Current state of semimetals with paramagnetic impurities

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The effect of paramagnetic impurities on the character of the coupling between the field and the current in a semimetal is considered. It is shown that allowance for the correlations, which lead to the Kondo effect, in the interaction of the impurity spin with the conduction electrons, leads to violation of Ohm's law at relatively large currents. The estimates give grounds for hoping that this effect can be experimentally observed not only in semimetals but also in certain metals, such as copper.

1. INTRODUCTION

The investigation of metals in the current states (CS) is of considerable interest in those cases when noticeable deviations from Ohm's law are produced. These deviations carry information concerning the subsystems that take part in the realization of the total energy balance[1,2]. As a rule, this gives rise to a nonlinearity due to energy drain (NED) from the electron subsystem. Relatively speaking, this drain proceeds in accordance with the following scheme: The electron acquires between the scattering acts an energy on the order of eEl (where E is the electric field and l is the mean free path). If this energy reaches a value ΔE that can be received by another subsystem, then inelastic scattering takes place, and is supplementary to the existing scattering mechanisms that are realized at energies below ΔE. Thus, the NED is due to the turning on of "new" scattering channels at definite field values. In the case of interaction with phonons, this proceeds continuously (temperature $T \neq 0$), and the NED will be strongly smoothed out on the current-voltage characteristic of the sample, and will manifest itself only when Cerenkov emission of phonons sets in at $v \gg s$ (v is the drift velocity of the electron gas and s is the speed of sound). In the case when there exists a discrete subsystem interacting with the electrons, the NED is more strongly pronounced because of the jumplike turning on of additional scattering mechanisms.

In addition to the NED, there exists also another mechanism that leads to violation of Ohm's law. This is the nonlinearity due to the change in the electron distribution (NCED), which generally speaking was investigated in many studies, but as a rule was connected with heating effects or with "hot electrons."

In this paper we consider a new NCED mechanism, which becomes most clearly manifest in the absence of electron heating, and consequently can be observed in pure form only in pulsed measurements. In essence, this mechanism becomes manifest in elastic scattering, and the role of the inelasticity reduces only to maintenance of the energy balance in the system. Before we examine how elastic scattering can lead to NCED, we must dwell on the actual meaning of the CS. In this paper we define the CS as an equilibrium homogeneous state that is realized because of the uniform distribution of the momentum by the electron subsystem from the external field. The only cause of such a distribution are the electron-electron collisions. It is necessary here that the rate of exchange of momentum between the carriers be much higher than between the carriers and the scattering centers. Quantitatively, this condition can be written in the form $\zeta = \tau_e/\tau_i \ll 1$ (where τ_e is the electron collision time and τ_i is the electron-impurity collision time). If $\zeta \gtrsim 1$, then the results obtained below are incorrect quantitatively, although qualitatively this effect is preserved. For a degenerate electron gas, the order of magnitude of ζ is $n_i\epsilon_0^2/nT^2$, where n_i is the impurity density, n is the carrier density, and ϵ_0 is the Fermi energy. It is easy to estimate that the condition $\zeta \ll 1$ can be realized at present only in semimetals or in degenerate semiconductors.

The reason for the appearance of NCED not due to heating can be understood from the following simple considerations: In the CS, the distribution function in momentum space is shifted by an amount $\Delta p = mv \sim eEl/v_0$ (v_0 is the electron velocity on the Fermi surface). If the state density has certain singularities in the vicinity of the Fermi energy, then a change in the conductivity will take place when the drift velocity is sufficient to permit some group of electrons to reach this singularity. In the isotropic case, this will occur first for electrons moving parallel or antiparallel to the field, depending on whether the singularity lies above or below the Fermi energy. A nonlinearity of the same type is produced also in the case of scattering by paramagnetic impurities. In this case, as is well known (see, e.g., the review of Abrikosov[3]), the static resistivity at $T \gg T_K$ (T_K is the Kondo temperature[3,4]) contains a term proportional to $\ln(T/\epsilon_0)$. The appearance of the logarithm is connected with the following factors:

1) the existence of an internal degree of freedom of the scattering center (impurity spin);

2) Fermi statistics of the carriers (the abrupt change of the electron density in the vicinity of the energy Fermi);

3) elastic scattering (a term $\ln(\Delta E/\epsilon_0)$ would appear in the case of inelastic scattering).

It must be emphasized that the temperature under the logarithm sign is due to the "smearing" of the electron distribution near the end-point energy in the energy region on the order of T. Inasmuch as a similar "smearing" takes place in the CS in an energy region on the order of $p_0v \approx eEl$, one should expect $\ln(eEl/\epsilon_0)$ to appear in this case instead of $\ln(T/\epsilon_0)$ (if $eEl > T$). We note that the smearing of the distribution in the CS differs significantly from the temperature smearing, since it has sharper upper and lower boundaries, and in addition each energy in the smearing region corresponds to a definite angle of electron motion relative to the current-flow direction. Nonetheless, the qualitative conclusion that the resistivity has a logarithmic dependence on the field is confirmed by a more rigorous calculation presented below.

2. GENERAL RELATIONS

We consider a system of electrons interacting with one another, with phonons, and with paramagnetic impurities. We assume that the system is in a potential field that varies slowly in space and in time, with potential $U(\mathbf{r}, t)$. The field will henceforth be regarded as constant and homogeneous, but up to a certain stage in the reasoning it is convenient to ignore this fact, so as to avoid formal difficulties connected with the fact that a constant field has an unlimited potential.

