Fluctuation-induced magnetic susceptibility of superconductors and normal metals

L. G. Aslamazov and A. I. Larkin

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
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It is shown that above the transition temperature the fluctuation-induced diamagnetic susceptibility of superconductors decreases slowly with increasing temperature according to a logarithmic law. In a normal metal the interaction of the electrons also leads to a temperature dependence of the magnetic susceptibility. The case of an anisotropic metal is investigated. The effect of the fluctuations is enhanced in layered metals, and the fluctuation-induced magnetic susceptibility may appreciably exceed the value of the Landau diamagnetism; in particular, layered normal metals should exhibit a strong electronic paramagnetism. The fluctuation-induced magnetic susceptibility has the same order of magnitude in quasi-one-dimensional systems as in an isotropic metal. Allowance for impurities weakens the temperature dependence of the susceptibility at temperatures which are smaller than the inverse time of the electron mean free path.

A fluctuation-induced diamagnetism in superconductors above the transition temperature has been observed in both bulk superconductors and in layered and quasi-one-dimensional superconductors (of the A-15 structure).[1-5] In the present article it is shown that, at temperatures appreciably greater than the critical temperature T_c the fluctuation-induced diamagnetic susceptibility in a weak magnetic field slowly decreases with increasing temperature in proportion to $\chi_L \ln^{-2}(T/T_c)$, where χ_L is the value of the diamagnetic Landau susceptibility. In connection with this, it can be observed far away from the critical temperature.

The case of a normal metal is also investigated in this article. Although repulsion dominates in the interaction of the electrons in a normal metal, a fluctuation pairing of electrons does occur. As a result a fluctuation-induced paramagnetic susceptibility appears, which increases with decreasing temperature according to the law $\chi(T) - \chi(0) \sim -|\chi_L| \ln^{-2}(T_F/T)$, T_F denotes the Fermi temperature of the metal.

In a dirty metal the logarithmic dependences of the susceptibility are important only in the temperature range $T > \hbar/\tau_{tr}$, where τ_{tr} is the transport time of the electrons' mean free path. In the range $T < \hbar/\tau_{tr}$ the fluctuation-induced susceptibility varies with the temperature in proportion to $\sqrt{T\tau_{tr}}$.

A number of new effects appear in strongly anisotropic metals, when the Fermi surface is close to a cylinder or to a system of planes. The effect of the fluctuations is enhanced in layered systems. In connection with a small probability for electron jumps between the layers, the fluctuation-induced succeptibility may considerably exceed the diamagnetic Landau susceptibility (the enhancement factor is T_F/T). In connection with this, layered normal metals should display strong paramagnetism of the conduction electrons.

In quasi-one-dimensional systems and for a small probability of electron hopping between the filaments, the Landau diamagnetism is small. However, the fluctuation-induced susceptibility has the same order of magnitude as the Landau diamagnetism in isotropic metals. We also note that in quasi-one-dimensional superconductors with A-15 structure, the susceptibility has a characteristic temperature dependence near T_c which allows us to estimate the probability for electron hopping.

1. ISOTROPIC METAL

In order to determine the fluctuation-induced susceptibility far from the transition temperature (at high temperatures or in strong magnetic fields) it is impossible to use the Ginzburg-Landau expression for the free energy of a superconductor since here the short-wave fluctuations become essential.

The appropriate theory, starting from a general formula for the expansion of the free energy in a series in powers of the interaction constant, was constructed by Ambegaokar, Kurkijärvi, and Eilenberger[6] and also by Lee and Payne.[7] The theory gave the behavior of the fluctuation-induced susceptibility of isotropic superconductors in strong magnetic fields. However, the magnetic field dependence and the temperature dependence of the susceptibility could be found only with the aid of numerical calculations.

In the present article the fluctuation-induced susceptibility is found by starting from a general expression for the polarization operator, which is evaluated by the diagram technique. The developed method also has the advantage that it allows us to investigate the case of anisotropic metals.

In the presence of a weak field **A**, the diamagnetic current density **j** is given by the formula

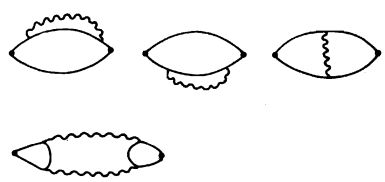
$$j_\alpha(\mathbf{k}) = -c^{-1} Q_{\alpha\delta}(\mathbf{k}) A_\delta(\mathbf{k}), \tag{1}$$

where $Q_{\alpha\delta}(\mathbf{k})$ is the polarization operator. Small values of k are essential in the presence of a homogeneous field, and we have

$$Q_{\alpha\delta} = \varepsilon_{\alpha\beta\mu}\varepsilon_{\delta\gamma\nu}\chi_{\mu\nu}k_\beta k_\gamma c^2/\hbar^2, \tag{2}$$

where $\epsilon_{\alpha\beta\mu}$ is the unit antisymmetric tensor, and $\chi_{\mu\nu}$ is the magnetic susceptibility which, generally speaking, is a tensor.

The four diagrams shown in the figure exist for Q upon taking account of the fluctuations to first order.



