Deformation thermomagnetic effect

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The appearance of a magnetic field around a deformed metallic sample through which heat flows is considered. The effect can be attributed to the presence of vortical electric currents in the sample. A qualitative analysis can be based on allowance for the deformation-induced changes of the kinetic coefficients that relate the charge- and heat-flux densities with the temperature gradient and the electric field. A phenomenological analysis of the effect and the results of calculations for the τ approximation and electron-boundary scattering are presented within the framework of the isotropic model.

The effect of the appearance of a magnetic field around an inhomogeneously deformed metallic sample through which heat flows was observed experimentally by Zavaritskiĭ[1], who investigated tin at low temperatures. This effect can be attributed, as proposed by A. F. Andreev (private communication), to the presence of an electric eddy current in the sample. In ordinary thermoelectric measurements no electric currents appear at all, but there appears an electric field that "cancels" the temperature gradient. If, however, the sample is unevenly deformed, then the current density must no longer be homogeneous (and therefore equal to zero), and eddy currents appear in the sample and give rise to the magnetic field.

The deformation of the sample cannot by itself produce eddy currents, so that for a quantitative analysis of the effect it is necessary to determine the change of the kinetic coefficients that describe the electric current following application of the deformation. We consider the coefficients that connect the heat and charge flux densities with the gradients of the temperature and of the effective electric field:

$$\begin{aligned} j_i &= -\sigma_{ik}\nabla_k\Phi + b_{ik}\nabla_k T, \\ q_i &= -c_{ik}\nabla_k\Phi + d_{ik}\nabla_k T. \end{aligned} \tag{1}$$

To find $\mathbf{j}$ it is necessary to solve the system (1) jointly with the conservation laws $\nabla\mathbf{j} = 0$ and $\nabla\mathbf{q} = 0$ and the boundary conditions. In all the calculations, the deformation will be assumed to be sufficiently small, so that the corrections to the kinetic coefficients, necessitated by the deformation, will be linear in the deformation. All the strain-caused corrections will be henceforth marked by a tilde.

We now consider phenomenologically the case when the ground state of the electrons (and of the phonons, if they make an appreciable contribution to the thermal conductivity) is a volume state, and the mean free path of the electrons is much shorter than the characteristic distance over which the deformation changes. In this case, the increments to the kinetic coefficients depend only on the strain, but not on its derivatives, i.e., say $\tilde{\sigma}_{ik} = \beta_{iknm}\nabla_n u_m$. Resolving the strain into symmetrical and antisymmetrical components u_{ik} and w_{ik}, we write $\tilde{\sigma}_{ik} = \beta^s_{iknm}u_{nm} + \beta^q_{iknm}w_n$, where $w_n = e_{nik}w_{ik}/2$ is the angle of rotation of the volume element.

We consider an isotropic metal, and also cubic and tetragonal crystals. The unperturbed kinetic coefficients in (1), which are second-rank tensors proportional to δ_{ik} for an isotropic metal or a cubic crystal, contain in the case of tetragonal symmetry also terms that are proportional to $m^3_i m^3_k$, where $\mathbf{m}^3$ is a unit vector directed along a fourfold axis of the tetragonal crystal. Throughout the article, $\mathbf{m}^1$ and $\mathbf{m}^2$ are unit vectors along twofold axes of a tetragonal crystal, and in the case of a cubic crystal $\mathbf{m}^p$ are directed along principal axes.

The dependence of the perturbed values of the kinetic coefficients on the rotation angle $\mathbf{w}$ is connected with the fact their form in the laboratory system differs from that in the coordinate frame connected with the lattice. A tensor proportional to δ_{ik} does not change form under rotation, so that there is no dependence on $\mathbf{w}$ in the isotropy case and for the case of cubic symmetry. For a tetragonal crystal, this correction is proportional to $w_{in}m^3_n m^3_k + w_{kn}m^3_i m^3_n$.

