Influence of exciton effect on electroabsorption in semiconductors

I. A. Merkulov

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences
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The influence of Coulomb interaction between an electron and a hole on the electric absorption coefficient in semiconductors is considered. Asymptotic expressions are obtained for the absorption coefficient and are valid far from the exciton maximum as well as in the vicinity of the maximum. It is demonstrated that, in a broad region of the photon energy deficit, allowance for the exciton effect results in a considerable increase of the absorption coefficient compared to that predicted by the Keldysh and Franz calculations. On the other hand, if the photon energy deficit exceeds the exciton binding energy by several units, the dependence of the absorption coefficient on the photon energy and the homogeneous electric field strength is mainly determined by the rapidly varying exponent in the Keldysh–Franz expression.

1. In the presence of an electric field, a superconductor can absorb light with a quantum energy $\hbar\omega$ smaller than the width E of the forbidden band (the Keldysh-Franz effect). This effect is connected with the tunneling of the electrons and holes in the forbidden band.

Keldysh[1] and Franz[2] obtained for the electroabsorption coefficient the asymptotic expression

$$K_{KF} \sim \frac{E}{E_g-\hbar\omega}\exp\left\{-\frac{4\sqrt{2\mu}}{3\hbar eE}(E_g-\hbar\omega)^{3/2}\right\}, \qquad (1)$$

which is valid if the argument of the exponential is large (the electron and hole must tunnel far in order to meet). Here μ is the reduced mass of the electron hole and E is the intensity of the applied electric field. In the derivation of (1), no account was taken of the Coulomb interaction between the electron and the hole (the exciton effect).

The influence of the exciton effect on electroabsorption has been the subject of many studies, in which either the problem was unjustifiably simplified, or else the problem was solved with a computer and the results presented in the form of plots (see the bibliography in[3,4]).

In this paper we consider the influence of the Coulomb interaction of an electron and a hole on electroabsorption in semiconductors. We obtain asymptotic expressions for the electroabsorption coefficient K, which are valid both far from the exciton maximum (16) and in the region of the maximum (36)[1]. The main condition for the validity of our results is the requirement that the electron and hole tunnel far enough towards each other. The quantitative criteria for this are the inequalities (3) and (4).

A comparison of the obtained expressions with the classical Keldysh-Franz formula shows that the exciton effect leads, in a wide range of photon-energy deficit values, to a considerable increase of the electroabsorption coefficient. The exciton peak has a Lorentz contour with half-width determined by the probability of ionization of the exciton in the presence of the electron field. The position of the peak is determined by the shift of the exciton level in the homogeneous electric field (the Stark effect). At a photon-energy deficit comparable with the exciton energy, the dependence of K on the photon energy differs significantly from that obtained by Keldysh and Franz (this fact was observed by Dow and Redfield in their numerical calculations[3]). However, if the photon energy deficit exceeds several

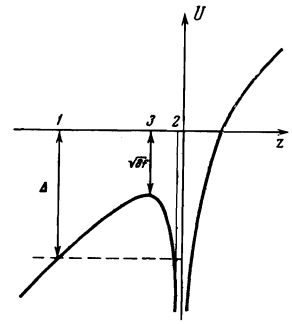

FIG. 1. Potential energy U as a function of the position on the z axis.

units of the exciton binding energy, then, even though the exciton effect does lead to a considerable increase of K in comparison with K_{KF}, the dependence of the electroabsorption coefficient on $\hbar\omega$ and E is determined mainly by the rapidly-varying exponential in the Keldysh-Franz expression (1). With increasing photon-energy deficit and with increasing electric-field intensity, the correction coefficient that relates K and K_F decreases slowly and becomes equal to unity ($K = K_{KF}$) in the limit of large values of $E_g - \hbar\omega$ and E. Thus the suggestion made by Dow and Redfield, that electroabsorption is never described by the Keldysh-Franz formula, turns out to be incorrect.