The vertex part of the electron-electron interaction is denoted by a wavy line:

$$L(\varepsilon, \mathbf{q})=[\lambda^{-1}+\Pi(\varepsilon, \mathbf{q})]^{-1}, \quad (3)$$

where λ is the coupling constant (in superconductors the electrons are attracted to each other and $\lambda < 0$, and in normal metals $\lambda > 0$), and as usual $\Pi(\epsilon, \mathbf{q})$ is given by the formula

$$\Pi(\varepsilon, \mathbf{q})=T\sum_{\omega}\int G_{\omega}(\mathbf{p})G_{-\omega+\varepsilon}(-\mathbf{p}+\mathbf{q})\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}, \quad (4)$$

$G_\omega(\mathbf{p})$ is the Green's function of a normal metal, which is denoted in the diagrams by a solid line. A vertex corresponds to the expression $ev_\alpha(\mathbf{p})$. We shall assume the external frequency to be equal to zero, and the external momentum to be small. As usual, the sum and integral are taken over the internal frequencies and momenta.[8]

Near the transition temperature the most important term is the last diagram, containing two wavy lines. However, for arbitrary temperatures it is necessary to take all diagrams into consideration. In this connection, in the pure case we obtain the following expression for Q at small $\mathbf{k}$:

$$Q_{\alpha\delta}(\mathbf{k})=-\frac{e^2T}{3}\sum_{\varepsilon}\int L^3\Pi_\alpha(2\Pi_{\beta\gamma}\Pi_\delta-\Pi_{\beta\delta}\Pi_\gamma-\Pi_{\gamma\delta}\Pi_\beta)k_\beta k_\gamma\frac{d^3\mathbf{q}}{(2\pi\hbar)^3}, \quad (5)$$

where the subscript on $\Pi(\epsilon, \mathbf{q})$ indicates differentiation with respect to the corresponding component of the vector $\mathbf{q}$. This formula enables us to determine the fluctuation-induced susceptibility over the entire range of temperatures in both isotropic and nonisotropic metals.

First of all let us investigate the case of an isotropic superconductor. Then it is obvious that Π does not depend on the direction of the vector $\mathbf{q}$. Introducing $\Pi'(\epsilon, q^2) = \partial\Pi(\epsilon, q^2)/\partial q^2$, from formulas (5) and (2) we obtain the following expression for the fluctuation-induced susceptibility:

$$\chi=-\frac{8}{9\pi^2}\frac{e^2T}{\hbar c^2}\sum_{\varepsilon}\int_0^{\infty}L^3q^4(\Pi')^3\,dq. \quad (6)$$

Near the transition temperature, we may keep only the term with $\epsilon = 0$ in the sum over ϵ, and the region of small q gives the essential contribution to the integral over q. In this region we obtain the usual expressions for Π' and L from Eqs. (3) and (4)

$$L=-\rho^{-1}(\tau+\eta q^2)^{-1}, \quad \Pi'=-\rho\eta, \quad (7)$$

where $\tau = (T - T_c)/T_c$, η is the parameter of the Ginzburg-Landau theory:

$$\eta=7\zeta(3)v_F^2/48\pi^2T_c^2,$$

and $\rho = mp_0/2\pi^2$ is the density of states.

Accordingly we obtain the following expression, first found by Schmid,[9] for the susceptibility:

$$\chi=-\frac{e^2T}{6\pi\hbar c^2}\left(\frac{\eta}{\tau}\right)^{1/2}=-0.83|\chi_L|\frac{1}{\tau^{1/2}}, \quad (8)$$

where $|\chi_L| = e^2v_F/12\pi^2\hbar c^2$ is the magnitude of the Landau diamagnetism.

At high temperatures, $T \gg T_c$, in formula (6) we may (with logarithmic accuracy) regard values of $\epsilon \sim vq \gg T$ as essential. In this region the expression for $\Pi(\epsilon, q)$ has the form

$$\Pi(\varepsilon, q)=\frac{1}{2}\left\langle\ln\frac{4\omega_D^2}{|\varepsilon|^2+(\mathbf{q}\partial\varepsilon_p/\partial\mathbf{p})^2}\right\rangle, \quad (9)$$

$$\langle\ldots\rangle=\int\ldots\delta(\varepsilon_p-\varepsilon_F)\frac{d^3\mathbf{p}}{(2\pi\hbar)^3},$$

where the angle brackets denote averaging over the Fermi surface. A formula can be immediately written down for the anisotropic case (ϵ_p denotes the spectrum of the electrons). In the isotropic case $\partial\epsilon_p/\partial\mathbf{p} = v_F\mathbf{p}/p_F$. Introducing the new variable α according to the formula $\alpha\epsilon = v_Fq$ and substituting expression (9) into formula (6), after the replacement of the sum over ϵ by an integral we have the following result (with logarithmic accuracy) for the susceptibility:

$$\chi=-c_0|\chi_L|\ln^{-2}(T/T_c), \quad (10)$$

where the constant c_0 is given by the formula

$$c_0=\frac{2}{3\pi}\int_0^{\infty}\frac{d\alpha}{\alpha^5}(\alpha-\operatorname{arctg}\alpha)^3=\frac{1}{4}\left(1-\frac{\pi^2}{12}\right). \quad (11)$$

Thus, we see that the reduction of the fluctuation-induced susceptibility with temperature takes place slowly, and this apparently explains the observed distant tails on the susceptibility.

The obtained formula is easily generalized to the case of a normal metal. Formulas (3)—(6) remain valid, and in place of formula (10) for the fluctuation-induced susceptibility over the entire range of temperatures we obtain the expression

$$\chi=-c_0|\chi_L|\left(\frac{1}{\lambda\rho}+\ln\frac{T_F}{T}\right)^{-2}=-c_0|\chi_L|\ln^{-2}\left(\frac{T^*}{T}\right), \quad (12)$$

where the unimportant term $c_0|\chi_L|(\lambda\rho)^2$, which does not depend on the temperature, has been omitted. Thus, in the case of a normal metal the temperature $T^* = T_Fe^{1/\lambda\rho}$ appears in the argument of the logarithm in place of the critical temperature T_c. We note that expression (12) vanishes at T = 0, where its derivative, however, is equal to infinity.

The derived formulas for the fluctuation-induced susceptibility assume that the interaction of the electrons is local. A nonlocal interaction leads to the result that the coupling constant λ depends on the angle between the electron momenta; then one can expand it in a series of Legendre polynomials. Only the zero-order term of this series, which determines the transition temperature, is important in superconductors near the critical temperature. In the general case the higher-order terms may become important; however, this will only affect the value of the coefficient c_0.

