPTIBILITIES OF AFMs

In the traditional experiment[3-6] one studies the nonlinear susceptibility χ of an antiferromagnet, defined by the equality

$$m_x(\omega_p)=\chi h(\omega_p),$$

where $m_x(\omega_p)$ and $h(\omega_p)$ are the Fourier components of $m_x(t)$ and $h(t)$. Using the expression for m_x in terms of the canonical variables a_k and b_0, and the equation of motion for b_0, we can obtain the simple formula

$$\chi=\chi_0+\frac{2}{h}\sum_k \tilde{V}_k\sigma_k, \tag{49}$$

in which χ_0 is the linear susceptibility of the uniform precession:

$$\chi_0=\frac{8U^2}{\Omega_0}f=\frac{M_0}{H_{ex}}f, \qquad f\equiv\frac{\Omega_0^2}{\Omega_0^2-\omega_p^2}; \tag{50}$$

Using the results of the preceding Sections, we shall calculate χ for different situations. In a pure AFM below the threshold for creation of a second pair ($\xi \ll \xi_2$, cf. (21)), we obtain

$$\chi=\chi_0+\frac{2}{h^2}[(S_0+S_2x_0^2)N^2+i\gamma_{x_0}N]. \tag{51}$$

Here x_0 and N are the solution of the cubic equation (19). It is interesting to note that for $H_D = 0$ ($CsMnF_3$) the susceptibility (51) does not depend on the field H_0. It is possible to see this if one takes into account that $\gamma_0 \sim H_0^2$, $S_1 \sim H_0$, $S_0 \sim H_0^2$ and γ_1 and S_2 do not depend on H_0 for a fixed frequency ω_k.

Beyond the threshold for creation of the second pair, taking (22) and (23) into account we obtain from (49):

$$\chi=\chi_0+\frac{2\tilde{V}^2}{P|S_0|}\{-P+\tilde{P}+i[\tilde{P}(P-\tilde{P})]^{1/2}\}, \tag{52}$$

where $P = h^2/h_1^2$ and $\tilde{P} \equiv (\tilde{\gamma}/\gamma_0)^2$; $\tilde{V}$, $\tilde{S}_0$ and $\tilde{\gamma}$ are determined by the formulas (12), (27) and (24) respectively. In analyzing (52) it is useful to bear in mind that

$$\frac{2\tilde{V}^2}{|S_0|}=\chi_0\left(\frac{H_D+2H_0}{H_0}\right),$$

so that for $H_D = 0$ the susceptibilities χ' and χ'' are practically independent of H_0. Figure 3 shows a graph of the dependence of χ'' on P for $CsMnF_3$. The arrow marks the threshold for the creation of the second group of SW pairs. An expression for the susceptibility χ, qualitatively similar to (52), was obtained by the authors and Kolganov in[22], in which for simplicity the dipole-dipole interaction was not taken into account, so that $S_{kk'} = S_0$. If $S_{kk'} = S_0$, then for any supercriticality one line of pairs ($x = 0$) is excited (this corresponds to the minimum damping) and the susceptibility is determined by formula (54). Thus, although the dipole-dipole interaction makes

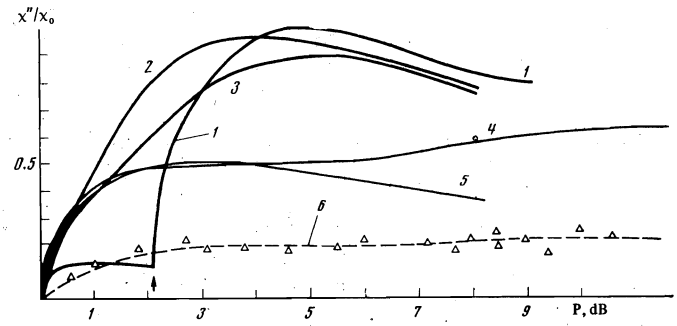

FIG. 3. Graphs of the dependence of the susceptibility χ''/χ_0 on the excess power P for $CsMnF_3$. The thick lines are the theoretical dependences for (1) $\epsilon = 0$, (2) $\epsilon = 0.05$, (3) $\epsilon = 0.25$, (4) $\epsilon = 0.5$, (5) for the pure AFM, (6) experimental data.

the main contribution to the interaction coefficients $S_{kk'}$, it nevertheless does not alter the order of magnitude of the nonlinear susceptibility of an EP AFM.