2. The wave function of an electron-hole pair produced in an external electric field upon absorption of a photon of energy $\hbar\omega$ satisfies the Schrödinger equation

$$(\nabla^2+2/r-fz)\Psi(\mathbf{r})=\Delta\Psi(\mathbf{r}), \qquad (2)$$

in which the length is measured in units of the exciton Bohr radius a, $\Delta = (E_g - \hbar\omega)/R$, R is the exciton Rydberg number, and $f = eEa/R$ is the electric field in dimensionless units (the z axis is chosen parallel to the homogeneous electric field). According to[6], the absorption coefficient is proportional to $|\Psi(0)|^2$.

The potential energy of the relative motion of the electron and of the hole as a function of the position on the z axis is shown schematically in Fig. 1. It is seen that the exciton effect leads to a lowering of the potential barrier under which the electron and hole must tunnel. In a wide range of values of Δ, this factor plays the principal role in the increase of the electroabsorption coefficient in comparison with the results of Keldysh and Franz. The presence of quasibound states in the Coulomb well leads to an additional increase of

the absorption coefficient when the photon-energy deficit is of the order of the exciton binding energy.

We first solve the problem under the assumption that the energy Δ is not too close to the exciton level. Then, at small distances from the center of the exciton, we neglect in (2) the homogeneous electric field and use the exact solution of the Coulomb problem. At large distances from the exciton center, we seek the wave function in the quasiclassical approximation. These two solutions must be matched in some region between the points 2 and 3 (Fig. 1) where, on the one hand, the quasiclassical approximation is valid and the asymptotic value of the Coulomb wave function has already been reached ($r \gg 1/\sqrt{\Delta}$), and on the other hand it is still possible to neglect the homogeneous electric field ($r \ll 1/\sqrt{f}$). Such a region exists if

$$f \ll \Delta. \tag{3}$$

In addition, as seen from Fig. 1, it becomes necessary for the electron and hole to tunnel towards each other if

$$\sqrt{8f} < \Delta. \tag{4}$$

The inequalities (3) and (4) are the fundamental conditions for the validity of all our results.

Out of all the solutions of (2), we are interested only in solutions that do not vanish identically on the z axis. Therefore, taking the cylindrical symmetry of the problem into account, we seek solutions that do not depend on the azimuthal angle (m = 0).

When constructing the quasiclassical wave function, it is necessary to solve the problem of particle motion in a force field in the form $F = f + 2r/r^3$. A decisive simplification is obtained because the only region that plays an important role in the determination of the absorption coefficient is a small region in the vicinity of the z axis.

As will be shown below, it suffices to find the solution only in a vicinity of the z axis in which

$$\frac{F_\perp}{F_\parallel} = \frac{2\rho}{fr^3 + 2|z|} \ll 1, \tag{5}$$

where ρ is a component of the radius vector $\mathbf{r}$ and is perpendicular to the field direction.

In the region where $r^2 \gg 1/f$, which includes the turning point 1, the condition (5) is satisfied at all ρ. In the matching region, where $1/\Delta \ll r^2 \ll 1/f$, the important role is played, as will be shown below, by the vicinity of the point z where $\rho \leq r^{1/2}/\Delta^{1/4}$ and the condition (5) is also satisfied. Thus, when solving the quasiclassical problem it can be assumed that the summary field is directed along the z axis, and consequently the motion is in a one-dimensional potential of the form

$$U = fz + 2/z. \tag{6}$$

To the left of the turning point 1 there are two types of quasiclassical solutions:

$$\varphi_1 = \frac{A(p_\perp)}{\sqrt{p}} \cos\left\{ \int_{z_1(p_\perp)}^{z} p\, dz - \frac{\pi}{4} \right\} J_0(p_\perp \rho), \tag{7}$$

$$\varphi_2 = \frac{B(p_\perp)}{\sqrt{p}} \sin\left\{ \int_{z_1(p_\perp)}^{z} p\, dz - \frac{\pi}{4} \right\} J_0(p_\perp \rho), \tag{8}$$

where

$$p = (-\Delta - fz - 2/z - p_\perp^2)^{1/2}$$

is the quasiclassical momentum component parallel to the field f, $p_\perp$ is the quasimomentum component perpendicular to the z axis,

