The correspondence principle in the problem of the self-energy of the electron

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It is shown that a correspondence exists between the quantum and the classical theories of the self-energy of the electron as $\hbar \to 0$. This result signifies, in particular, that the divergences in classical and in quantum electrodynamics have a common nature. The apparent violation of the correspondence principle in the problem of the self-energy, noted in the literature, can be ascribed to the inadequacy of the usual expansion procedure in terms of the parameter $e^2/\hbar c$ as $\hbar \to 0$.

1. INTRODUCTION

In the literature (cf., for example,[1]) there is a widely held opinion that the correspondence principle is not satisfied in the problem of the self-energy of the electron and that the self-energy problem in quantum theory has an entirely different nature than in classical theory. Such an opinion originated on the basis of a comparison of the expression for the proper mass of the electron in quantum electrodynamics in second order of perturbation theory[2-4]

$$\delta m_{qu}=m_0\left[\frac{3e^2}{2\pi\hbar c}\left(\ln\frac{\hbar}{r_0 m_0 c}+\frac{1}{4}\right)+O\left(\frac{e^4}{\hbar^2c^2}\right)\right] \qquad (1)$$

(m_0 is the initial mass, r_0 is the cutoff distance) with the well-known expression for the proper mass in classical electrodynamics[1,5]

$$\delta m_{cl}=e^2/2r_0c^2. \qquad (2)$$

Actually, however, in order to clarify the question of the correspondence for $\hbar \to 0$ between the quantum electrodynamical and the classical theories of proper energy it is not possible to base the discussion on the results of perturbation theory, since the expansion parameter $e^2/\hbar c$ in this limit becomes large.

In the present paper we shall directly, without utilizing perturbation theory, investigate the limiting value of the quantum electrodynamical operator for the electron mass as $\hbar \to 0$ and we shall show that it agrees with the classical expression (2). The method utilized here is close to the method of Thirring[6] developed in order to prove the validity of the Thompson formula for the exact cross section for the Compton effect in the limit of soft photons. The result obtained in particular means that the divergences in classical and quantum electrodynamics have a common nature[1)]. This conclusion is significant in seeking a generalization of electrodynamics associated with the introduction into the theory of an additional fundamental constant (possibly of the gravitational constant[9]) which in a fundamental manner removes divergences at small distances, since it becomes sensible and heuristically useful to begin the search for such a generalization at the level of the classical ($\hbar = 0$) field theory in order to generalize the ideology and the method developed here to quantum theory.

2. SELF-ENERGY OF THE ELECTRON AS $\hbar \to 0$

The self-energy (mass) of the electron in quantum electrodynamics is determined by the value of the mass operator $i\Sigma(p)$ on the mass surface[2)]

$$i\Sigma(p_0)=\frac{ie_0^2}{4\pi^3\hbar c}\int\gamma_\mu G(p_0-q)\Gamma_\nu(p_0-q,p_0)D_{\mu\nu}(q)\,d^4q, \qquad (3)$$

$p_0^2 = -m^2c^2$, ($\hat{p}_0 \to imc$ after carrying all the matrix operations). Here e_0 is the initial charge, m is the physical mass defined as the pole of the electron Green's function G(p).

Our problem is to find the value of the mass operator (3) in the limit as $\hbar \to 0$ and to compare this value with the classical result (2). In order that such a comparison should be sensible it is necessary first of all to guarantee that the cut-off parameter should be introduced in quantum and in classical theories in an equivalent manner. In order to guarantee convergence of the integral in (3) we adopt the introduction of the usual Feynman modification of the free propagation function for the photon

$$1/q^2\to\Lambda^2/q^2(q^2+\Lambda^2). \qquad (4)$$

Further, we introduce the cutoff length $r_0 = \hbar/\Omega$ corresponding to the cutoff momentum Λ. With this the replacement in (4) assumes the form

$$1/q^2\to 1/q^2[(r_0q/\hbar)^2+1]=1/\hbar^2k^2(r_0^2k^2+1), \qquad (5)$$

where $k = q/\hbar$ is the propagation vector of the photon. In classical electrodynamics it is evidently necessary to modify in an equivalent manner the Green's function for the wave equation:

$$D(k^2)=1/k^2\to 1/k^2(r_0^2k^2+1). \qquad (6)$$

We first obtain the form of the classical proper energy of the electron when the cutoff is introduced by the method of (6). In classical electrodynamics for an electron at rest at the origin we have: $\rho(\mathbf{x}) = e\delta(\mathbf{x})$,

$$\delta m_{cl}=\frac{1}{2c^2}\int\rho(\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x}=\frac{e}{2c^2}\varphi(0); \qquad (7)$$

$$\varphi(\mathbf{x})=4\pi\int D(\mathbf{x}-\mathbf{y},x_0-y_0)\rho(\mathbf{y})\,d^4y,$$
$$\varphi(0)=4\pi e\int D(0,x_0-y_0)\,dy_0=\frac{e}{4\pi^3}\int D(k^2)e^{ik_0(x_0-y_0)}\,d^4k\,dy_0$$
$$=\frac{e}{2\pi^2}\int D(k^2)\delta(k_0)\,d^4k. \qquad (8)$$

Substituting this expression into (7) and taking (6) into account we obtain

$$\delta m_{cl}=\frac{e^2}{4\pi^2c^2}\int\frac{\delta(k_0)}{k^2}\frac{d^4k}{r_0^2k^2+1}=\frac{e^2}{2r_0c^2}, \qquad (9)$$

i.e., we obtain exactly formula (2).

We now show that the quantum electrodynamical mass operator (3) with the regularization (5) gives in the limit as $\hbar \to 0$ the same result (9). In order to accomplish this we go over in (3) to integration over the propagation vector:

$$i\Sigma(p_0)=\frac{ie_0^2\hbar^3}{4\pi^3c}\int\gamma_\mu G(p_0-\hbar k)\Gamma_\nu(p_0-\hbar k,p_0)D_{\mu\nu}(\hbar k)\,d^4k \qquad (10)$$

and examine the limit of this expression as $\hbar \to 0$. Since the integral in (10) converges after introduction of the regularization (5) we can carry out the transition to the limit under the integral sign. Due to the presence in (10) of the factor $\hbar^3$ it is evidently sufficient to pick out in the integrand only the terms that are most singular as $\hbar \to 0$ and which, as we shall see, behave as $\sim\hbar^{-3}$.

The vertex function does not have a singularity as $\hbar \to 0$[4] [3)]:

$$\lim \Gamma_\nu(p_0-\hbar k,\ p_0)=\lim \Gamma_\nu(p_0,\ p_0)=\gamma_\nu \lim Z_1^{-1}. \tag{11}$$

But, as may easily be seen, the Green's functions $G(p_0 - \hbar k)$ and $D_{\mu\nu}(\hbar k)$ have at the point $\hbar = 0$ poles respectively of the first and second order. Indeed, in accordance with the spectral properties of the electron Green's function the following representation holds[4]:

$$G(p)=Z_2 S_c(p)\,[1+C_1(p^2)+\hat{p}C_2(p^2)],$$

where $S_c(p) = (-\hat{p} + imc)^{-1}$, while $C_1(p^2)$ and $C_2(p^2)$ are scalar functions having zeros on the mass surface: $C_1(p_0^2) = C_2(p_0^2) = 0$. Taking these properties into account we obtain retaining in the limit only the most singular term:

$$\lim G(p_0-\hbar k)=\lim Z_2(\hbar\hat{k}-\hat{p}_0+imc)^{-1}=Z_2^0(\hat{p}_0+imc)/2\hbar(kp_0), \tag{12}$$

where $Z_2^0 = \lim Z_2$.

Similarly for the photon Green's function in virtue of its spectral properties and taking the Feynman modification (5) into account we have (since the quantity (10) is gauge-independent, we choose the simplest gauge for $D_{\mu\nu}$)[4]:

$$D_{\mu\nu}(\hbar k)=\delta_{\mu\nu}\frac{Z_3}{i\hbar^2k^2}\frac{1}{r_0^2k^2+1}[1+d(\hbar^2k^2)],$$

where the scalar function $d(q^2)$ has a zero at $q^2 = 0$: $d(0) = 0$. As a result, retaining only the most singular term, we obtain

$$\lim D_{\mu\nu}(\hbar k)=\frac{Z_3^0\delta_{\mu\nu}}{i\hbar^2k^2(r_0^2k^2+1)}, \tag{13}$$

where $Z_3^0 = \lim Z_3$.