We now consider β^s_{iknm}. It is symmetrical in the last two indices and is a polar tensor, so it can be constructed with the aid of δ_{ik} and m^p_i. Symmetry considerations for a cubic crystal lead to the following independent components:

$$\delta_{ik}\delta_{nm}, \quad \delta_{im}\delta_{kn}+\delta_{in}\delta_{km}, \quad \sum_{p=1}^{3} m_i^p m_k^p m_n^p m_m^p, \tag{2}$$

In the isotropic case there is no tensor made up of the products m^p_i. For a tetragonal crystal (without allowance for symmetry and accurate to permutation of the indices), the following independent fourth-rank polar tensors exist:

$$\begin{aligned} &\delta_{ik}\delta_{nm}, \quad m_i^1 m_k^1 m_n^1 m_m^1 + m_i^2 m_k^2 m_n^2 m_m^2, \ \delta_{ik}m_n^3 m_m^3, \\ &m_i^1 m_k^1 m_n^2 m_m^2 + m_i^2 m_k^2 m_n^1 m_m^1. \end{aligned} \tag{3}$$

In particular, allowance for the symmetry yields for the elastic-moduli tensor the following independent components:

$$\begin{aligned} &\delta_{ik}\delta_{nm}, \quad (m_i^1 m_k^2 + m_i^2 m_k^1)(m_n^1 m_m^2 + m_n^2 m_m^1), \\ &\delta_{in}\delta_{km}+\delta_{im}\delta_{kn}, \quad m_i^1 m_k^1 m_n^1 m_m^1 + m_i^2 m_k^2 m_n^2 m_m^2, \\ &\delta_{ik}m_n^3 m_m^3 + m_i^3 m_k^3 \delta_{nm}, \quad m_i^1 m_k^1 m_n^2 m_m^2 + m_i^2 m_k^2 m_n^1 m_m^1. \end{aligned} \tag{4}$$

The concrete calculations will be carried out from now on for a cylindrical sample of length L and radius R, with a heat flux Q produced along its axis. The cylindrical sample is deformed by twisting through a small angle α. The unit vector $\boldsymbol{\nu}$ is directed along the cylinder axis. We introduce the symbols $\mathbf{z} = \boldsymbol{\nu}$, $\mathbf{x}$ and $\mathbf{r} = \mathbf{x} - \nu\mathbf{x}$, where $\mathbf{x}$ is the radius vector. If the torsion is isotropic in the plane of the cylinder cross section, then $\mathbf{u} = L^{-1}\alpha z\boldsymbol{\nu}\times\mathbf{r}$, whence

$$u_{ik} = \frac{\alpha}{2L}(\nu_i[\nu\times\mathbf{r}]_k + \nu_k[\nu\times\mathbf{r}]_i), \quad \mathbf{w} = \frac{\alpha}{2L}(2z\nu - \mathbf{r}), \tag{5}$$

For a cubic crystal, the independent components of

the elastic-moduli tensor coincide with (2), It is easy to verify that the components of the stress tensor Σ_{ik} calculated with the aid of (2) and (5) satisfy the equilibrium conditions $\nabla_i\Sigma_{ik} = 0$, i.e., Eq. (5) indeed describes the deformation of a cubic (and isotropic) sample. As to the tetragonal crystal, we can verify, by using (4), that the equilibrium conditions are satisfied if $\nu \cdot \mathbf{m}^3 = 0$, i.e., in this case (5) describes the deformation of the tetragonal crystal.