2. INFLUENCE OF IMPURITIES

At sufficiently high temperatures, $T > \hbar/\tau_{tr}$, the effect of impurities is small and the formulas derived in the pure case remain valid. However, for $T < \hbar/\tau_{tr}$ the fluctuation-induced susceptibility weakly depends on the temperature. For its determination in this range we shall utilize the following general formula for the fluctuation free energy in a magnetic field:[6]

$$F(H)=\frac{eHT}{\pi\hbar c}\sum_{n,\omega_m}\int\frac{dk}{2\pi}\ln[1+\lambda\Pi(n, k, \omega_m)], \quad (13)$$

where $\omega_m = 2\pi mT$ are the Matsubara frequencies, k is the longitudinal momentum, and n is the transverse quantum number.

In the limit of a large number of impurities, $\omega_m\tau_{tr}\hbar^{-1} \ll 1$ and $k\,l_{tr}\hbar^{-1} \ll 1$, the expression for $\Pi(n, k, \omega_m)$ has the form[10]

$$\Pi(n, k, \omega_m)=-\frac{1}{\lambda}-\rho\left\{\ln\frac{T}{T_0}+\psi\left(\frac{1}{2}+\frac{|\omega_m|+D[4eH(n+1/2)c^{-1}+k^2\hbar^{-1}]}{4\pi T}\right)-\psi\left(\frac{1}{2}\right)\right\}, \quad (13a)$$

where $\psi(z)$ denotes the logarithmic derivative of the gamma function, $D = v_F^2\tau_{tr}/3$ is the diffusion coefficient, $T_0 = T^*$ in a normal metal, and $T_0 = T_c$ in a superconductor.

For the determination of the fluctuation-induced susceptibility $\chi = -\partial^2 F/\partial H^2$ in the presence of a weak magnetic field H, we shall use the well known formula

$$H\sum_{n=0}^{\infty} f\left[H\left(n+\frac{1}{2}\right)\right] = H\int_{-1/2}^{\infty} f\left[H\left(n+\frac{1}{2}\right)\right]dn - \frac{H^2}{24}[f'(\infty)-f'(0)],$$

which gives the expansion in powers of the field correct to terms of order H^2. By introducing the new variable $x = H(n + 1/2)$, one can easily see that the term containing the integral over n does not depend on the magnetic field and therefore does not give a contribution to the susceptibility. As a result we obtain

$$\chi = -\frac{eT}{24\pi\hbar^2 c}\sum_{\omega_m}\int dk\,\frac{\Pi'(-1/2, k, \omega_m)}{1/\lambda+\Pi(-1/2, k, \omega_m)}, \qquad (14)$$

where Π' denotes the derivative with respect to the variable $x = H(n + 1/2)$. Values $\omega_m\tau_{tr}\hbar^{-1} \sim 1$ and $k l_{tr}\hbar^{-1} \sim 1$ are important in this formula. However, for a determination of the order of magnitude of the fluctuation-induced susceptibility one can use expression (13a) for $\Pi(n, k, \omega_m)$ and replace the sum over ω_m in formula (14) by an integral. In the range $T_c \ll T < \hbar/\tau_{tr}$ (only the right-hand inequality must be satisfied in a normal metal) here we have $\chi \approx -|\chi_L|\ln^{-1}\hbar/T_0\tau_{tr}$ with logarithmic accuracy.

In order to determine the temperature dependence of the susceptibility, we subtract from expression (14) its value at low temperatures ($T \sim T_c$ in superconductors and $T = 0$ in a normal metal). In this difference the essential values of ω_m and $Dk^2\hbar^{-1}$ still are of the order of $T \ll \hbar/\tau_{tr}$ and, using formula (13a) for $\Pi(n, k, \omega_m)$ in the limit of a large number of impurities, with logarithmic accuracy we obtain

$$\Delta\chi = c|\chi_L|\sqrt{\tau_{tr}T\hbar^{-1}}\ln^{-1}(T/T_0), \qquad (15)$$

$$c = -\frac{2}{\sqrt{3}\pi}\int_0^{\infty} dx\left\{\sum_m \psi'\left(\frac{1}{2}+\frac{|m|}{2}+x^2\right) - \int_{-\infty}^{+\infty}\frac{dm}{|m|/2+x^2}\right\} \approx 2.$$

In the terminology of Maki,[10] this formula corresponds to the contribution of the thermal fluctuations and the zero-point vibrations to the fluctuation-induced susceptibility. However, in contrast to Maki's assertion the zero-point vibrations give a contribution $\Delta\chi_0$ to the temperature-dependent part of the susceptibility which is of the same order of magnitude as that of the thermal fluctuations (namely, $\Delta\chi_0 = 4(1 - 1/\sqrt{2})\Delta\chi \approx 1.2\,\Delta\chi$, and the contribution of the thermal fluctuations $\Delta\chi_T = -(3 - 2\sqrt{2})\Delta\chi \approx -0.2\Delta\chi$).

Thus, in a normal metal the susceptibility decreases with increasing temperature at first according to the square-root law (15) and later exhibits the logarithmic dependence (12). In a dirty superconductor the susceptibility falls with decreasing temperature first according to the logarithmic law (10), but then in the range $T_c \ll T < \hbar/\tau_{tr}$ where logarithmic accuracy is valid, it falls according to the square-root dependence (15).

Near the critical temperature T_c the susceptibility can be found according to formula (8), where in the case of dirty metals $\eta = \psi'(1/2)v_F^2\tau_{tr}/12\pi\hbar T_c$. Obviously, the coefficient associated with $1/\sqrt{\tau}$ will also be proportional to $(\hbar^{-1}\tau_{tr}T_c)^{1/2}$.