$$z_1(p_\perp) = [-(\Delta + p_\perp^2) - \sqrt{(\Delta + p_\perp^2)^2 + 8f}]/2f$$

is the quasiclassical turning point, J_0 is a Bessel function, and $A(p_\perp)$ and $B(p_\perp)$ are normalization constants. The functions φ_2 under the barrier correspond to exponentially increasing solutions, while the functions φ_1 correspond to exponentially decreasing solutions. If the photon-energy deficit Δ is not too close to the exciton-level energy, then it can be assumed that the wave function has no components that increase exponentially under the barrier. Then in the z-axis vicinity defined by the condition (5) the quasiclassical solution under the barrier takes the form

$$\Psi_3^-(\Delta, p_\perp, r) = \frac{A(p_\perp)}{2\sqrt{|p|}} \exp\left\{ -\int_{z_1(p_\perp)}^{z} |p|\, dz \right\} J_0(p_\perp \rho), \tag{9}$$

where

$$|p| = (\Delta + fz + 2/z + p_\perp^2)^{1/2} = p_0 + p_\perp^2/2p_0,$$
$$p_0 = (\Delta + fz + 2/z)^{1/2},$$

and the constant $A(p_\perp)$ is determined from the corresponding normalization of the wave function.

We assume $z = -r[1 - (\pi - \theta)^2/2]$, $\rho = r(\pi - \theta)$ near the z axis. Then in the matching region, where $1/\sqrt{\Delta} \ll r \ll 1/\sqrt{f}$, we have

$$\Psi_3^-(\Delta, p_\perp, r) \approx \frac{A(p_\perp)}{2\Delta^{1/4}} \exp\left\{ -\int_{z_1(0)}^{-r} p_0\, dz \right.$$
$$\left. - \frac{r\sqrt{\Delta}}{2}(\pi - \theta)^2 - p_\perp^2 \int_{z_1(0)}^{-r} \frac{dz}{2p_0} \right\} J_0(p_\perp r(\pi - \theta)). \tag{10}$$

Owing to the presence in (10) of a rapidly decreasing exponential, it can be assumed that $p_\perp \leq f^{1/2}/\Delta^{1/4}$ in the argument of the bessel function, and therefore $p_\perp r(\pi - \theta) \leq f^{1/2}/\Delta^{3/4} \ll 1$. We can therefore replace J_0 by unity.

We now proceed to determine the solution of the region $r \ll 1/\sqrt{f}$, where the external electric field can be neglected. In this region we expand the solution in spherical harmonics. It is seen from (10) that the expansion includes all the terms with m = 0, but we are interested in only one of them, the spherically symmetrical one ($l = 0$), inasmuch as all the others vanish at the center of the exciton. The sought S-function, which is regular at the origin, takes the form (see[7])

$$\Psi_2(r) = \Psi(0) e^{-r\sqrt{\Delta}} F(1 - 1/\sqrt{\Delta},\ 2,\ 2r/\sqrt{\Delta}), \tag{11}$$

where F is a confluent hypergometric function.

At $r \gg 1/\sqrt{\Delta}$, using the asymptotic form of the function F, we obtain

$$\Psi_2^+(r) = \Psi(0) e^{r\sqrt{\Delta}} / \Gamma(1 - 1/\sqrt{\Delta})\, (2r\sqrt{\Delta})^{1 + 1/\sqrt{\Delta}}. \tag{12}$$

In this expression we have neglected the damped part of the wave function, a neglect justified if $\hbar\omega$ is not too close to the exciton peak. The appropriate criteria will be derived below (see (20) and (21)).