We introduce the nonequilibrium causal Green's function

$$g(1.1')=\frac{1}{2i}\sum_{\sigma}\langle T\Psi_{\sigma}(1)\Psi_{\sigma}^{+}(1')\rangle \tag{1}$$

and the correlation functions

$$\begin{aligned} g^{>}(1.1')&=\frac{1}{2i}\sum_{\sigma}\langle \Psi_{\sigma}(1)\Psi_{\sigma}^{+}(1')\rangle,\\ g^{<}(1.1')&=-\frac{1}{2i}\sum_{\sigma}\langle \Psi_{\sigma}^{+}(1')\Psi_{\sigma}(1)\rangle. \end{aligned} \tag{2}$$

In relations (1) and (2), $\Psi_{\sigma}^{+}(1)$ and $\Psi_{\sigma}(1)$ are the field operators of the creation and annihilation of electrons with spin σ after the space-time point $\mathbf{r}$, t (the time dependence is taken in the Heisenberg representation), the angle brackets $\langle\ldots\rangle$ denote thermodynamic averaging, and the superscripts > and < indicate that $g^{>}$ coincides with g at $t_1 > t_1'$ and $g^{<}$ coincides with g at $t_1 < t_1'$. It is necessary to introduce the functions $g^{\gtrless}$ because there are no non-equilibrium equations for g in closed form. In essence, it suffices to consider only the two functions $g^{\gtrless}$, since g can be obtained from them in trivial fashion.

Taking into account the slow variation of the potential $U(\mathbf{r}, t)$, it is convenient to change over to the new variables $\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_1'$, $\tau = t_1 - t_1'$; $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_1')/2$, $t = (t_1 + t_1')/2$, and to take the Fourier transforms with respect to the variables $\mathbf{x}$ and τ:

$$g^{\gtrless}(\mathbf{p},\omega;\mathbf{r},t)=\pm i\int d\mathbf{x}\int d\tau g^{\gtrless}(\mathbf{x},\tau;r,t)\exp[-i\mathbf{p}\mathbf{x}+i\omega\tau],$$

$$g(\mathbf{p},\omega;\mathbf{r},t)=\int d\mathbf{x}\int d\tau g(\mathbf{x},\tau;\mathbf{r},t)\exp[-i\mathbf{p}\mathbf{x}+i\omega\tau],$$

where the plus sign is taken for $g^{>}$ and the minus sign for $g^{<}$. The dependence of the functions $g^{\gtrless}$ on the variables $\mathbf{p}$, ω, $\mathbf{r}$, and t, is determined by a system of two equations, one of which is of the form (for a detailed derivation of these equations see[5])

$$\begin{aligned} &\left[\omega-\frac{p^2}{2m}-U(\mathbf{r},t)-\operatorname{Re}\Sigma(\mathbf{p},\omega;\mathbf{r},t),\ g^{<}(\mathbf{p},\omega;\mathbf{r},t)\right]\\ &\qquad+[\operatorname{Re}g(\mathbf{p},\omega;\mathbf{r},t),\ \Sigma^{<}(\mathbf{p},\omega;\mathbf{r},t)]\\ &=\Sigma^{<}(\mathbf{p},\omega;\mathbf{r},t)g^{>}(\mathbf{p},\omega;\mathbf{r},t)-\Sigma^{>}(\mathbf{p},\omega;\mathbf{r},t)g^{<}(\mathbf{p},\omega;\mathbf{r},t). \end{aligned} \tag{3}$$

The second equation is obtained from this by replacing $g^{<}$ by $g^{>}$ and $\Sigma^{<}$ by $\Sigma^{>}$.

The square brackets in (3) are generalized Poisson brackets, defined as

$$[A,B]=\frac{\partial A}{\partial\omega}\frac{\partial B}{\partial t}-\frac{\partial A}{\partial t}\frac{\partial B}{\partial\omega}-\frac{\partial A}{\partial\mathbf{p}}\frac{\partial B}{\partial\mathbf{r}}+\frac{\partial A}{\partial\mathbf{r}}\frac{\partial B}{\partial\mathbf{p}},$$

$\Sigma(\mathbf{p}, \omega; \mathbf{r}, t)$ is the electronic self-energy part. As is well known, the function $\Sigma(1, 1')$ has a δ-like singularity as $t_1 \to t_1'$. If we separate the singularity, then the remaining part consists of two analytic functions $\Sigma^{>}(1, 1') = \Sigma(1, 1')$ for $t_1 > t_1'$ and $\Sigma^{<}(1, 1') = \Sigma(1, 1')$ for $t_1 < t_1'$. The Fourier transforms of the functions $\Sigma^{\lessgtr}$ with respect to the variables $\mathbf{x}$ and τ are specified in the following manner:

$$\Sigma^{\gtrless}(\mathbf{p},\omega;\mathbf{r},t)=\pm i\int d\mathbf{x}\,d\tau\,\Sigma^{\gtrless}(\mathbf{x},\tau;\mathbf{r},t)\exp[-i\mathbf{p}\mathbf{x}+i\omega\tau].$$

The quantities $\Sigma^{\gtrless}$ in Eq. (3) must be expressed in terms of the functions $g^{\gtrless}$ with the aid of some approximation. In the simplest case this can be an expansion in terms of the coupling constants and the exact functions $g^{\gtrless}$. It is necessary to add one more initial condition. As seen from the definitions (1) and (2), this condition should be in the form $g^{\gtrless} = G^{\gtrless}$ prior to the turning on of the interaction ($G^{\gtrless}$ are the equilibrium values of the correlators (2)).