Let us find the dependence of the effect on the orientation of the cylinder axis relative to the crystal axis for cubic and tetragonal symmetry: in this case ν is assumed to lie in a plane perpendicular to $\mathbf{m}^3$, namely $\nu = \cos\chi\mathbf{m}^1 + \sin\chi\mathbf{m}^2$, i.e., the deformation is described by (5). If we recognize now that the independent components β^S_{iknm} for a cubic crystal are the components (2) then, upon substitution of (5) we obtain, by perturbing the coefficients (1) (A_1 and A_2 and constants),

$$\tilde{\mathbf{j}} = A_1[\nu\times\mathbf{r}] + A_2 x \sin 2\chi(\mathbf{v}\cos 2\chi + \Pi\sin 2\chi) - \sigma\nabla\tilde{\Phi} + b\nabla\tilde{T} \qquad (6)$$

and an analogous expression for $\tilde{\mathbf{q}}$. In this formula $\Pi = \nu \times \mathbf{m}^3$ and $x = \mathbf{m}^3 \cdot \mathbf{x}$. Thus, in the laboratory frame (which is connected with unit vectors $\mathbf{m}^3$, Π, and ν) the dependence of the right-hand side of (6) on χ is determined only by the factors $\cos 2\chi \sin 2\chi$ and $\sin^2 2\chi$.

To find $\tilde{\Phi}$ and $\tilde{T}$, it is necessary to solve $\nabla\tilde{\mathbf{j}} = 0$ and $\nabla\mathbf{q} = 0$ after substituting (6) with the conditions that the normal components of $\tilde{\mathbf{j}}$ and $\tilde{\mathbf{q}}$ vanish on the sample boundaries, By virtue of the linearity of these equations and owing to the zero boundary conditions, the dependence of $\tilde{\Phi}$ and of $\tilde{T}$ on χ is determined by the same factors, i.e., when $\tilde{\Phi}$ and $\tilde{T}$ are substituted in (6) we find that the dependence of $\tilde{\mathbf{j}}$, and hence also of the magnetic field, on χ takes the following form ($\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$ are constants):

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2\sin(4\chi) + \mathbf{H}_3\cos(4\chi). \qquad (7)$$

The same program can be carried out for a tetragonal crystal. The independent components β^S_{iknm} are obtained from (3), and after substituting (5) we obtain the general form of the corrections to the kinetic coefficients. Writing down the perturbed equations (1) in the laboratory system and reasoning in similar fashion, we obtain the same result (7).

We proceed now to estimate the effect by using the kinetic equation. If the correction to the electron distribution is obtained in the form

$$f_1 = e(\nabla\Phi)\psi\frac{\partial n_F}{\partial\varepsilon} - \nabla T\varphi\frac{\partial n_F}{\partial T},$$

then the kinetic coefficients which will be needed later on are expressed as follows[2]:

$$\sigma_{ik} = -\frac{2e^2}{(2\pi\hbar)^3}\int d^3p\frac{\partial n_F}{\partial\varepsilon}v_i\psi_k,$$
$$b_{ik} = -\frac{2e}{(2\pi\hbar)^3}\int d^3p\frac{\partial n_F}{\partial\varepsilon}v_i\varphi_k(\varepsilon-\zeta). \qquad (8)$$

If φ and ψ have no singularities at $\epsilon = \zeta$ (ζ is the chemical potential), then we have in the first approximation that does not vanish in the temperature

$$\sigma_{ik} = \frac{2e^2}{(2\pi\hbar)^3}\int d^3p\,\delta(\varepsilon-\zeta)v_i\psi_k, \qquad (9)$$

$$b_{ik} = \frac{2\pi^2 e}{(2\pi\hbar)^3}T\int d^3p\frac{\partial}{\partial\zeta}\delta(\varepsilon-\zeta)v_i\varphi_k. \qquad (9')$$

In the τ approximation, which is directly applicable if the principal electron scattering mechanism is scattering by impurities, the solution of the Boltzmann equation is

$$\varphi = \psi = l(\mathbf{p})\mathbf{n}, \qquad (10)$$

where $\mathbf{n}$ is a unit vector in the direction of the electron velocity and $l(\mathbf{p})$ is the mean free path. In this case φ does not depend explicitly on ζ, and therefore the differentiation in (9′) can be taken outside the integral sign, after which we obtain

$$b_{ik} = \frac{\pi^2 T}{3e}\frac{\partial}{\partial\zeta}\sigma_{ik}. \qquad (11)$$