3. LAYERED CONDUCTORS

Now let us determine the fluctuation-induced susceptibility of layered metals. Such systems usually consist of metallic layers each having a thickness of several Angstrom units and separated by layers of organic compounds of roughly the same thickness. If the probability for electron hopping between the layers is small, then the Fermi surface is a corrugated cylinder:

$$\varepsilon_p - \varepsilon_F = v_0(|p_\perp| - p_0) + w\cos(p_z a/\hbar), \qquad (16)$$

where $p_\perp$ is the projection of the electron momentum in the plane of the layer, v_0 and p_0 denote the velocity and Fermi momentum in this direction, p_z is the momentum in the perpendicular direction, a is the distance between layers, and w/ϵ_F is a small parameter characterizing the corrugations.

The value of the fluctuation-induced susceptibility significantly depends on the direction of the magnetic field. Let us first determine the temperature behavior of the susceptibility in a field perpendicular to the plane of the layers. Directing the magnetic field along the z axis, from formulas (2) and (5) we have the following expression for the susceptibility:

$$\chi_\perp = -\frac{2}{3}\frac{e^2T}{\hbar c^2}\sum_\varepsilon\int\frac{d^3q}{(2\pi)^3}L^2\Pi_x(\Pi_x\Pi_{yy}-\Pi_y\Pi_{xy}), \qquad (17)$$

where the expression for Π is found according to formula (4) with the anisotropic spectrum (16).

First let us consider the case of a layered superconductor. Near T_c we have the following expression for the fluctuation-induced susceptibility:

$$\chi_\perp = -\frac{9}{4\pi^2}\frac{e^2T}{\hbar c^2}\int_0^{\infty}\int_{-\pi\hbar/a}^{\pi\hbar/a}\eta^3 q_\perp{}^3\left(\tau+\frac{3}{2}\eta q_\perp{}^2+6\eta\frac{w^2}{v_0{}^2}\sin^2\frac{q_z a}{2\hbar}\right)^{-3}dq_\perp\,dq_z. \qquad (18)$$

After integration we obtain

$$\chi_\perp = -\frac{1}{2\pi}\frac{Te^2\eta}{ac^2}\frac{1}{(\tau^2+6\eta w^2\tau/v_0{}^2)^{1/2}}. \qquad (19)$$

This formula agrees with Yamaji's result, which he obtained phenomenologically by starting from the Ginzburg-Landau expression for the free energy of a layered superconductor with the Josephson energy taken into account.[11] However, $\ln(T/T_c)$ appears in Yamaji's formula in place of τ, which is an excess of accuracy since this formula is only valid near T_c.

The temperature behavior of the susceptibility significantly depends on the value of the parameter w. Let us investigate limiting cases for various relationships between the parameters T, w, and T_c. If $w > T_c$, then up to a temperature of the order of w the temperature dependence is the same as in the three-dimensional case. In fact, in the region $\tau < 1$ and for $w > T_c\sqrt{\tau}$ we obtain

$$\chi = -\frac{1}{2\sqrt{6}\pi}\frac{e^2T}{c^2}\frac{v_0}{wa}\left(\frac{\eta}{\tau}\right)^{1/2}. \qquad (20)$$

from formula (19). In the region $T \gg T_c$ one can find the susceptibility according to formulas (9) and (17) in almost the same way as in the isotropic case. In this connection the velocity at the Fermi surface, $v_F = \partial\epsilon_p/\partial p$, is found from formula (16). Introducing the new variables $v_0 q_\perp = \alpha_1\epsilon$ and $waq_z/\hbar = \alpha_2\epsilon$, with logarithmic accuracy we have

$$\chi_\perp = -\frac{c_2}{12\pi^6}\frac{e^2v_0}{c^2}\frac{v_0}{wa}\ln^{-2}\left(\frac{T}{T_c}\right), \qquad (21)$$

$$c_2 = \int_0^{\infty} d\alpha_1\,d\alpha_2\left[\int_0^{\pi}\frac{i\cos x\,dx}{\sqrt{(1+i\alpha_1\cos x)^2+\alpha_2{}^2}}\right]^3 = 1{,}3.$$

Just as in the three-dimensional case, the behavior of the susceptibility down to temperatures $T \sim w$ is explained by the fact that the size of a pair in the direction perpendicular to the layers, $\xi_\perp \sim \hbar v/T \sim wa/T$, is bigger at these temperatures than the distance a between the layers. The increase of the susceptibility by a factor of $\hbar v_0/wa$ in comparison with the three-dimensional case is related to the enhancement of fluctuations in two-dimensional systems.

In the temperature range $T > w$ the size of a pair becomes smaller than the distance between the layers, and they behave as purely two-dimensional systems. In this case the values $\epsilon \sim vq \sim T$ are essential in formula (17) and, determining Π according to the general formula (4), after the change of variable $vq = \alpha T$ we have (with logarithmic accuracy)

$$\chi_\perp = -\frac{2c_2'}{3\pi^3}\frac{e^2v_0}{c^2}\frac{v_0}{aT}\ln^{-3}\left(\frac{T}{T_c}\right), \qquad (22)$$

$$c_2' = \int_0^\infty \left[\left(\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{(n^2+\alpha^2)^{1/2}}\right)^3 + \sum_{m>0}\left(\sum_{n>m}^{\infty}\frac{1}{(n^2+\alpha^2)^{1/2}}\right)^3\right]\alpha^3\,d\alpha = 0.06$$

(n is even if m is odd, and vice versa). Thus, for $T > w$ the susceptibility will fall with increasing temperature faster than in the three-dimensional case.

However, if the parameter w is very small, $w < T_c$, even near T_c the susceptibility ceases to depend on w for $\tau > w^2/T_c^2$ and is given by the formula

$$\chi_\perp = -\frac{1}{2\pi}\frac{e^2T}{c^2}\frac{\eta}{a\tau}. \qquad (23)$$

Formula (22) is then valid over the entire region $T \gg T_c$. In this case also, there is an enhancement of the fluctuations by a factor of ξ/a in comparison with the three-dimensional case ($\xi \sim \hbar v_0/T\sqrt{1 - T_c/T}$). Only two regions exist in a normal metal. For $T > w$ formula (22) is valid with the replacement of T_c by the temperature T*, which is determined by formula (12). The fluctuation-induced susceptibility is paramagnetic ($\ln(T/T^*) < 0$) and increases rapidly with decreasing temperature. The value of the fluctuation-induced susceptibility is larger than the ordinary susceptibility of a normal metal by roughly ϵ_F/T times.