Equation (3) and its complement have one exact integral

$$\begin{aligned} g(\mathbf{p},z;\mathbf{r},t)&=\int\frac{d\omega}{2\pi}\frac{g^{>}(\mathbf{p},\omega;\mathbf{r},t)+g^{<}(\mathbf{p},\omega;\mathbf{r},t)}{z-\omega}\\ &=[z-\xi_{\mathbf{p}}-U(\mathbf{r},t)-\Sigma(\mathbf{p},z;\mathbf{r},t)]^{-1}. \end{aligned} \tag{4}$$

Here z is the complex frequency and ξ_p is the energy of an electron with momentum p and is reckoned from the Fermi level.

Equation (3) together with relation (4) comprise the generalized Boltzmann equation. The latter can be obtained under certain additional simplifying assumptions (see[5]). The left-hand side of (3) corresponds to the field part of the Boltzmann equation, but with allowance for the kinetic effects of the interaction, i.e., with allowance for the change in the connection between the energy and the momentum as a result of the interaction. The right-hand side of this equation is the analog of the collision integral. Since we are interested in the limit $\zeta \ll 1$, we divide the collision integral into three parts: electron-electron ($\Sigma_{e}^{\gtrless}$), impurity ($\Sigma_{i}^{\gtrless}$), and phonon ($\Sigma_{f}^{\gtrless}$). The impurity and the phonon parts of the collision integral coincide in order of magnitude with the field part, while the electron-electron part is ζ^{-1} times larger$^{1)}$. In the zero-order approximation in ζ we therefore obtain the equation

$$\Sigma_{e}^{>}(\mathbf{p},\omega;\mathbf{r},t)g_0^{<}(\mathbf{p},\omega;\mathbf{r},t)-\Sigma_{e}^{<}(\mathbf{p},\omega;\mathbf{r},t)g_0^{>}(\mathbf{p},\omega;\mathbf{r},t)=0.$$

The solution of this equation consists of the functions

$$g_0^{\gtrless}(\mathbf{p},\omega;\mathbf{r},t)=G^{\gtrless}(\mathbf{p}-m\mathbf{v}(\mathbf{r},t),\bar{\omega}_{\mathbf{p}};\beta(\mathbf{r},t),\ \mu(\mathbf{r},t)). \tag{5}$$

Here $\mathbf{v}(\mathbf{r}, t)$ is the local drift velocity, $\beta(\mathbf{r}, t)$ is the local reciprocal temperature, $\mu(\mathbf{r}, t)$ is the local chemical potential, and

$$\bar{\omega}_{\mathbf{p}}=\omega-\mathbf{p}\mathbf{v}(\mathbf{r},t)+{}^1/_2 mv^2(\mathbf{r},t)+U(\mathbf{r},t)+\varepsilon_0-\mu(\mathbf{r},t).$$

The arbitrary functions $\mathbf{v}(\mathbf{r}, t)$, $\beta(\mathbf{r}, t)$, and $\mu(\mathbf{r}, t)$ are determined from the balance conditions for the energy, the number of particles, and momentum. Let us set up the momentum balance equation; to this end, we multiply (3) by $\mathbf{p}$ and integrate with respect to $\mathbf{p}$ and ω. After performing these operations, we can assume that the electric field is constant and homogeneous. Then the momentum-balance takes the form

$$\begin{aligned} e\mathbf{E}n=2\int\frac{d\omega\,d\mathbf{p}}{(2\pi)^4}\mathbf{p}\{&[\Sigma_i^{>}(\mathbf{p},\omega)+\Sigma_f^{>}(\mathbf{p},\omega)]g_0^{<}(\mathbf{p},\omega)-[\Sigma_i^{<}(\mathbf{p},\omega)\\ &+\Sigma_f^{<}(\mathbf{p},\omega)]g_0^{>}(\mathbf{p},\omega)\}, \end{aligned} \tag{6}$$

where n is the electron density:

$$n=2\int\frac{d\omega\,d\mathbf{p}}{(2\pi)^4}g^{<}(\mathbf{p},\omega).$$

In the derivation of (6) we made use of the fact that the term containing $\Sigma_{e}^{\gtrless}$ vanishes (without allowance for

Umklapp processes), since the total momentum in electron-electron collisions remains unchanged. The formal dependence on r in relation (6) and in the expression for n drops out, because the balance condition for the particle number yields $\epsilon_0 = \mu(\mathbf{r}) - U(\mathbf{r})$. Actually, Eq. (6) connects v with the intensity of the field E, i.e., it gives the current-voltage characteristic. Generally speaking, it is necessary here to write down the energy balance equation, from which it is possible to obtain β as a function of E and v. Therefore, a dependence on E also appears in the right-hand side of (6). In this paper the $\beta(E, v)$ relation will not be taken into account, since it gives rise to heat-induced nonlinear effects that can be excluded by using a pulsed measurement technique. We shall therefore not need the energy-balance equation. In addition, we shall disregard the terms containing $\Sigma_f^{\gtrless}$. Allowance for these terms is important in the energy-balance equation, but in relation (6) they lead, as shown in[1], to a noticeable NED only for $v \gg s$. We shall consider below the NCED that results from the terms with $\Sigma_i^{\gtrless}$ at $v \approx T/p_0$, which is much less than s, if $T \ll \Theta_0$ (p_0 is the Fermi momentum and Θ_0 is the Debye temperature).