If we now consider the isotropic case $\epsilon = \epsilon(\mathbf{p})$ and $l = l(\mathbf{p})$, then (9) yields, after substituting (10),

$$\sigma_{ik} = \frac{e^2 p_F^2}{3\pi^2\hbar^3}l(p_F)\delta_{ik}.$$

The values of b_{ik} are now obtained from (11), $d_{ik} = -(\pi^2 T/3e^2)\sigma_{ik}$, and the coefficient c_{ik} can be neglected in this approximation with respect to temperature. Solving now the system (1) for a cylindrical sample, we obtain $\mathbf{j} = 0$ and $\mathbf{q} = q\nu$, and for the gradient we get

$$\nabla T = -\frac{9\hbar^3 q}{p_F^2 l T}\mathbf{v}, \quad \nabla\Phi = -\frac{3\pi^2\hbar^3 q}{p_F^4 l^2 v_F e}(p_F^2 l)'\mathbf{v}, \qquad (12)$$

where the prime denotes differentiation with respect to p_F, and $l = l(p_F)$.

All the formulas derived above describe the unperturbed case. If we now introduce a strain, then the corrections to the kinetic coefficients are obtained by separating from (9) and (9′) the part linear in the strain. Thus, the expression for the correction to the conductivity tensor takes the form

$$\tilde{\sigma}_{ik} = \frac{2e^2}{(2\pi\hbar)^3}\int d^3p\,\delta(\varepsilon-\zeta)\left[v_i(\tilde{l}n_k + l\tilde{n}_k) - (\tilde{\varepsilon}-\tilde{\zeta})\frac{\partial}{\partial p_i}(ln_k)\right].$$

Isotropic compression cannot lead to eddy currents (in the isotropic case this is manifest in the fact that the corrections to the kinetic coefficients that are proportional to u_{nn} are proportional to δ_{ik}), so that to simplify the intermediate formulas we put $u_{nn} = 0$. In this case we have for the isotropic model

$$\tilde{\varepsilon} = \lambda(p)n_i n_k u_{ik}, \quad \tilde{\zeta} = 0, \quad \tilde{l} = 0,$$
$$\tilde{n}_k = \frac{2\lambda}{pv}(u_{kn}n_n - n_k n_n n_m u_{nm}).$$

The value $\tilde{\xi} = 0$ follows from the condition that the charge density remain unchanged, and l does not change since it is determined by the impurity concentration which does not change because $u_{nn} = 0$. Substitution of these expressions leads to

$$\tilde{\sigma}_{ik} = \frac{2e^2 p_F^2}{15\pi^2\hbar^3}\left(4\frac{\lambda l}{p_F v_F} - \frac{\lambda}{v_F}l'\right)u_{ik}. \qquad (13)$$

To estimate the effects we can use the expression $\lambda = pv$ (which is obtained if it is recognized that the change in the electron dispersion law is due only to deformation of the Brillouin zone[3]). If we put $\tilde{\Phi} = \tilde{T} = 0$ and substitute this expression for λ jointly with (5) into the perturbed expression (1), then we get

$$\tilde{\mathbf{j}} = \frac{eq\alpha}{5L}\frac{\partial}{\partial\zeta}\left(p_F\frac{l'}{l}\right)[\nu\times\mathbf{r}]. \qquad (14)$$

Thus, the assumption $\tilde{\Phi} = \tilde{T} = 0$ is self-consistent since (14) and the expression analogous to it for $\mathbf{q}$ satisfy both the conservation laws and the boundary conditions. Formula (14) enables us to calculate the magnetic moments of the cylinder

$$\mathbf{M} = \frac{QR^2 e\alpha}{20c}\frac{\partial}{\partial\zeta}\left(p_F\frac{l'}{l}\right)\nu. \qquad (15)$$