Expression (12) must be matched to the spherically symmetrical part of (10). Averaging Ψ_3^- over the angles, we obtain

$$\frac{1}{4\pi} \int \Psi_3^-\, d\Omega = \frac{A(p_\perp)}{2\Delta^{1/4}(2r\sqrt{\Delta})} \exp\left\{ -\int_{z_1(0)}^{-r} p_0\, dz - p_\perp^2 \int_{z_1(0)}^{-r} \frac{dz}{2p_0} \right\}. \tag{13}$$

We note that the essential region in the integration of of the angles is $(\pi - \theta) \leq 1/r^{1/2}\Delta^{1/4}$, and consequently $\rho \leq r^{1/2}/\Delta^{1/4}$. This justifies the use of a one-dimensional potential in the matching region

Comparing now (13) with (12), we obtain the connection between $A(p_\perp)$ and the value of the wave function at the center of the exciton:

$$\Psi(\Delta, p_\perp, 0) = \frac{A(p_\perp)\Gamma(1-1/\sqrt{\Delta})}{2\Delta^{1/4}} \exp\left\{-\Phi(\Delta, f) - \frac{\alpha(\Delta, f)}{2} p_\perp^2\right\} \quad (14)$$

where

$$\Phi(\Delta, f) = \int_{z_1}^{z_2} \left(\Delta + \frac{2}{z} + fz\right)^{1/2} dz + \frac{1}{\sqrt{\Delta}} \ln \sqrt{\Delta} + \frac{1}{\sqrt{\Delta}},$$

$$\alpha(\Delta, f) = \int_{z_1}^{z_2} \frac{dz}{(\Delta + fz + 2/z)^{1/2}}.$$

We call attention to the fact that the principal term in the expression for Φ is the tunnel integral taken on a trajectory that coincides with the z axis (Fig. 1) and is evaluated between the turning points 1 and 2. Using the fact that $\Delta \gg f$, we obtain

$$\Phi(\Delta, f) \approx \frac{2\Delta^{3/2}}{3f} - \frac{1}{\sqrt{\Delta}} \ln\left(\frac{8\Delta^{3/2}}{f}\right) \qquad \alpha(\Delta, f) \approx 2\Delta^{1/2}/f. \quad (15)$$

The first term in the expression for Φ corresponds to the result of the Keldysh-Franz theory, while the second is a correction due to the lowering of the potential barrier as a result of the Coulomb interaction of the electron and hole. The correction term in (15) is much smaller in magnitude than the principal term. However, being in the argument of the exponential, it can greatly increase the value of $\Psi(\Delta, p_\perp, 0)$ (by several orders of magnitude). The corrections to the value of α turn out to be insignificant, inasmuch as the quantity α goes over from the argument of the exponential to the preexponential factor after summing over all the values of the perpendicular momentum.

The value of the constant $A(p_\perp)$ in (14) can be determined from the renormalization of the wave function. Since the exciton effect has in this case a small influence on the form of the wave function to the left of the turning point 1, it is obvious that the corrections to $A(p_\perp)$ resulting from the Coulomb interaction are small and have practically no influence on the answer. Then, integrating $|\Psi(\Delta, p_\perp, 0)|$ over the values of the perpendicular momentum, we obtain a final expression for the electroabsorption coefficient

$$K = K_{KF}\left[\Gamma(1-1/\sqrt{\Delta}) \exp\left(\frac{1}{\sqrt{\Delta}} \ln \frac{8\Delta^{3/2}}{f}\right)\right]^2, \quad (16)$$

where K is the electroabsorption coefficient with allowance for the electron-hole interaction, and K_{KF} is the usual Keldysh-Franz absorption coefficient.

$$K_{KF} = \frac{C}{32\pi^2} \frac{f}{\Delta} \exp\left[-\frac{4\Delta^{3/2}}{3f}\right],$$

C is a proportionality constant relating the square of the modulus of the wave function at the center of the exciton with the transition probability. We recall that Δ is the photon-energy deficit measured in units of the Bohr energy of the exciton, and f is the electric field in units of the Bohr energy, divided by the Bohr radius.