3. IMPURITY SELF-ENERGY PART

To obtain the explicit form of the function E(v) it is necessary to know the values of $\Sigma_i^{\gtrless}$ as functionals of the functions $g^{\gtrless}$. It suffices here to know the functional dependence of the equilibrium $\Sigma_i^{\gtrless}$ on $G^{\gtrless}$, inasmuch as the form of the functional connection does not change in the case of slowly-varying perturbations (allowance for these changes leads to corrections on the order of $\tau_1\epsilon_0$ and $\lambda_1 p_0$ in Eq. (3), where τ_1 and λ_1 are the characteristic time and the characteristic distance over which the perturbation changes). The sought relation can be obtained rather simply by perturbation theory in terms of the coupling constant with exact G-functions in the s-d exchange model[6], if we confine ourselves to logarithmic accuracy in the region $T > T_K$. In the more general case, the determination of $\Sigma_i^{\gtrless}$ entails considerable difficulties. In studies based on the s-d exchange model it is possible to separate three basic approaches: the Abrikosov method of selective summation of the perturbation-theory[6], the Nagaoka method of equations of motion[7] and its later modification by Abrikosov[8], and the Suhl-Maleev S-matrix method. As to perturbation theory, its results are valid if $T > T_K$ and coincide in this region with the results of the remaining methods. The Suhl-Maleev method and the formulation of the theory in terms of the Nagaoka equations of motion, as shown by Zittartz[11], are fully equivalent, it must be emphasized, however, that in the S-matrix method, and consequently also in the Nagaoka method, no account is taken of the intermediate multiparticle states (see[10,12]), and no one has estimated the resultant error. The formulation of the method of equations of motion, proposed by Abrikosov, leads to results that differ from those of the remaining methods at $T \ll T_K$ and seem to agree better with the experimental results in the limit of very low temperatures. Unfortunately, the results of Abrikosov's theory cannot be directly applied to the technique of the generalized kinetic equations, which is used here, inasmuch as in this theory the only two correlators $g^{\gtrless}$ will not suffice in the non-equilibrium case, but it is necessary to introduce eight additional correlators. We therefore confine ourselves in this paper to the accuracy obtained with the Nagaoka method. It is also attractive because the equations of motion have an exact equilibrium solution for definite types of state densities $N(\omega)$. The latter was first demonstrated by Zittartz and by Muller-Hartmann[13]. This solution for the t matrix on the upper and lower (±) edges of the cut along the real ω axis is of the form

$$t(\omega_\pm) = \pm\frac{1}{2\pi i N(\omega)}\left\{1 - \frac{X(\omega_\pm)\exp[\mp i\varphi(\omega)]}{[|X(\omega_\pm)|^2 + (\pi J N(\omega))^2 S(S+1)]^{1/2}}\right\}. \tag{7}$$

Here $\omega_\pm = \omega \pm i\delta$ ($\delta = +0$),

$$X(z) = 1 + JR(z) - |F(z)|^2 (J/2)^2 S(S+1),$$

$$F(z) = \frac{1}{N}\sum_{\mathbf{k}} \frac{1}{z - \xi_{\mathbf{k}}} = \int_{-\infty}^{\infty} d\xi \frac{N(\xi)}{z - \xi},$$

J is the interaction constant in the local s-d exchange model, N is the number of electrons,

$$R(z) = \frac{1}{2N}\sum_{\mathbf{k}} \frac{2f_{\mathbf{k}} - 1}{z - \xi_{\mathbf{k}}} = -\frac{1}{2}\int_{-\infty}^{\infty} d\xi \frac{N(\xi)}{z - \xi} \operatorname{th}\frac{\beta\xi}{2}, \tag{8}$$

$$\varphi(\omega) = \frac{1}{2\pi} \mathrm{P}\int_{-\infty}^{\infty} \frac{d\Omega}{\omega - \Omega} \ln[|X(\Omega_+)|^2 + (\pi J N(\Omega))^2 S(S+1)],$$
$$f_{\mathbf{k}} = [1 + \exp(\beta\xi_{\mathbf{k}})]^{-1}, \tag{9}$$

where P is the symbol of the principal value of the integral, and S is the spin of the impurity. Relation (7) was obtained for the case when the electrons interact only with one impurity, and the two matrix was introduced here with the aid of the formal relation

$$G_{\mathbf{k}\mathbf{k}'}(z) = G_{\mathbf{k}}^0(z)\delta_{\mathbf{k}\mathbf{k}'} + N^{-1} G_{\mathbf{k}}^0(z) t(z) G_{\mathbf{k}'}^0(z),$$

where $G_{\mathbf{k}\mathbf{k}'}$ is the equilibrium Green's function and $G_{\mathbf{k}}^0$ is the Green's function for the noninteracting electrons.

In the case when the impurity density is finite, it is necessary to average over their positions. This operation can be easily carried out by neglecting the interaction between impurities and the correlations in different scattering acts. We then obtain in place of $G_{\mathbf{k}\mathbf{k}'}$ a Green's function that is diagonal in the momentum, and whose self-energy part is given by the relation

$$\Sigma(\omega_\pm) = c\,t(\omega_\pm), \quad c = n_i/n. \tag{10}$$

This relation, unfortunately, still does not solve the problem of finding the values of $\Sigma^{\gtrless}$ as functionals of $G^{\gtrless}$, since it is the result of solving the equations of motion. To find them, we proceed in the following manner. We define the discontinuity of Σ on the cut:

$$\Gamma(\omega) = i[\Sigma(\omega_+) - \Sigma(\omega_-)].$$

We calculate $\Gamma(\omega)$ by perturbation theory accurate to terms of third order in J, using relations (7) and (10):

$$\Gamma(\omega) \approx \frac{\pi c}{2N} J^2 S(S+1) N(\omega)[1 - 2J \operatorname{Re} R(\omega)]. \tag{11}$$