The solution of the kinetic equation is particularly simple in the case when the principal electron scattering is due to the presence of the sample boundaries. Let us estimate the effect in this case, assuming the condition $L \gg R$ to be satisfied. In the Boltzmann equation, scattering by the boundaries corresponds to a possibility of neglecting the collision integral with comparison with the term $\mathbf{v} \cdot \nabla f_1$. In addition, we assume that ∇T and $\nabla\Phi$ are homogeneous and do not depend on u_{ik}, and see that this leads to a result that satisfies both the conservation laws and the boundary conditions. Taking the homogeneity of the gradients into account, we obtain for φ and ψ the equations

$$\begin{aligned} \varphi(\mathbf{v}\nabla)\ln\frac{\partial n_F}{\partial T}+(\mathbf{v}\nabla)\varphi=\mathbf{v},\\ \psi(\mathbf{v}\nabla)\ln\frac{\partial n_F}{\partial \varepsilon}+(\mathbf{v}\nabla)\psi=\mathbf{v}. \end{aligned} \tag{16}$$

In the absence of strain, the solution of this system with allowance for diffuse boundary conditions is

$$\varphi=\psi=\mathbf{x}-\mathbf{x}_f, \tag{17}$$

where $\mathbf{x}_f$ is a point on the boundary such that $\mathbf{x}-\mathbf{x}_f$ is directed along n. Introducing $\xi(\mathbf{n}, \mathbf{x}) = |\mathbf{x} - \mathbf{x}_f|$, and linearizing (16), we obtain equations whose solutions take the form

$$\begin{aligned} \tilde{\varphi}=\int_{x_f}^{x} d\xi\left[\frac{\tilde{\mathbf{v}}}{v}+\mathbf{n}\xi\left(\frac{1}{T}\operatorname{th}\frac{\varepsilon-\zeta}{2T}-\frac{1}{\varepsilon-\zeta}\right)\frac{d}{d\xi}(\tilde{\varepsilon}-\tilde{\zeta})-\mathbf{n}\left(\frac{\tilde{\mathbf{v}}}{v}\nabla\right)\xi\right],\\ \tilde{\psi}=\int_{x_f}^{x} d\xi\left[\frac{\tilde{\mathbf{v}}}{v}+\mathbf{n}\xi\frac{1}{T}\operatorname{th}\frac{\varepsilon-\zeta}{2T}\frac{d}{d\xi}(\tilde{\varepsilon}-\tilde{\zeta})-\mathbf{n}\left(\frac{\tilde{\mathbf{v}}}{v}\nabla\right)\xi\right], \end{aligned} \tag{18}$$

where the integration is with respect to $d\xi \equiv \mathbf{n} \cdot d\mathbf{x}$ along a straight line from $\mathbf{x}_f$ to $\mathbf{x}$.

In the case of an infinitely long cylinder, the function $\xi(\mathbf{x}, \mathbf{n})$ is given by

$$\xi=R\frac{(1-\eta^2\sin^2\varphi)^{1/2}+\eta\cos\varphi}{\sin\theta}, \tag{19}$$

where $\eta = r/R$, while θ and φ are the azimuthal and polar angles with unit vectors $\mathbf{r}/r$, μ, and ν. Using this expression and formulas (9) and (17), we obtain in the isotropic case

$$\sigma_{ik}=\frac{p_F^2Re^2}{8\pi^2\hbar^3}\left(J_1\frac{r_ir_k}{r^2}+J_2\mu_i\mu_k+4E\nu_i\nu_k\right), \tag{20}$$

where

$$\begin{aligned} J_1(\eta)=\frac{4}{3}\left[\left(\frac{1}{\eta^2}+1\right)E(\eta)-\left(\frac{1}{\eta^2}-1\right)K(\eta)\right],\\ J_2(\eta)=\frac{4}{3}\left[\left(2-\frac{1}{\eta^2}\right)E(\eta)+\left(\frac{1}{\eta^2}-1\right)K(\eta)\right], \quad \mu=\left[\mathbf{v}\frac{\mathbf{r}}{r}\right], \end{aligned}$$