We see that there are two factors that increase the absorption. One is the Γ function describing the variation of Ψ near the Coulomb center, and the other is the exponential factor due to the lowering of the potential barrier as a result of the Coulomb interaction. The Γ function makes an appreciable contribution to the increase of the absorption coefficient if the photon energy deficit is comparable in magnitude with the exciton energy. Abrupt changes of the Γ function in this region cause the dependence of K on Δ to differ strongly from the results of Keldysh and Franz. On the other hand, if the photon energy $\hbar\omega$ is far enough from the exciton peak in absorption (Δ amounts to several units), then the principal role in the increase of the absorption coefficient is played by the lowering of the potential barrier. The correction coefficient in (16) then changes much more slowly than K_{KF}, and the dependence of K on the photonenergy and on the electric field intensity is determined mainly by the exponential in the Keldysh-Franz expression.

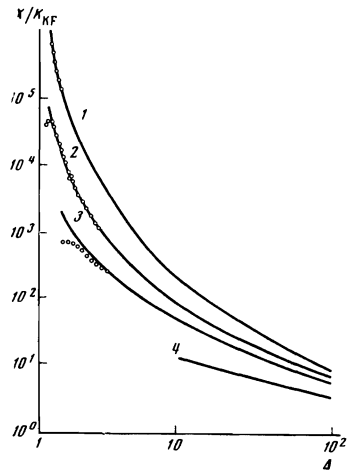

FIG. 2. The ratio K/K_F as a function of the quantum energy deficit Δ: 1–f = 0.1; 2–f = 0.4; 3–f = 1.0; 4–f = 10. The points show the results of the calculations by Dow and Redfield [3].

The values of the ratio K/K_{KF} calculated with the aid of (16) are shown in Fig. 2. We see that the correction coefficient reaches several orders of magnitude in a wide range of values of the photon-energy deficit. At the same time, with increasing value of Δ, the ratio K/K_{KF} decreases slowly and in the limit of large Δ we have $K = K_{KF}$.

The numerical calculations of Dow and Redfield[3], which were made in a narrow range of photon energy deficits ($1 \leq \Delta \leq 3$), are in good agreement with our results. However, the hypothesis advanced in[3] that the electroabsorption coefficient is never described by the Keldysh-Franz formula is incorrect.

3. As Δ approaches the bound-exciton energy, the value of K calculated from formula (6) increases without limit. As already noted, this is due to the fact that our assumptions are not valid near the exciton level.

At this point we consider the electroabsorption in the case when

$$|\Delta - 1| \ll 1. \quad (17)$$

The condition (3) then leads to $f \ll 1$.

To solve the problem in this case it is necessary to take into account two factors. First, when (17) is satisfied the Γ function in the denominator of the exponentially-increasing asymptotic form of the S wave in (12) causes the latter to decrease strongly (at $\Delta = 1$ there is no exponentially growing part in the asymptotic form at all). Thus, in the region where the solutions are

matched, the decreasing part of the asymptotic Coulomb wave function may turn out to be comparable with or even larger than the increasing part. We, on the other hand, did not take into account the decreasing asymptotic form in the derivation of (16), and this has led to a divergence in the expression for the electroabsorption coefficient.

Second, owing to the smallness of the coefficient of the increasing asymptotic expression for the Coulomb wave function, even small corrections to the wave function, due to the electric field, can significantly alter its form. In particular, a change takes place in the value of Δ at which the increasing asymptotic expression vanishes (the Stark effect).

Thus, in the analysis of electroabsorption near the exciton peak we must take additional account of the resonant scattering and of the influence of the homogeneous field on the form of the wave function in the vicinity of the Coulomb center. The corrections that must be introduced in the Coulomb wave functions to account for the homogeneous field can be determined by successive approximations. However, the corrections of interest to us can be obtained also in a simpler manner. Assuming that in the region $r \ll 1/\sqrt{f}$, $r \ll 1/|\Delta - 1|$ the wave function depends analytically on Δ and f, we can write for the spherically-symmetrical part of the solution

$$\Psi_2(\Delta, f, p_\perp, \mathbf{r}) \approx \Psi_2(1, 0, p_\perp, r) + \frac{\partial \Psi_2}{\partial \Delta}(\Delta - 1) + \frac{\partial \Psi_2}{\partial f} f + \frac{1}{2}\frac{\partial^2 \Psi_2}{\partial f^2} f^2, \quad (18)$$

where the partial derivatives are taken at $\Delta = 1$ and $f = 0$.