In a normal metal in the temperature range $T < w$, the fluctuation-induced susceptibility slowly increases with decreasing temperature, and as $T \to 0$ it tends to the constant limit:

$$\chi_\perp = \frac{c_2}{12\pi^6}\frac{e^2v_0^2}{wac^2}\left(\frac{1}{\ln^2(T^*/w)} - \frac{1}{\ln^2(T^*/T)}\right). \qquad (24)$$

Thus, a large paramagnetism should be observed in layered normal metals.

As we have already noted, the derived results pertain to the temperature behavior of the susceptibility in the presence of a magnetic field perpendicular to the layers. Shielding diamagnetic currents generally do not arise in a parallel field without electron hopping between the layers, and therefore there is no fluctuation-induced susceptibility either.

It turns out that the susceptibility in a parallel field is proportional to the small parameter w. This is related to the fact that the susceptibility is proportional to the square of the velocity of electron hopping $v_\perp \sim w$ (two jumps are required in order to produce a closed current loop) and inversely proportional to w due to the enhancement of fluctuations in two-dimensional systems.

Impurities in layered superconductors not only scatter electrons in the plane of the layers, but also facilitate electron transitions between the layers. If the transition probability is large, the qualitative picture of the behavior of the susceptibility does not differ from the isotropic case. For a small transition probability, the two-dimensional picture is preserved; however, the impurity scattering in the plane of the layers leads to a weakening of the temperature dependence for $T < \hbar/\tau_{tr}$.

As shown by Bulaevskiĭ,[12] the susceptibility depends on the temperature very weakly in the range $T_c \ll T < \hbar/\tau_{tr}$, like $\ln\ln(T/T_c)$.

Formula (19) is valid in a superconductor near the transition temperature, where the expression for η is taken for a very dirty metal.

4. QUASI-ONE-DIMENSIONAL METALS

Now let us proceed to an investigation of quasi-one-dimensional superconductors. An example of such are the compounds with A-15 structure of the type V_3X or Nb_3X, in which the atoms of vanadium or niobium are distributed along straight lines, forming three parallel sets along three mutually perpendicular directions. In this connection the s electrons, as usual, undergo collective motion, and the motion of the d electrons along the chains of vanadium or niobium atoms has a quasi-one-dimensional character.

The d electrons give the major contribution to the fluctuation-induced diamagnetic susceptibility, since their density of states is large. In the first approximation, the Fermi surface of the d electrons is formed by six planes: For each system of parallel filaments there are two parallel planes at a distance $2p_F$ from each other and positioned perpendicular to the corresponding direction.

Electron hopping between the filaments or the exchange interaction between the s and d electrons lead to the result that these planes are slightly warped. The behavior of the fluctuation-induced susceptibility essentially depends on the relationship between the temperature and the parameter characterizing the quasi-one-dimensionality:

$$w = \epsilon_F p_F/p_F', \qquad (25)$$

where p_F' is the radius of curvature of the Fermi surface.

If w is small in comparison with the transition temperature T_c, a "parquet" situation arises for $T \gg T_c$. The fluctuation-induced diamagnetic susceptibility turns out to be small in this region. However, in the case $w < T_c$ the region near T_c does not yield to quantitative calculations since here it is generally impossible to isolate the important diagrams.

We shall confine the investigation to the case $T_c < w < \epsilon_F$, which is satisfied for a number of compounds. Here the Landau diamagnetism is proportional to the square of the parameter w and, as a consequence of this, is small. However, in the temperature range $T < w$ the fluctuation-induced susceptibility turns out to generally not depend on the parameter w as a consequence of the enhancement of the fluctuations in one-dimensional systems.

As has already been mentioned, in the A-15 structure the d electrons primarily move along filaments which form three parallel sets. In each set of parallel fila-

ments one can change to the momentum representation and assume that the Green's function only depends on the electron momentum. Then for the entire structure (if it is assumed that the electron transitions between the sets of filaments are small) the Green's function is a matrix whose subscripts indicate the labels of the sets to which the electron belongs:

$$G_{ik}(p)=\langle a_i^+(p)a_k(p)\rangle, \tag{26}$$

where a_i^+ is the creation operator for an electron in the i-th set, a_k is the annihilation operator for an electron in the k-th set, and moreover these operators depend on the electron momentum in each set, and the subscripts i and k take the three values corresponding to the labels of the sets.

In this case formula (5) for the fluctuation-induced diamagnetic response of the system has the form

$$Q_{\alpha\delta}(k)=-\frac{e^2T}{3}\sum_{\varepsilon}\int\frac{d^3q}{(2\pi\hbar)^3}L^{ik}\Pi_{\alpha}^{kl}(2L^{lm}\Pi_{\beta\gamma}^{mn}L^{nj}\Pi_{\delta}^{ji}$$
$$-L^{lm}\Pi_{\beta\delta}^{mn}L^{nj}\Pi_{\gamma}^{ji}-L^{lm}\Pi_{\gamma\delta}^{mn}L^{nj}\Pi_{\beta}^{ji})k_{\beta}k_{\gamma}, \tag{27}$$

$$\Pi^{ik}=T\sum_{\omega}\int G_{\omega}^{ik}(p)G_{-\omega+\varepsilon}^{ik}(-p+q)\frac{d^3p}{(2\pi\hbar)^3},\quad L^{ik}=(\lambda^{-1}+\Pi^{ik})^{-1}, \tag{28}$$

and the lower subscripts on Π denote differentiation with respect to the corresponding component of the vector $\mathbf{q}$. The difference from the isotropic case here consists in the fact that it is now impossible to take the matrix L outside the brackets and it is necessary to place it between the appropriate matrices Π. After multiplication of these matrices, the sum of the diagonal elements is taken.