In a crystal with inhomogeneities in the case $\gamma_{imp} > \langle\gamma\rangle$ ($\tilde{\gamma}_{imp}$ is arbitrary), we obtain, using formulas (49) and (42)–(44),

$$\chi''=\frac{2\tilde{V}^2}{|S_0|}\frac{m}{P}=\frac{2\tilde{V}}{|S_0|}\frac{\sqrt{P-1}}{P}\sqrt{\frac{1+2Dm^2}{2B+A^2m^2}}$$
$$\chi'=\chi_0+\frac{2\tilde{V}^2}{|S_0|}\left(\frac{1}{P}-1\right)\left[\frac{1+c^2(S_0/S_0)A^2m^2}{2B+A^2m^2}\right], \tag{53}$$

where

$$\frac{2\tilde{V}^2}{|S_0|}=\chi_0\frac{2[2H_0+H_D]^2}{H_0(4H_0+H_D)}$$

and m is a dimensionless amplitude ~N (the solution of the biquadratic equation (44)).

It can be seen from these formulas that impurities do not affect the order of magnitude of χ' and χ'', or the character of their dependence on P. For example, for $\epsilon = 0$ we have

$$\chi=\chi_0+\frac{2\tilde{V}^2}{|S_0|P}[-P+1+i\sqrt{P-1}]. \tag{54}$$

The nonlinear part of the susceptibility has been reduced by a factor of $(\tilde{S}_0/S_0)^{-1} = (4H_0 + H_D)/(2H_0 + H_D)$ compared with (52) for pure AFMs. Allowance for the deviation of ϵ from zero decreases the susceptibility χ'' at low supercriticality, and increases it at high supercriticality and displaces the maximum of the dependence $\chi''(P)$ from $P = 2$ (3 db) to the right. These features of the behavior of $\chi''(P)$ are clearly visible in Fig. 3, where the curves of χ'' are plotted against P for $\epsilon = 0.05$, 0.25 and 0.5 for $H_D = 0$ ($CsMnF_3$).

First of all we shall compare the dependences $\chi'(P)$ and $\chi''(P)$ obtained for pure AFMs (cf. Fig. 3) with experiment[5,6]. It can be seen that they correctly predict the qualitative behavior of the dependences of χ' and χ'' on P: $(\chi' - \chi_0)$ is of the same order as χ'') the maximum of χ'' is in the range (3–5) dB, the dependence of χ' and χ'' on the field H_0 is weaker in $CsMnF_3$ than in $MnCO_3$, and so on. However, the absolute values of $(\chi' - \chi_0)$ and χ'' near the maximum are found to be three or four times too high in the theory (which does not take the impurities into account). Here it is pertinent to recall that the characteristic damping $\gamma_k(T)$ was found to be smaller than the experimental value of $h_1\tilde{V}$, especially at low temperatures (Fig. 1). In Sec. 3 we associated this discrepancy with the effect of im-

purities and described the temperature dependence $h_1(T)$ with the aid of a phenomenological parameter $\tilde{\gamma}_{imp}$ characterizing the impurities. Another impurity parameter $\epsilon < 1$ appears in the formulas for the susceptibilities beyond the threshold. It can be seen by comparing the theoretical dependences of Fig. 3 with experiment that allowance for the impurities makes it possible to understand the experiments better, both qualitatively and quantitatively, even with $\epsilon = 0.05$. The corresponding curves (4 and 6) have equally extended plateaux, which we explain by the extended nature of the transition $\tilde{S}_0 \to S_0$ for small ϵ, and the curves are similar when the excess power is not too small ($\chi''_{theor}/\chi''_{exp} \approx 2-2.5$). Evidently, this difference cannot be associated entirely with a possible systematic error in the experiment. It implies, then, that there must exist an additional mechanism limiting the amplitude—one that does not appear in our simple theory, which takes into account only the interaction of the parametric waves amongst themselves. In addition, we have not explained the phenomenon of "hard" excitation of SW, observed in the experiment of[6]. We suppose that these effects are at least partly associated with the interaction of the parametric SW with thermal SW; this interaction, as we have shown in[13], alters the amplitude of the thermal SW in such a way that the four-magnon contribution to the damping of the parametric waves is increased and the three-magnon contribution is decreased.

For small N[13] we have

$$\gamma(N)=\gamma\left[1-\frac{N}{N_{cr}}+\left(\frac{N}{N_s}\right)^2\right].$$

Estimates of the values of the characteristic amplitudes N_{cr} and N_S show that the nonlinear-damping mechanisms that we have proposed should indeed be observed in experiment.

[1)] In the following we shall assume that $|\mathbf{k}|$ is determined by the relation (14), so that all the coefficients of the problem ($\gamma_{\mathbf{k}}$, $P_{\mathbf{k}}$, etc.) depend only on the angular variable Ω.

[2)] A detailed analysis shows that it is not necessary that $sk < \omega_0$ for S_3 to be unimportant. For distributions N_Ω possessing a fourth-order symmetry axis this term vanishes on averaging. It is important that such a distribution satisfies the equations of motion.

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