Using the known expression for the level shift in a homogeneous electric field, we find that at $\Delta = 1 + 9f^2/2$ the increasing asymptotic of the function Ψ_2 should vanish. Consequently, $\partial\Psi_2/\partial f$ should not have a growing asymptotic form, and the growing parts in the asymptotic forms $-9\partial\Psi_2/\partial\Delta$ and $\partial^2\Psi_2/\partial f^2$ should be identical. Substituting in (18) the explicit forms of Ψ_2 and $\partial\Psi_2/\partial\Delta$, and retaining only the most significant terms in the asymptotic form of the obtained expression, we find that the spherically symmetrical part of the Coulomb solution takes in the matching region the form

$$\Psi_2(\Delta, f, p_\perp, \mathbf{r}) \approx \left[e^{-r\sqrt{\Delta}} + \frac{e^{r\sqrt{\Delta}}}{2(2r\sqrt{\Delta})^2}\left(\Delta - 1 - \frac{9f^2}{2}\right)\right]\Psi(\Delta, f, p_\perp, 0). \quad (19)$$

If

$$|\Delta - 1| \gg {}^9/_2 f^2, \quad (20)$$

then expression (19) coincides, under the condition (17), with the previously used asymptotic form of the S wave (12). Thus, the condition (20) is a criterion for the validity of (16) and supplements (3) and (4).

If the inequality (20) is not satisfied, but

$$|\Delta - 1 - {}^9/_2 f^2| \gg f e^{-2/rf}, \quad (21)$$

then the decreasing part of the asymptotic form of (19) can still be neglected, but the correction quadratic in f will now play an essential role. Allowance for this correction leads in fact to the corresponding Stark shift of the level and to a change of the argument of the function in (16).

Finally, if condition (21) is also not satisfied, then it is necessary to take into account both the increasing and the decreasing parts in (19). Naturally, the quasiclassical wave function now contains not only the functions φ_1 [Eq. (7)], which are joined to the growing asymptotic form of the Coulomb S wave, but also the functions φ_2 [Eq. (8)], which are joined to the exponentially decreasing part of expression (20). Owing to the strong scattering, the perpendicular momentum is generally speaking not conserved.

To the left of the turning point 1, we seek a solution in the form

$$\Psi_1(\Delta, p_\perp, \mathbf{r}) = \int \left\{ \frac{A(p_\perp)\delta(p_\perp - q)}{\sqrt{p}} \cos\left[\int_{z_1(p_\perp)}^{z} p\, dz - \frac{\pi}{4}\right] + \frac{B(p_\perp, q)}{\sqrt{p}} \sin\left[\int_{z_1(q)}^{z} p\, dz - \frac{\pi}{4}\right]\right\} J_0(q\rho)\, dq. \quad (22)$$

The first term in this expression corresponds to the usual nonresonant process, while the second is due to resonant scattering. Using the results (18) obtained above, we find directly

$$A(p_\perp) = \Psi(\Delta, p_\perp, 0)\left(\Delta - 1 - \frac{9f^2}{2}\right)\exp\left[\Phi(\Delta, f) + \frac{\alpha(\Delta, f)}{2} p_\perp^2\right]. \quad (23)$$

We now determine the expression for $B(p_\perp, q)$. The part of the quasiclassical solution that increases under the barrier takes in the vicinity of the z axis the form

$$\Psi_2^+(\Delta, p_\perp, \mathbf{r}) = \int \frac{B(p_\perp, q)}{\sqrt{p_0}} \exp\left\{\int_{z_1(q)}^{z} p_0\, dz + q^2 \int_{z_1(q)}^{z} \frac{dz}{2p_0}\right\} J_0(q\rho)\, dq. \quad (24)$$

At the same time, the exponentially decreasing part of the asymptotic form of the S wave (19) is represented in the vicinity of the z axis ($z < 0$) in the form

$$\Psi_2^-(\Delta, p_\perp, \mathbf{r}) \approx \Psi(\Delta, p_\perp, 0)\exp\{z\sqrt{\Delta} + \rho^2\sqrt{\Delta}/2z\}. \quad (25)$$