First let us consider the case when T is close to T_c. Then we have the following expression for the matrix L^{-1}:

$$L^{-1}=\lambda^{-1}+\Pi$$
$$=-\rho\begin{pmatrix}\tau+2\alpha+\eta q_1^2+\beta(q_2^2+q_3^2); & -\alpha; & -\alpha\\ -\alpha; & \tau+2\alpha+\eta q_2^2+\beta(q_1^2+q_3^2); & -\alpha\\ -\alpha; & -\alpha; & \tau+2\alpha+\eta q_3^2+\beta(q_1^2+q_2^2)\end{pmatrix}. \tag{29}$$

The diagonal elements of this matrix correspond to linear Ginzburg-Landau equations for each set of filaments. The parameter $\tau=(T-T_c)/T_c$, η is the usual parameter of the Ginzburg-Landau theory which is proportional to v_F^2/T_c^2, β is a small parameter proportional to the square of the radius of curvature p_F' of the Fermi surface, and ρ is the density of states. The off-diagonal elements arise as a consequence of the electron jumps between different sets of filaments: α is a small parameter proportional to the radius of curvature of the Fermi surface at the point of intersection of the planes. The parameter α in the diagonal terms arises as a consequence of the variation of the transition temperature due to the electron hopping.

In formula (27) for T close to T_c, as usual let us keep the first term and set $\epsilon=0$. Due to cubic symmetry the fluctuation-induced susceptibility has the form $\chi_{\mu\nu}=\chi\delta_{\mu\nu}$ and, directing the magnetic field along the z axis, we obtain from formula (27)

$$\chi=-\frac{2e^2T}{3}\int\frac{d^3q}{(2\pi\hbar)^3}[(L^{11})^3(\Pi_x^{11})^2\Pi_{yy}^{11}+(L^{22})^3(\Pi_x^{22})^2\Pi_{yy}^{22} \tag{30}$$
$$+L^{11}L^{12}L^{21}(\Pi_x^{11})^2\Pi_{yy}^{22}],$$

where the subscripts x and y denote differentiation with respect to the corresponding components of the vector $\mathbf{q}$. Here only the terms corresponding to the contribution from the two sets of filaments which are perpendicular to the magnetic field are kept, since the contribution from the third set, which is directed along the field, is small.

We find the components of the matrix L and the derivatives of the matrix Π from formula (28). Substituting these results into formula (30), we find after integration that for $\tau\ll\alpha$ only the second term is important, and moreover in this region we have the three-dimensional result:

$$\chi=-\frac{e^2T}{6\pi\hbar c^2}\left(\frac{\eta}{\tau}\right)^{1/2}. \tag{31}$$

On the other hand, the first term gives the major contribution for $\tau\gg\alpha$

$$\chi=-\frac{e^2T}{3\pi\hbar c^2}\left(\frac{\eta}{\tau}\right)^{1/2}. \tag{32}$$

This unusual temperature behavior of the susceptibility is associated with the fact that, for $\tau\gg\alpha$ the probability for transitions of superconducting pairs from one system of filaments to the other is small, and each system, which is perpendicular to the field, gives its own independent contribution to the susceptibility. On the other hand, in the region $\tau\ll\alpha$ the pairs undergo collective motion, and we obtain the same answer as in the three-dimensional case. We note that experimental data for the temperature dependence of the susceptibility near T_c would enable us to estimate the value of the parameter α.

In the region of high temperatures, $T\gg T_c$, each system of filaments gives an independent contribution to the susceptibility, and it is sufficient for us to find the fluctuation-induced susceptibility of a single system of filaments which are perpendicular to the magnetic field. In this connection it turns out that for $T_c\ll T\ll w$ the temperature behavior of the susceptibility is the same as in the three-dimensional case:

$$\chi=-c_1|\chi_L|\ln^{-2}(T/T_c), \tag{33}$$

where the constant c_1 is a positive number of the order of unity, but depending on the shape of the system's Fermi surface, and $|\chi_L|$ is the magnitude of the Landau diamagnetism in an isotropic metal.

Before beginning the derivation of formula (33), let us give a qualitative explanation of it. The result is related to the fact that, in this range of temperatures the size of a pair in the direction perpendicular to a filament, $\xi_\perp\sim\hbar v_\perp/T\sim wa/T$, exceeds the distance between the filaments. However, in contrast to the case of a layered superconductor, here the constant c_1 is of the same order of magnitude as in the three-dimensional case, and there is no enhancement of the effect, namely, for a system of filaments the fluctuation-induced susceptibility is proportional to the square of the electron hopping velocity, $v_\perp\sim w$, and inversely proportional to w^2 owing to the enhancement of fluctuations in one-dimensional systems.

Let us derive derive formula (33) assuming that the hopping of the electrons between filaments occurs only as a consequence of the weak exchange interaction between the s and d electrons. Then we have the following system of self-consistent equations for the electron Green's functions $G_{ik}^d(p)$ and $G_k^s(p)$:

$$(\omega-\omega_i)G_{ik}^d+uG_k^s=\delta_{ik},$$
$$(\omega-\omega_s(p))G_k^s+u\sum G_{ik}^d=0, \tag{34}$$

where ω is the energy variable, $\omega_i = v_F(|p_i| - p_F)$, p_i is the component of the electron's momentum in the direction of the i-th set of filaments, $\omega_s(p)$ is the spectrum of the s electrons, and u is the small coupling constant of the exchange interaction ($u \ll \epsilon_F$).

Solving the system (34), one can find the Green's function of the d electrons:

$$G_{ik}^{d}(\omega, p) = \left[\delta_{ik} - W\left(\frac{1}{\omega-\omega_k} - \delta_{ik}\sum_j \frac{1}{\omega-\omega_j}\right)\right] \times \left[(\omega-\omega_i)\left(1 + W\sum_j \frac{1}{\omega-\omega_j}\right)\right]^{-1}, \quad (35)$$

where $W = u^2/(\omega_s(p) - \omega)$, and then from formula (28) one can find the matrices Π^{ik} and L^{ik}.