and $E(\eta)$ and $K(\eta)$ are elliptic functions. Similar expressions can be obtained for the remaining kinetic coefficients, from which we see that the unperturbed current density is equal to zero, and $\mathbf{q} \sim E(\eta)\boldsymbol{\nu}$. If we express q in terms of the total heat flux, then we obtain $\mathbf{q} = (3Q/2\pi R^2)E(\eta)\boldsymbol{\nu}$, from which we get, by substituting in (1),

$$\nabla T=-\frac{9Q\hbar^3}{\pi R^3Tp_F^2}\mathbf{v}, \quad \nabla\Phi=-\frac{6\pi Q\hbar^3}{eR^3v_Fp_F^3}\mathbf{v}. \tag{21}$$

We now proceed to calculate $\tilde{\sigma}_{ik}$ and $\tilde{b}_{ik}$. The expressions (18) used in this case have a singularity at $\epsilon = \xi$, and therefore the perturbed values of the kinetic coefficients should be obtained by substituting (18) directly in (8). Substituting in (8) also (17) and retaining the first term of the expansion of the temperature, we can get

$$\begin{aligned} \tilde{\sigma}_{ik}=\frac{2e^2}{(2\pi\hbar)^3}\int d^3p\,\delta(\varepsilon-\zeta)\left[-(\tilde{\varepsilon}-\tilde{\zeta})\frac{\partial}{\partial p_i}(\xi n_k)+n_j\int d\xi\tilde{v}_k\right.\\ \left.-n_in_k\int d\xi\tilde{\mathbf{v}}\nabla\xi+\frac{\partial}{\partial p_i}\left(n_k\int\xi d(\tilde{\varepsilon}-\tilde{\zeta})\right)\right], \end{aligned} \tag{22}$$

where the integration is the same as in (18). The expression for $\tilde{b}_{ik}$ differs from (22) in that the δ function in the integrand is replaced by its derivative and by the same coefficient as in (11).

We now consider the isotropic case under the same assumptions as in the analysis of the scattering by impurities. Then $\tilde{\zeta} = 0$, the integrand in (22) does not depend explicitly on ξ (with the exception of the δ function) and therefore to calculate $\tilde{b}_{ik}$ we can use (11). On the other hand, the expression for $\tilde{\sigma}_{ik}$, obtained by substituting in (22) all the expressions that hold in the isotropy case, and by integrating over momentum space, takes the form

$$\tilde{\sigma}_{ik}=\frac{e^2\lambda\alpha R^2p_F}{2\pi^2\hbar^3Lv_F}\eta\left[\left(E-\frac{1}{8}J_1\right)(\mu_i\nu_k+\mu_k\nu_i)-\frac{1}{4}J_1\nu_i\mu_k\right]. \tag{23}$$

Calculating $\tilde{b}_{ik}$ from (11) and substituting in (1) with allowance for $\tilde{\Phi} = \tilde{T} = 0$, we get

$$\tilde{\mathbf{j}}=-\frac{3e\alpha Q}{2\pi LR}\frac{\partial}{\partial\zeta}\left(\frac{\lambda}{p_Fv_F}\right)\eta\left(E-\frac{1}{8}J_1\right)\mu. \tag{24}$$

Recalling the definition of μ, we see that (24) describes a circular current which satisfies the conservation laws and the boundary conditions (the same pertains to $\tilde{\mathbf{q}}$), i.e., the assumption $\tilde{\Phi} = \tilde{T} = 0$ is indeed self-consistent. Using (24) we easily obtain the magnetic moment of the cylinder

$$\mathbf{M}=-\frac{e\alpha QR^2}{12c}\frac{\partial}{\partial\zeta}\left(\frac{\lambda}{p_Fv_F}\right)\mathbf{v}. \tag{25}$$

We note that the effect does not appear in the approximation $\lambda = pv$.

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