At the same time, it is easy to calculate $\Gamma(\omega)$ with the same accuracy, by means of a formal expansion in J and the exact G functions:

$$\Gamma(\omega) \approx \frac{\pi e}{2N} J^2 S(S+1) N(\omega)\left[1 + \frac{J}{n} \mathrm{P}\int \frac{d\Omega}{2\pi} \cdot \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{G^>(\mathbf{p}, \Omega) - G^<(\mathbf{p}, \Omega)}{\omega - \Omega}\right] \tag{12}$$

Comparing (11) and (12), we obtain R(z) as a functional of $G^{\gtrless}$:

$$R(z;\ G^>;\ G^<) = -\frac{1}{2n}\int \frac{d\Omega}{2\pi}\int \frac{d\mathbf{p}}{(2\pi)^3} \frac{G^>(\mathbf{p}, \Omega) - G^<(\mathbf{p}, \Omega)}{z - \Omega}. \tag{13}$$

Performing similar operations for the quantities $\Sigma^{\gtrless}$, we obtain their functional connection with $G^{\gtrless}$:

$$\Sigma^{\gtrless}(\omega; G^{>}; G^{<}) = \int \frac{d\mathbf{p}}{(2\pi)^3} G^{\gtrless}(\mathbf{p}, \omega) \frac{\Gamma(\omega; G^{>}; G^{<})}{2\pi n N(\omega)}. \tag{14}$$

In this relation, $\Gamma(\omega; G^{>}; G^{<})$ is a functional obtained as a result for substituting in (7) the functional $R(z; G^{>}; G^{<})$ in place of $R(z)$.

4. DIFFERENTIAL RESISTANCE

Relation (14) allows us to calculate with the aid of formula (6) the connection between the electric field and the drift velocity of the carriers. To this end it is necessary to replace $G^{\gtrless}$ in (14) by $g_0^{\gtrless}$, which can be represented, as seen from (5) and from the boundary conditions for the equilibrium function G, in the form

$$\begin{aligned} g_0^{>}(\mathbf{p}, \omega) &= A(\mathbf{p}-m\mathbf{v}, \bar{\omega}_\mathbf{p})[1-f(\bar{\omega}_\mathbf{p})], \\ g_0^{<}(\mathbf{p}, \omega) &= A(\mathbf{p}-m\mathbf{v}, \bar{\omega}_\mathbf{p}) f(\bar{\omega}_\mathbf{p}), \end{aligned} \tag{15}$$

where $A(p, \omega)$ is the spectral intensity, which is easily determined from relation (4):

$$A(\mathbf{p}, \omega) = \frac{\Gamma(\omega)}{[\omega - \xi_\mathbf{p} - \operatorname{Re}\Sigma(\omega)]^2 + \Gamma^2(\omega)/4}, \quad f(\omega) = [1+e^{\omega/T}]^{-1}.$$

With the aid of formulas (15) we can calculate the function $R(z)$ in the current state if we use the following interpolation procedure. The integrands are represented in a form such that the Fermi function $f(\omega)$ is everywhere replaced by the derivative $\partial f/\partial\omega$. This quantity is a sharp function with a sharp maximum at $\omega = 0$, but the functions that enter together with it in the integrands also undergo noticeable changes in the vicinity of this point, and consequently cannot be taken outside the integral sign. As an approximation we can therefore replace $\partial f/\partial\omega$ by a Lorentzian having the same smearing width and area under the curve. This approximation is very good because of the slow variation of the logarithmic functions. As a result of this approximation we get for $R(z)$ the expression

$$\operatorname{Re} R(\omega) = N(0)\psi(\omega), \quad \operatorname{Im} R(\omega_\pm) = \pm N(0)\chi(\omega). \tag{16}$$

Here

$$\psi(\omega) = \frac{1}{2p_0 v}\left\{\bar{\omega}_+ \ln\frac{C}{(\bar{\omega}_+^2+\gamma^2T^2)^{1/2}} - \bar{\omega}_- \ln\frac{C}{(\bar{\omega}_-^2+\gamma^2T^2)^{1/2}} + \gamma T\left(\operatorname{arc\,tg}\frac{\bar{\omega}_-}{\gamma T} - \operatorname{arc\,tg}\frac{\bar{\omega}_+}{\gamma T}\right)\right\},$$

$$\chi(\omega) = \frac{1}{2p_0 v}\left\{\bar{\omega}_+ \operatorname{arc\,tg}\frac{\bar{\omega}_+}{\gamma T} - \bar{\omega}_- \operatorname{arc\,tg}\frac{\bar{\omega}_-}{\gamma T} - \frac{\gamma T}{2}\ln\frac{\bar{\omega}_+^2+\gamma^2T^2}{\bar{\omega}_-^2+\gamma^2T^2}\right\},$$

where $C = 2e\alpha D\gamma/\pi$, D is the width of the d-band ($D \sim \epsilon_0$), while α and γ are constants ($\ln\alpha = 0.577$ is the Euler constant, $\gamma = 4/\pi$), $\bar{\omega}_\pm = \omega \pm p_0 v$, and e is the base of the natural logarithm.

It should be noted that the imaginary part of R, i.e., $\chi(\omega)$ can be calculated without using the interpolation approximation, and it is then equal to

$$\chi(\omega) = \frac{\pi T}{2p_0 v}\ln\left[\frac{\operatorname{ch}(\bar{\omega}_+/2T)}{\operatorname{ch}(\bar{\omega}_-/2T)}\right].$$

This function, however, hardly differs from that given in (16), so that within the limits of the accuracy assumed in[13], the final results will be identical. Still, it is more consistent to use the values of $\psi(\omega)$ and $\chi(\omega)$ obtained within the limits of one and the same approximation, since they are connected by the dispersion relations that follow from (13).