As has already been mentioned, for $T - T_c \gg \alpha T_c$, where α is a small parameter determined by the off-diagonal elements of the matrix G_{ik}, each set of filaments gives its own independent contribution to the susceptibility. This contribution is determined by the corresponding diagonal element G_{ii} of the matrix Green's function.

We shall assume that the magnetic field is directed along the z axis and along the third set of filaments, and we shall determine the fluctuation-induced magnetic susceptibility of, for example, the first set, which is directed along the x axis. The corresponding diagonal element G_{11} is found from formula (35) and has a pole at ω close to ω_1 and also near ω_2 and ω_3. However, the residues at the poles close to ω_2 and ω_3 are proportional to u^2/ϵ_F^2 and, as a consequence of this, are small. Therefore, in connection with the determination of $\Pi^{11}(\epsilon, q)$ according to formula (28), the region of p close to the segments of the Fermi surface corresponding to the first set of filaments, $|p_x| = p_F$, gives the major contribution to the integral. From formula (35) we obtain the following expression for G_{11} near the pole at ω_i:

$$G_{11} = \left\{\omega - \xi - w\left(1 - \frac{p_y^2 + p_z^2}{p_0^2}\right)^{-1}\right\}^{-1}, \quad (36)$$

where $\xi = \omega_1 = v_F(|p_x| - p_F)$, the parameter $w = u^2 p_{FS}^2/p_0^2\epsilon_F$ as usual determines the curvature of the Fermi surface, $p_0 = \sqrt{|p_{FS}^2 - p_F^2|}$, and p_{FS} and p_F denote the Fermi momenta of the s and d electrons.

For high temperatures, $T \gg T_c$, as usual the sum over ω in formula (28) for $\Pi(\epsilon, q)$ can be replaced by an integral. Using in this connection expression (36) for G_{11}, after the integrations over ω and ξ we have the following result for Π^{11}:

$$\Pi^{11}(\varepsilon, q) = \frac{\rho}{2}\left[\ln\frac{4\omega_D^2}{|\varepsilon|^2} - \int_{-k_0/p_0}^{k_0/p_0} \ln\left[1 + \left(\alpha_1 + \frac{\alpha^2 y}{(1-y^2-z^2)^2} + \frac{\alpha_3 z}{(1-y^2-z^2)^2}\right)^2\right]\frac{dy\,dz}{4k_0^2/p_0^2}\right], \quad (37)$$

where the following notation has been introduced: $v_F q_x = \alpha_1\epsilon$, $2wq_y/p_0 = \alpha_2\epsilon$, $2wq_z/p_0 = \alpha_3\epsilon$, and the following change of variables has been made: $y = p_y/p_0$, $z = p_z/p_0$; k_0 is the magnitude of the reciprocal lattice vector which determines the region of the integration with respect to p_y and p_z.

In order to determine the fluctuation-induced susceptibility for $T \gg T_c$, we shall utilize formula (27) where all terms are still important. However, after multiplication of the matrices we may keep only the terms corresponding to the contribution from the first set of filaments:

$$\chi = \frac{2e^2T}{3}\sum_\epsilon \int \frac{d^3q}{(2\pi\hbar)^3}\left[(L^{11})^3\Pi_x^{11}(\Pi_x^{11}\Pi_{yy}^{11} - \Pi_y^{11}\Pi_{xy}^{11})\right]. \quad (38)$$

The derivatives Π^{11} are found with the aid of formula (37), and L^{11} is determined by formula (28). In this connection small values of $q \sim p_0 T/w \ll k_0$ are important in formula (38) in the case $T < w$, and the integral over q can be regarded as extending to infinity. Changing to the dimensionless variables α_1, α_2, and α_3 we obtain (with logarithmic accuracy) formula (33) for the susceptibility, where c_1 is a function of the dimensionless variable k_0/p_0.

Values of $q \sim k_0$ are essential in formula (38) for $T > w$, and the integrand is proportional to w^2. Therefore, the susceptibility turns out to be proportional to $(w/T)^2$. Thus, in the region $T > w$ the fluctuation-induced susceptibility of the system of filaments falls with temperature faster than for isotropic or layered superconductors.

Let us also determine the value of the parameter α, which determines the range of temperatures near T_c in which the transitions of superconducting fluctuating pairs from one set of filaments to another are important. For this purpose it is necessary, as is clear from formula (29), to calculate the off-diagonal elements of the matrix Π^{ik}, for example, the element Π^{12} corresponding to the transitions of pairs from the first set to the second.

From formula (35) for G_{12} one can see that this function, just like G_{11}, has poles for ω close to ω_1, ω_2, and ω_3. However, the residue at the last pole is always small, but at the poles close to ω_1 and ω_2 it isn't small provided that ω_1 is very close to ω_2. In this connection, in the neighborhood of the pole E we have the following expression for G_{12}:

$$G_{12} = -\frac{W}{[(\omega_1-\omega_2)^2 + 4W^2]^{1/2}}\frac{1}{\omega - E}, \quad (39)$$

where W can be regarded as independent of p_x and p_y, having set $|p_x| = |p_y| = p_F$.