Expanding Ψ_2^- in Bessel functions and comparing the obtained expression with (24), we find

$$B(p_\perp, q) = {}^1/_2 \exp[-\Phi(\Delta, f) - \alpha(\Delta, f) q^2/2] q \Psi(\Delta, p_\perp, 0). \quad (26)$$

We note than the function Ψ_2^- decreases rapidly with increasing distance from the z axis. Thus, an important role in the matching is played by the region $\rho \leq 2r^{1/2}/\Delta^{1/4}$, in which the complicated potential can be approximately replaced by a one-dimensional one.

Using (22), (23), and (26), we obtain the explicit form of the wave function to the left of the point 1:

$$\Psi_1(\Delta, p_\perp, \mathbf{r}) = \Psi_1(\Delta, p_\perp, 0) \int \left\{\left(\Delta - 1 - \frac{9f^2}{2}\right)\exp[\Phi(\Delta, f) + \alpha(\Delta, f) p_\perp^2/2] \times \frac{\delta(p_\perp - q)}{\sqrt{p}} \cos\left[\int_{z_1(p_\perp)}^{z} p\, dz - \frac{\pi}{4}\right] + \frac{\exp[-\Phi(\Delta, f) - \alpha(\Delta, f) q^2/2]}{2\sqrt{p}} \times q \sin\left[\int_{z_1(q)} p\, dz - \frac{\pi}{4}\right]\right\} J_0(q\rho)\, dq. \quad (27)$$

It is seen from this expression that $p_\perp$ is no longer a "good" quantum number here, since the wave functions with different values of $p_\perp$ are not orthogonal:

$$\int \Psi(\Delta_1, p_1, \mathbf{r})\Psi(\Delta_2, p_2, \mathbf{r})\, d^3r = [\beta(p_1)\delta(p_1 - p_2) + \gamma]\delta(\Delta_1 - \Delta_2), \quad (28)$$

where

$$\beta(p_1) = \frac{2\pi^2 \exp[2\Phi(\Delta, f) + \alpha(\Delta, f) p_1^2]}{p_1}\left[\Delta - 1 - \frac{9f^2}{2}\right], \quad \gamma = \frac{\pi^2 \exp[-2\Phi(\Delta, f)]}{4\alpha(\Delta, f)} \quad (29)$$

Here and below we assume for simplicity that $\Psi(\Delta, p_\perp, 0) = 1$.

To find the absorption coefficient, it would be necessary to construct out of the functions $\Psi(\Delta, p_\parallel, \mathbf{r})$ an orthonormalized set of functions $\chi(\Delta, \nu, \mathbf{r})$ where ν is a certain new quantum number that labels the corresponding degenerate states. However, the answer can be obtained also without resorting to orthonormalization in explicit form. Assume that we have found an orthonormal set $\chi(\Delta, \nu, \mathbf{r})$. We then have for the absorption coefficient

$$K(\Delta)=C\int|\chi(\Delta,\nu,0)|^2\,d\nu,$$

where C is the proportionality coefficient which was already used by us in (16).

Let us consider the more general expression

$$G(\Delta,\mathbf{r},\mathbf{r}')=\int\chi(\Delta,\nu,\mathbf{r})\chi(\Delta,\nu,\mathbf{r}')\,d\nu. \tag{30}$$

The function G is in fact independent of the concrete choice of the orthornomal set χ. A direct consequence of the completeness of the set χ in the class of the solutions of the wave equation (2), which are of interest to us, is the relation

$$\int G(\Delta,\mathbf{r},\mathbf{r}')\Psi(\Delta_1,p_1,\mathbf{r}')\,d^3\mathbf{r}'=\Psi(\Delta_1,p_1,\mathbf{r})\,\delta(\Delta-\Delta_1). \tag{31}$$