We break up the electric field in (6) into two parts: $E = E_1 + E_2 \cdot E_1$ contains, from the right side of the equation, only the scattering by the paramagnetic impurities, while E_2 will contain the remaining part, which is connected with the phonon and potential scattering. So far we have considered paramagnetic scattering without allowance for the potential scattering. As shown by Schotte[14], the potential scattering interferes strongly with the paramagnetic scattering and suppresses it noticeably. In the case when only s-scattering with phase-shift δ_0 is possible, the paramagnetic contribution to the scattering can be obtained from the presented formulas by multiplying the t-matrix in (7) by $\exp(2i\delta_0)$ and replacing the constant J by $J\cos^2\delta_0$. Taking these remarks into account, it is easy to obtain with the aid of relations (14)—(16) the expressions for E_1 in the form

$$E_1 = \frac{1}{2p_0 v^2 e}\int_{-p_0 v}^{p_0 v} d\omega \overline{\Gamma}(\omega)[(p_0 v)^2 - \omega^2]. \tag{17}$$

Here

$$\overline{\Gamma}(\omega) = \frac{c\cos 2\delta_0}{\pi}\left\{1 - \frac{\cos\bar{\varphi}(\omega)\operatorname{Re}\overline{X}(\omega) + \sin\bar{\varphi}(\omega)\operatorname{Im}\overline{X}(\omega)}{[|\overline{X}(\omega)|^2 + (\pi\lambda)^2 S(S+1)]^{1/2}}\right\},$$

$$\operatorname{Re}\overline{X}(\omega) = 1 + \lambda\psi(\omega) - (\pi\lambda)^2 S(S+1)/4; \quad \operatorname{Im}\overline{X} = \lambda\chi(\omega);$$

$$\bar{\varphi}(\omega) = \frac{1}{2\pi}\mathrm{P}\int\frac{d\Omega}{\omega-\Omega}\ln[|\overline{X}(\Omega)|^2 + (\pi\lambda)^2 S(S+1)]; \quad \lambda = JN(0)\cos^2\delta_0,$$

N(0) is the density of states on the Fermi surface.

Further simplifications of relation (17) will be carried out in the following manner. Recognizing that the function $\overline{\Gamma}(\omega)$ is even in ω, we integrate in (17) from zero to $p_0 v$, and in this interval $\overline{\Gamma}(\omega)$ decreases monotonically with increasing ω ($p_0 v \ll \epsilon_0$), so that we can use the mean-value theorem and take $\overline{\Gamma}$ out at a certain average point $\omega_{av} = \kappa p_0 v$ ($0 < \kappa < 1$). To determine κ, we expand $\overline{\Gamma}(\omega)$ in the vicinity of ω_{av}, after which the integral in (17) can be easily calculated. Comparing the result obtained in this manner with the result given by the mean-value theorem, we obtain an equation for the determination of κ:

$$\sum_{n=1}^{\infty}\frac{C_n(\varkappa)}{n!}\frac{\partial^n}{\partial\varkappa^n}\overline{\Gamma}(\varkappa p_0 v) = 0,$$

where

$$C_n(\varkappa) = \frac{2}{n+1}\left[\frac{1}{n+3} + \sum_{m=1}^{n}\binom{n+1}{m}\frac{(-\varkappa)^m}{n+3-m}\right].$$

Naturally, the exact solution of this equation is impossible, since it is equivalent to an exact calculation of the integral in (17). However, since the terms of these equations decrease quite rapidly with increasing n, we can construct a perturbation theory for κ. It is easy to find that

$$\varkappa \approx 0.375 + 0.029 f(p_0 v/T),$$

where $f > 0$ for $p_0 v < T$ and $f < 0$ for $p_0 v > T$, and with a modulus on the order of $|f| \sim 1$.

Thus, we can write for E_1 the relation

$$E_1 = 2p_0^2\overline{\Gamma}(\varkappa p_0 v)v/3e. \tag{18}$$

The nonlinearity becomes most clearly pronounced in the case of an antiferromagnetic coupling constant $J < 0$. The reason is that for $J < 0$ there exists a small temperature scale $T_K \ll \epsilon_0$. On the other hand, if $J > 0$, then $T_K \gg \epsilon_0$, where T_K is so defined that $\overline{X}(0) = 0$ at $T = T_K$. We shall therefore consider below only the case $J < 0$.[2]

Using expression (18), we can easily obtain the contribution made to the differential resistivity ρ_S by the scattering from the paramagnetic impurities. Equations that are lucid enough are obtained only in limiting cases. If we consider the limit when $p_0v \ll T$, then in this case we obtain for the phase shift φ the expression

$$\varphi=\varkappa\frac{p_0v}{\gamma T}\frac{\Lambda_T}{\Lambda_T^2+\pi^2S(S+1)},$$

where $\Lambda_T = \ln(T_K/T)$. We then find from (17) and (18) that

$$\rho_s(T, N)=\rho(T)+\Delta\rho(T, v). \qquad (19)$$

Here $\rho(T)$ is the constant temperature-dependent part of the resistivity (see, e.g.,[12]):

$$\rho(T)=\rho_0\cos 2\delta_0\left\{1+\frac{\Lambda_T}{[\Lambda_T^2+\pi^2S(S+1)]^{1/2}}\right\},$$

and $\Delta\rho(T, v)$ is the nonlinear increment produced in the current state:

$$\Delta\rho(T, v)=-\frac{\rho_0\cos 2\delta_0\pi^2S(S+1)\varkappa}{3[\Lambda_T^2+\pi^2S(S+1)]^{3/2}}\left(\frac{p_0v}{\gamma T}\right)^2 \times\left[5\left(1+\frac{\varkappa^2}{3}\right)-\frac{\varkappa\Lambda_T}{\Lambda_T^2+\pi^2S(S+1)}\right], \qquad (20)$$

where $\rho_0 = mc/\pi e^2 nN(0)$.