Only the range of values of p close to the lines of intersection of the Fermi surfaces of the corresponding sets ($|p_x| = |p_y| = |p_z|$) gives an important contribution to the integral over p in formula (28) for $\Pi^{12}(\epsilon, q)$. Using formula (39), it is not difficult for $q = \epsilon = 0$ to obtain

$$\Pi^{12} = \rho' \ln\frac{1.14\omega_D}{T},$$
$$\rho' = \frac{w' p_0'}{4\pi^2 v_F^2}\operatorname{arctg}\frac{k_0}{p_0'},$$
$$p_0' = (|2p_F^2 - p_{Fs}^2|)^{1/2}, \quad w' = \frac{u^2}{\varepsilon_F}\left(\frac{p_{Fs}}{p_0'}\right)^2. \quad (40)$$

Accordingly for the parameter α we have

$$\alpha = \frac{1}{\rho}\Pi^{12} = \frac{w'}{\varepsilon_F}\frac{p_F p_0'}{16\pi^2 k_0^2}\operatorname{arctg}\frac{k_0}{p_0'}\ln\frac{1.14\omega_D}{T_c}, \quad (41)$$

and therefore α is the same order of magnitude as the cited curvature of the Fermi surface, p_F/p_F'.

In a normal quasi-one-dimensional metal at low temperatures, $T < w$, the temperature-dependent part of the fluctuation-induced susceptibility is given by formula (33) with the replacement of T_c by T^*. For $T > w$ the fluctuation-induced susceptibility is proportional to $(w/T)^2$.

In the physically interesting region $T < w$, where the fluctuation-induced susceptibility is not small, the impurities smooth out the temperature dependences for $T < \hbar/\tau_{tr}$, just as in the isotropic case.

5. DISCUSSION OF THE RESULTS

The large value for the fluctuation-induced susceptibility of a metal has a simple physical explanation. Although the density of fluctuating pairs is small, they have large dimensions and give an appreciable contribution to the magnetic susceptibility. One can use the well known Langevin formula for the diamagnetism of atoms [13] for a qualitative interpretation of the obtained results:

$$\chi \sim e^2 \langle r^2 \rangle n/mc^2, \tag{42}$$

where in the case of superconducting pairs r has the meaning of the size of the pair ξ, and n and m are their density and mass. In this connection the superconducting pairs obey Bose statistics:

$$n_p = [e^{\varepsilon(p)/T} - 1]^{-1} \approx T/\varepsilon(p).$$

In the anisotropic case the size of a pair depends on the direction. In order of magnitude

$$\xi_{x,y,z} \sim \hbar v^F_{x,y,z}/T\sqrt{1-T_c/T}, \tag{43}$$

where $V^F_{x,y,z}$ are the Fermi velocities in the corresponding directions. Choosing the z axis along the direction of the field H, we have

$$\langle r^2 \rangle \sim \xi_x \xi_y, \quad m = \sqrt{m_x m_y}, \quad m^{-1}_{x,y} = \partial^2 \varepsilon(p)/\partial p^2_{x,y},$$

and the spectrum $\epsilon(p)$ is a function of the dimensionless variables $\hbar^{-1}p_x\xi_x$, $\hbar^{-1}p_y\xi_y$, and $\hbar^{-1}p_z\xi_z$.

For the fluctuation-induced susceptibility we have

$$\chi \sim \frac{e^2T}{\hbar^3c^2}\xi_x^2\xi_y^2 \int \varphi(\hbar^{-1}p_x\xi_x, \hbar^{-1}p_y\xi_y, \hbar^{-1}p_z\xi_z)\, dp_x\, dp_y\, dp_z, \tag{44}$$

where φ is a certain function of the order of unity for $\hbar^{-1}p_x\xi_x$, $\hbar^{-1}p_y\xi_y$, $\hbar^{-1}p_z\xi_z \sim 1$, and the integral is taken over all values of the momenta within the limits of the reciprocal lattice.

The order of magnitude of the integral in formula (44) is determined by the size of the region of important values of the variables $\tilde{p}_x$, $\tilde{p}_y$, and $\tilde{p}_z$; here, for example, $\tilde{p}_x \sim \hbar/\xi_x$ if the size of a pair is larger than the interatomic distance a, and $\tilde{p}_x \sim \hbar/a$ in the opposite case. As a result we obtain the formula

$$\chi \sim \frac{e^2T}{\hbar^2c^2} \frac{\xi_x^2\xi_y^2}{(\xi_x+a)(\xi_y+a)(\xi_z+a)}, \tag{45}$$

which gives the correct order of magnitude for the fluctuation-induced susceptibility of an anisotropic superconductor over the entire temperature range (although the slowly varying logarithmic dependences are not taken into consideration).

As is obvious, the enhancement of the effect of fluctuations in a layered superconductor takes place due to the following reasons. The size and mass of the pairs in the plane of the layers remain the same as in the three-dimensional case. However, the size of a pair ξ_z in the perpendicular direction may become small. This increases the density of the fluctuating pairs and leads to an appreciable increase of the fluctuation-induced susceptibility.

In the presence of impurities the temperature dependent part of the fluctuation-induced susceptibility can also be found according to formula (45) with the replacement of ξ by $\xi_{eff} = \sqrt{\xi l}$, where l is the mean free path. In this case there is also a large, temperature independent term which one can obtain by the same method if the contribution of the zero-point vibrations to the density of fluctuating pairs is taken into account.

In a normal metal the repulsion of the electrons leads to a decrease of the diamagnetism (the fluctuation-induced susceptibility turns out to be paramagnetic). One can easily understand this in an isotropic metal, when the fluctuation-induced susceptibility is a small correction to the Landau diamagnetism. The fluctuation-induced susceptibility is proportional to an odd power of the coupling constant and changes sign together with it. In layered normal metals the fluctuation-induced susceptibility may appreciably exceed the Landau diamagnetism, and they should exhibit a strong paramagnetism. For arbitrary anisotropy, formula (45) with $T_c = 0$ again gives the correct order of magnitude for the fluctuation-induced susceptibility in a normal metal.

We note that if in superconductors the susceptibility monotonically decreases with decreasing temperature as a consequence of fluctuation effects (they become more diamagnetic), in normal metals the susceptibility will, on the other hand, increase.

Let us also present a conjectured restriction on the magnetic field: $H < \Phi_0/\xi_x\xi_y$, where $\Phi_0 = \hbar c/2e$ is the flux quantum.

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