The validity of this relation can be easily verified by expanding $\Psi(\Delta, p_1, \mathbf{r})$ in terms of $\chi(\Delta, \nu, \mathbf{r})$. From (31) and (28) it follows that

$$\begin{aligned}&\iint G(\Delta,\mathbf{r},\mathbf{r}')\Psi(\Delta_1,p_1,\mathbf{r})\Psi(\Delta_2,p_2,\mathbf{r}')\,d^3r\,d^3r'\\&=[\beta(p_1)\delta(p_1-p_2)+\gamma]\delta(\Delta_1-\Delta_2)\delta(\Delta-\Delta_2).\end{aligned} \tag{32}$$

Another expression for the left-hand side of (32) can be obtained by using the fact that the system of functions Ψ, while not orthonormalized, is complete, i.e., each function χ can be expanded in terms of the functions Ψ. Then

$$G(\Delta,\mathbf{r},\mathbf{r}')=\iint F(\Delta,p_1,p_2)\Psi(\Delta,p_1,\mathbf{r})\Psi(\Delta,p_2,\mathbf{r}')\,dp_1\,dp_2, \tag{33}$$

where $F(\Delta, p_1, p_2)$ is a certain function that does not depend on the choice of the orthonormal set χ. Since $K \sim G(\Delta, 0, 0)$ and $\Psi(\Delta, p_\perp, 0) = 1$, the absorption coefficient can be easily expressed in terms of F:

$$K(\Delta)=C\iint F(\Delta,p_1,p_2)\,dp_1\,dp_2.$$

On the other hand, if we express G in terms of the integral of Ψ [Eq. (33)] and use (28), then

$$\begin{aligned}&\iint G(\Delta,r,r')\Psi(\Delta_1,p_1,r)\Psi(\Delta_2,p_2,r')\,d^3r\,d^3r'\\&=\Big\{\beta(p_1)\beta(p_2)F(\Delta_1,p_1,p_2)+\gamma\Big(\beta(p_1)\int F(\Delta,p_1,q)\,dq\end{aligned} \tag{34}$$

$$+\beta(p_2)\int F(\Delta,p_2,q)\,dq\Big)+\gamma^2\iint F(\Delta,q_1,q_2)\,dq_1\,dq_2\Big\}\delta(\Delta-\Delta_1)\delta(\Delta_1-\Delta_2).$$

Equating the right-hand sides of (32) and (34), we obtain an integral equation for $F(\Delta, p_1, p_2)$. Dividing the right- and left-hand parts of this equation by $\beta(p_1)\beta(p_2)$ and integrating with respect to p_1 and p_2, we get

$$K(\Delta)=C\iint F(\Delta,p_1,p_2)\,dp_1\,dp_2=C\int_0^\infty\frac{dq}{\beta(q)}\left(1+\gamma\int_0^\infty\frac{dq}{\beta(q)}\right)^{-1}. \tag{35}$$

Substituting expression (29) for $\beta(q)$ in (35) and evaluating the integrals, we obtain ultimately

$$K(\Delta)=\frac{Cx}{\pi^2(\delta^2x^2+1)}, \tag{36}$$

where

$$\delta=\Delta-1-\frac{9f^2}{2},\qquad x=\frac{8}{f}\exp\left\{\frac{4}{3}\frac{\Delta^{3/2}}{f}-\frac{2}{\sqrt{\Delta}}\ln\left(\frac{8\Delta^{3/2}}{f}\right)\right\},$$

We see that the exciton peak in electroabsorption has a simple Lorentz shape. The line half-width, as expected, is determined by the ionization probability of the quasibound state. We note that if the inequalities (17) and (20) are simultaneously satisfied then expressions (36) and (16) coincide.

Summarizing, we can state that the expressions obtained by us describe electroabsorption both in the case of energy deficits close to the exciton-level energy [Eq. (36)], and far from the level [Eq. (16)]. There exists a region, defined by inequalities (17) and (20), in which both expressions are valid simultaneously, i.e., the regions of applicability of formulas (16) and (36) overlap. The only condition for our results to be valid is the deep-tunneling requirement (3) and (4).

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[1)]The results for the case when the photon energy lies far enough from the exciton maximum were reported in a brief communication [5].

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