As seen from (20), the characteristic scale of variation of the drift velocity is T/p_0. To compare the orders of magnitude, we recall that the resistivity due to the potential scattering for the s wave is equal to $\rho_i = 2\rho_0\sin^2\delta_0$. Thus, the coefficient of the dimensionless ratio can be of the order of ρ_i. The differential resistance as $v \to 0$ is maximal or minimal, depending on the sign of $\cos 2\delta_0$. The latter agrees with the qualitative reasoning presented in the introduction. According to this reasoning, turning on the field is similar to a certain extent to raising the temperature (without taking the heating effects into account). Indeed, if $\cos 2\delta_0 > 0$, then $\rho(T)$ increases with decreasing temperature, and when $\cos 2\delta_0 < 0$ then $\rho(T)$ decreases.

We consider now the case when $p_0v \gg T$. It is then easy to obtain for φ the expression

$$\varphi(\varkappa p_0v)\approx-\frac{1}{4}\left\{\varkappa+(1-\varkappa^2)\ln\frac{1+\varkappa}{1-\varkappa}\right\}\Big/[\Lambda_v^2+\pi^2S(S+1)],$$

where $\Lambda_v = \ln(\gamma T_K/p_0v)$. In this case we can put $\varphi = 0$ in the expression for $\Gamma(\kappa p_0v)$. It must be emphasized that when $p_0v \ll T$, then φ is also small, but the phase makes a contribution of the same order to $\Delta\rho(T, v)$ as the corrections due to the remaining terms. If $p_0v \gg T$, then the main contribution, ~1, is made by the real part of $\overline{X}$, and φ provides increments on the order of $10^{-2}/S(S+1)$. Thus, the principal term in the resistivity ρ_S takes the form

$$\rho_s(v)=\rho_0\cos 2\delta_0\left\{1+\frac{1}{[\Lambda_v^2+\pi^2\varkappa^{-2}S(S+1)]^{1/2}}\left[\Lambda_v+\frac{\pi^2S(S+1)\varkappa^{-2}}{\Lambda_v^2+\pi^2\varkappa^{-2}S(S+1)}\right]\right\}. \qquad (21)$$

In this case the characteristic scale of variation of the drift velocity is T_K/p_0.

A simple expression for $\rho_S(T, v)$, can be obtained also for arbitrary ratios of p_0v and T, but in the weak-coupling limit $|\lambda| \ll 1$. This approximation conforms with the results obtained by Abrikosov by the selected summation method[6] at $v = 0$:

$$\rho_s(T, v)=\rho_c\frac{\pi^2}{4}S(S+1)\frac{\cos 2\delta_0}{\Lambda_{vT}^2}\left[1+\frac{2B(v, T)}{\Lambda_{vT}}\right], \qquad (22)$$

where

$$\Lambda_{vT}=\ln\left\{\frac{\gamma T_K}{[(\varkappa_+^2p_0^2v^2+\gamma^2T^2)(\varkappa_-^2p_0^2v^2+\gamma^2T^2)]^{1/4}}\right\}$$

$$B(v, T)=1+\frac{\gamma T}{2p_0v}\left\{\operatorname{arctg}\left(\frac{\varkappa_-p_0v}{\gamma T}\right)-\operatorname{arctg}\left(\frac{\varkappa_+p_0v}{\gamma T}\right)\right\},$$

$$\varkappa_\pm=\varkappa\pm 1.$$

In the derivation of (22) it was assumed that $|\Lambda_{vT}| \gg 1$. The term containing $B(v, T)$ was retained because of its relatively rapid variation with v, so that it can be revealed experimentally in an investigation of the derivatives of the resistivity with respect to the current. If we put $\delta_0 = 0$, then the principal part of ρ_S in (22) can be expressed in the more usual form

$$\rho_s=\rho_B\left\{1+|J|N(0)\ln\frac{[(\varkappa_+^2p_0^2v^2+\gamma^2T^2)(\varkappa_-^2p_0^2v^2+\gamma^2T^2)]^{1/4}}{\gamma D}\right\}, \qquad (23)$$

where ρ_B is the first Born approximation for the resistivity due to scattering by paramagnetic impurities, $\rho_B = \frac{1}{4}\rho_0\pi J^2S(S+1)$. The condition $|\Lambda_{vT}| \gg 1$ then takes the form

$$\left\{1+|J|N(0)\ln\frac{\max(p_0v, T)}{D}\right\}\gg|J|N(0).$$

The figures shows the dependence of the relative change of the resistivity $\Delta\rho_S(T, v)/\rho_S(T, 0)$ as a function of the parameter $k = p_0v/\gamma T_K$ at different values of the temperature $y = T/T_K$. It is seen from this figure that the deviations from Ohm's law become more clearly manifest with decreasing temperature.