Substituting the limiting expressions (11)—(13) into (10) we obtain after taking into account that $Z_1 = Z_2$:

$$\lim i\Sigma(p_0)=\lim_{\hbar\to 0}\frac{e_0^2Z_3\hbar}{4\pi^3c^2}\int \gamma_\mu(\hbar\hat{k}-\hat{p}_0+imc)^{-1}\gamma_\mu\frac{d^4k}{k^2(r_0^2k^2+1)}. \tag{14}$$

Before going over to the final limit $\hbar = 0$ in (14) we note that expression (14) arising as $\hbar \to 0$ without using perturbation theory formally coincides with the expression for the proper energy of an electron in the first nonvanishing order of the renormalized perturbation theory. This result is analogous to the well-known result of Thirring[6,4] which states that the exact matrix element for the Compton effect in the soft photon limit coincides with the matrix element in the first nonvanishing order of the renormalized perturbation theory.

Going over in (14) to the limit $\hbar = 0$ in accordance with (12) and making (after carrying out all the matrix operations in (14)) the substitution $\hat{p}_0 \to imc$ we obtain the integral

$$\lim i\Sigma(p_0)=\frac{e_0^2Z_3^0}{4\pi^3c^2}\int\frac{imc}{kp_0}\frac{d^4k}{k^2(r_0^2k^2+1)}, \tag{15}$$

in which the usual (Feynman) rule for going around the poles is assumed. Since expression (15) is Lorentz-invariant we evaluate it in the rest system of the electron. In doing this we utilize

$$\frac{imc}{kp_0+io}\to\frac{i}{-k_0+io}=\frac{1}{i}\mathrm{P}\frac{1}{k_0}+\pi\delta(k_0). \tag{16}$$

The first term in (16) gives zero contribution to the integral in (15) due to the fact that the integrand is odd in the variable k_0. The second term leads to the integral

$$\lim i\Sigma(p_0)=\frac{e_0^2Z_3^0}{4\pi^3c^2}\int\frac{\delta(k_0)\,d^4k}{k^2(r_0^2k^2+1)}=\frac{e_0^2Z_3^0}{2r_0c^2}, \tag{17}$$

which coincides with expression (9) arising in classical electrodynamics for $e_0^2Z_3^0 = e^2$.[4)]

Thus, the correspondence principle is satisfied in the problem of the self-energy of the electron.

In conclusion we note that in the case $\hbar = 0$ the integral (14) for $r_0 \to 0$ diverges logarithmically in accordance with (1), while its limiting form (15) for $\hbar = 0$ diverges linearly. This is associated with the fact that in the limit as $\hbar \to 0$ the degree of the variable in the denominator of (14) is reduced by unity. As can be easily verified, as $\hbar \to 0$ the region $k > mc/\hbar$, in which one can not neglect in the denominator the term $\hbar^2k^2$ compared to $\hbar(kp_0)$, introduces into the integral (14) a relative contribution $\sim\hbar c/e^2$ which tends to zero in the limit under consideration.

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[1)]We note that earlier with the aid of a definite departure beyond the framework of perturbation theory it was shown [7,8] that in quantum electrodynamics, just as in classical electrodynamics, the true divergence is entirely contained in the proper energy of the electron and has a linear nature (and not a logarithmic one to which perturbation theory leads); the renormalization constants $Z_{1,2,3}$ are finite, with $0 < Z_3 < 1$.

[2)]We shall basically utilize the notation adopted in the text [4].

[3)]Here and later lim denotes the limit as $\hbar \to 0$.

[4)]It can be shown (for example, based on the results of Schwinger's article [10] that $Z_3^0 = \lim Z_3 = 1$, so that, strictly speaking, in the limit as $\hbar \to 0$ one should assume that $e^2 \to e_0^2$.

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