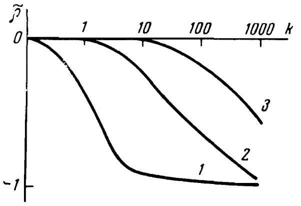
5. CONCLUSION

Let us estimate the conditions under which it is possible to observe the effect in question. As seen from the presented calculation, a quantitative comparison of the obtained data with experiments is possible only when the fastest relaxation process is the relaxation in the electronic subsystem. The attuning of the electrons to the impurities and the phonons is the next stage in the establishment of the balance in the complete system. It must be emphasized that exchange of virtual phonons between the electrons can greatly influence the character of the electron-electron relaxation. This was demonstrated recently in[15]. When the virtual exchange is taken into account, the condition $\zeta \ll 1$ must be replaced by $\zeta \ll \epsilon^2(2p_0, 0)/\epsilon^2(2p_0, T)$, where $\epsilon(p, \omega)$ is the dielectric constant of the metal as a function of the wave vector p and the frequency ω. This condition is much less stringent than $\zeta \ll 1$ for certain metals. The physical reason for the appearance of the square of the ratio $\epsilon(p, \omega)$ at $\omega = 0$ and $\omega = T$ is that in the electron-electron interaction the individual electron can change its energy by an amount ~T, whereas in elastic scattering by an impurity the energy remains unchanged, and consequently $\omega = 0$.

For semimetals such as bismuth, with low Fermi energy ($\epsilon_0 \sim 10^{-2}$ eV), the current state at $T \sim 10°$K can be realized at concentrations $c \lesssim 10^{-2}$ ($c = n_i/n$). The same order of c can be obtained also for copper, if it is recognized that

$$\varepsilon^2(2p_0, 0)/\varepsilon^2(2p_0, T)=(T/\Theta_A)^4,$$

when $T \gg \Theta_A$, where Θ_A is the value of the pole of the



Plot of $\rho = [\rho_S(v, T)-\rho_S(0, T)]/\rho_S(0, T)$ against $k = p_0v/\gamma T_K$ in the weak-coupling limit $|\lambda| \ll 1$ for different values of the temperature: $y = T/T_K = 2$ (curve 1), $y = 10$ (curve 2), and $y = 100$ (curve 3).

dielectric constant at $p = 2p_0$ (see[15]). It must be emphasized that this situation is typical only of metals for which $\Theta_A \ll \Theta_0$. According to the data given in[15] for copper we have $\Theta_A^2 \approx 10^{-4}\Theta_0^2$, so that the condition $\Theta_0^2 \gg T^2 \gg \Theta_A^2$ is satisfied at $T \sim 10^\circ$K.

The order of magnitude of the field-dependent part of the resistivity can be easily determined by using expression (23). In the case when the principal contribution to the resistivity is due to the potential scattering, the drift velocity can be expressed in terms of the electric field: $v \sim eEl/p_0$, and we then obtain for $|\lambda| \ll 1$ and $eEl > T$

$$\Delta\rho_s(E)/\rho_B \cong |\lambda| \ln(\varepsilon_0/eEl).$$

As seen from this formula, when the field changes by a factor e, the relative change of the resistivity amounts to $\sim|\lambda|$. It is also easy to estimate that discernible deviations from Ohm's law can be expected in bismuth at sufficiently small current densities $\sim 10^5$ A/cm^2. Caution must be exerted, however, to prevent heating effects from coming into play. To this end it is desirable to perform the experiment under pulsed measurement conditions and with the most favorable heat-exchange conditions.

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[1] Yu. M. Ivanchenko, Zh. Eksp. Teor. Fiz. 59, 820 (1970) [Sov. Phys.-JETP 32, 449 (1971)].
[2] L. A. Zil'berman and Yu. M. Ivanchenko, Fiz. Tverd. Tela 14, 1355 (1972) [Sov. Phys.-Solid State 14, 1164 (1972)].
[3] A. A. Abrikosov, Usp. Fiz. Nauk 97, 403 (1969) [Sov. Phys.-Uspekhi 12, 168 (1969)].
[4] J. J. Kondo, Progr. Theor. Phys., 32, 37 (1964).
[5] L. Kadanoff and T. Baym, Quantum Statistical Mechanics, Benjamin, 1962.
[6] A. A. Abrikosov, Physics 2, 5 (1965).
[7] Y. Nagaoka, Phys. Rev., 138, A1112 (1965).
[8] A. A. Abrikosov, Zh. Eksp. Teor. Fiz. 53, 1078, 2109 (1967) [Sov. Phys.-JETP 26, 641, 1192 (1968)].
[9] H. Suhl, Phys. Rev., 138, A515 (1965).
[10] S. V. Maleev, Zh. Eksp. Teor. Fiz. 51, 1940 (1965); 53, 1038 (1967) [Sov. Phys.-JETP 24, 1300 (1966); 26, 620 (1968)].
[11] J. Zittartz, Zs. Phys., 217, 43 (1968).
[12] K. Fischer, Phys. Stat. Sol., 46, 11 (1971).
[13] J. Zittartz and E. Müller-Hartmann, Zs. Phys., 212, 380 (1968).
[14] D. Schotte, Zs. Phys., 212, 467 (1968).
[15] A. I. Akhiezer, I. A. Akhiezer, and V. G. Bar'yakhtar, Zh. Eksp. Teor. Fiz. 65, 342 (1973) [Sov. Phys.-JETP 38, 167 (1974)].

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1) Such a subdivision implies that $\Sigma_i^{\gtrless}$ and $\Sigma_f^{\gtrless}$ contain the renormalized interaction constants due to the long-range part of the Coulomb interaction, while $\Sigma_e^{\gtrless}$ contains only the short-range part of the interaction, which appears when the polarization of the medium is taken into account.

2) For weak coupling $|\lambda| \ll 1$ we have $T_K = (2\alpha D/\pi)\, e^{[-1/|\lambda|]}$.